Fat and Thin Points for Tensor Decomposition

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FAT PART

$V_1, \dots, V_k$ finite dimensional vector spaces over $\mathbb{C}$, $(\mathbb{R})$

$$F \in V_1 \otimes \cdots \otimes V_k$$

$$F = \sum_{i=1}^{r} \lambda_i v_{1,i} \otimes \cdots \otimes v_{k,i}, \tag{1}$$

$v_{j,i} \in V_i$, $\lambda_i \in \mathbb{C}$.
$v_{1,i} \otimes \cdots \otimes v_{k,i}$ rank one element.

Definition

The minimum r s.t. F can be written as in (1) is the *rank* of F.

Linear and Multilinear algebra
Structured Tensor decomposition

One can ask to F and to its decomposition to have certain structure:

- $F \in S^d V$, $F = \sum_{i=1}^r \lambda_i v_i^{\otimes d}$,
  i.e. $F \in \mathbb{C}[x_0, \dots, x_n]_d$, $F = \sum_{i=1}^r \lambda_i L_i^d$ with L_i linear forms
- $F \in \bigwedge^d V$, $F = \sum_{i=1}^r \lambda_i v_{1,i} \wedge \cdots \wedge v_{d,i}$,
- $F \in S^{d_1} V_1 \otimes \cdots \otimes S^{d_k} V_k$, $F = \sum_{i=1}^r \lambda_i v_{1,i}^{d_1} \otimes \cdots \otimes v_{k,i}^{d_k}$
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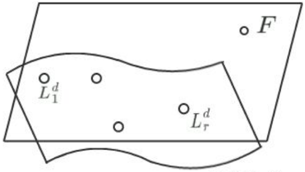
Rank of the generic element

Problem (Big Waring Problem)

F generic in $S^d V^* = K[x_0, \ldots, x_n]_d$, find the minimum r s.t. $F = \sum_{i=1}^r \lambda_i L_i^d$ with L_i linear forms.

$X_{n,d} = \{[L^d] \mid L \in S^1 V^*\} = \{[v^{\otimes d}] \mid v \in V\} \subset \mathbb{P}(S^d V^*)$ Veronese



$$\sigma_r(X_{n,d}) := \overline{\bigcup_{[L_1^d],\ldots,[L_r^d] \in X_{n,d}} \langle [L_1^d], \ldots, [L_r^d] \rangle}$$

$\mathrm{expdim}(\sigma_r(X_{n,d})) = \min\{rn + r - 1, \binom{n+d}{d} - 1\}$, [A,H '99]

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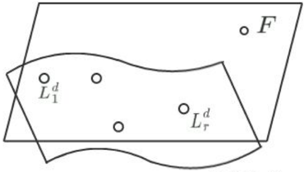

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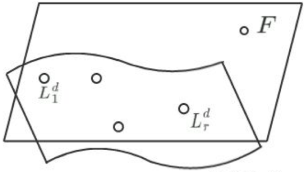

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$$\dim(\sigma_r(X_{n,d})) + 1 = \dim T_Q(\sigma_r(X_{n,d}))$$

$$\overset{\text{Terracini}}{=} \dim\langle T_{L_1^d}(X_{n,d}), \dots, T_{L_r^d}(X_{n,d})\rangle$$

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Apolarity

Apolarity for symmetric tensors (Veronese):
$S = K[x_0, \dots, x_n]$, $T = K[y_0, \dots, y_n]$

$$y^\alpha\left(x^{[\beta]}\right) = \begin{cases} x^{[\beta-\alpha]} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

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$$= \dim I(Z)_d = H(I(Z), d) \text{ with } Z = \wp_1^2 \cap \cdots \cap \wp_r^2$$

Rank of the generic element [Alexander, Hirschowitz '99]

Theorem (Alexander-Hirschowitz)

$H(K[x_0, \ldots, x_n]/\wp_1^2 \cap \cdots \cap \wp_s^2, d)$ (or $\dim(\sigma_s(\nu_d(\mathbb{P}^n)))$) is as expected except for:

- $d = 2$, $n \geq 2$, $s \leq n$;
- $d = 3$, $n = 4$, $s = 7$, $(\delta = 1)$;
- $d = 4$, $n = 2$, $s = 5$, $(\delta = 1)$;
- $d = 4$, $n = 3$, $s = 9$, $(\delta = 2)$;
- $d = 4$, $n = 4$, $s = 14$, $(\delta = 1)$.

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More than one Fat Point

Example

$P_1, P_2 \in \mathbb{P}^2$, $P_i \leftrightarrow \wp_i$, $I = \wp_1^2 \cap \wp_2^2$. Is the Hilbert function of I equal to the Hilbert function of 6 points of $\mathbb{P}^2$ in general position? No: the Hilbert function of 6 general points of $\mathbb{P}^2$ is $[1, 3, 6, 6 \ldots]$ and this would mean that I does not contain conics, but.........

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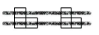


SHGH Conjecture

Conjecture (Segre, 1961)

If $\mathcal{L}_{2,d}(-\sum_{i=1}^{s} m_i P_i)$ is a special linear system, then there is a fixed double component for all curves through the scheme defined by $\wp_1^{\alpha_1} \cap \cdots \cap \wp_s^{\alpha_s}$.

Conjecture (Gimigliano, 1987)

For the linear system $\mathcal{L}_{2,d}(-\sum_{i=1}^{s} m_i P_i)$ one has the following possibilities:

1. *the system is non-special and its general member is irreducible;*
2. *the system is non-special, its general member is non-reduced, reducible, its fixed components are all rational curves, except for at most one (this may occur only if the system has dimension 0), and the general member of its movable part is either irreducible or composed of rational curves in a pencil;*
3. *the system is non-special of dimension 0 and consists of a unique multiple elliptic curve;*
4. **the system is special** *and it has some multiple rational curve as a fixed component.*

SHGH Conjecture

This problem is related to the question of what self-intersections occur for reduced irreducible curves on the surface X_s obtained by blowing up the projective plane at the s points.

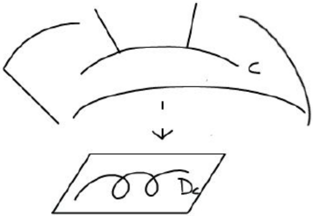
Blowing up the points introduces rational curves (infinitely many when $s > 8$) of self-intersection -1.

Each curve $C \subset X_s$ corresponds to a curve $D_C \subset \mathbb{P}^2$ of some degree d vanishing to orders m_i at the s points:

$$X_s \dashrightarrow \mathbb{P}^2$$

$$C \mapsto D_C$$

the self-intersection $C^2 = d^2 - m_1^2 - \cdots - m_n^2$.



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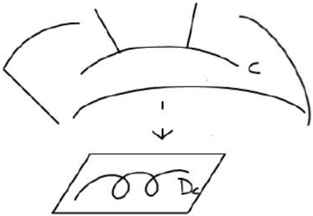

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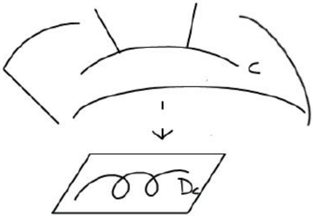

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An example of a curve D_C corresponding to a curve C s.t. $C^2 = -1$ on X_s is the line through two of the points; in this case, $d = 1$, $m_1 = m_2 = 1$ and $m_i = 0$ for $i > 2$, so we have $d^2 - m_1^2 - m_2^2 = 1 - 1 - 1 = -1$.

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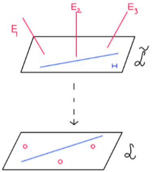
Blow-up $\pi : \widetilde{\mathbb{P}}^2 \dashrightarrow \mathbb{P}^2$ at $P_1, \dots, P_s$. Let $E_1, \dots, E_s$ be the exceptional divisors and let H be the pull-back of a general line of $\mathbb{P}^2$. The strict transform of the system $\mathcal{L} := \mathcal{L}_{2,d}(\sum_{i=1}^s m_i P_i)$ is the system $\widetilde{\mathcal{L}} = |dH - \sum_{i=1}^s m_i E_i|$.



Consider two linear systems $\mathcal{L} := \mathcal{L}_{2,d}(\sum_{i=1}^s m_i P_i)$, $\mathcal{L}' := \mathcal{L}_{2,d}(\sum_{i=1}^s m_i' P_i)$. Define intersection by using the intersection of their strict transforms on $\widetilde{\mathbb{P}}^2$:

$$\mathcal{L} \cdot \mathcal{L}' = \widetilde{\mathcal{L}} \cdot \widetilde{\mathcal{L}}' = dd' - \sum_{i=1}^s m_i m_i'.$$

Anticanonical class $-K := -K^2_{\widetilde{\mathbb{P}}^2}$ of $\widetilde{\mathbb{P}}^2$ corresponding to the linear system $\mathcal{L}_{2,3}(-\sum_{i=1}^s P_i)$, which we also denote by $-K$. The adjunction formula gives arithmetic genus $p_a(\widetilde{\mathcal{L}})$ of a curve in $\widetilde{\mathcal{L}}$ is:
$p_a(\widetilde{\mathcal{L}}) = \frac{\mathcal{L}\cdot(\mathcal{L}+K)}{2} + 1 = \binom{d-1}{2} \sum_{i=1}^s \binom{m_i}{2}$ which one defines to be $g_{\mathcal{L}}$ = the geometric genus of $\mathcal{L}$.

SHGH Conjecture

The theorem of Riemann-Roch then says that

$$\dim(\mathcal{L}) = \dim(\widetilde{\mathcal{L}}) = \mathcal{L} \cdot (\mathcal{L} - K) + h^1(\widetilde{\mathbb{P}}^2, \widetilde{L}) - h^2(\widetilde{\mathbb{P}}^2, \widetilde{L}) =$$

$$= \mathcal{L}^2 - g_{\mathcal{L}} + 1 + h^1(\mathbb{P}^2, \widetilde{\mathcal{L}}) = exp\dim(\mathcal{L}) + h^1(\mathbb{P}^2, \widetilde{\mathcal{L}})$$

because clearly $h^2(\widetilde{\mathbb{P}}^2, \widetilde{\mathcal{L}}) = 0$. Hence:

$$\mathcal{L} \text{ is non-special if and only if } h^0(\widetilde{\mathbb{P}}^2, \widetilde{\mathcal{L}}) \cdot h^1(\widetilde{\mathbb{P}}^2, \widetilde{\mathcal{L}}) = 0.$$

SHGH Conjecture

Now we can see how, in this setting, special systems naturally arise. Let us look for an irreducible curve C on $\widetilde{\mathbb{P}}^2$, corresponding to a linear system $\mathcal{L}$ on $\mathbb{P}^2$, which is expected to exist but (for ex) its double is not expected to exist. It translates in the following set of inequalities:

$$\begin{cases} exp\dim(\mathcal{L}) \geq 0 \\ g_{\mathcal{L}} \geq 0 \\ exp\dim(2\mathcal{L}) \leq -1 \end{cases}$$

This system is equivalent to:

$$\begin{cases} C^2 - C \cdot K \geq 0 \\ C^2 + C \cdot K \geq -2 \\ 2C^2 - C \cdot K \leq 0 \end{cases}$$

and it has the only solution:

$$C^2 = C \cdot K = -1$$

which makes all the above inequalities equalities. Accordingly C is a rational curve, i.e. a curve of genus 0, with self-intersection -1. (i.e. (-1)-curves). A famous theorem of Castelnuovo's says that these are the only curves that can be contracted to smooth points via a birational morphism of the surface on which they lie to another surface. By abusing terminology the curve $\Gamma \subset \mathbb{P}^2$ corresponding to C is also called a (-1)-curve.

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More than one Fat Point

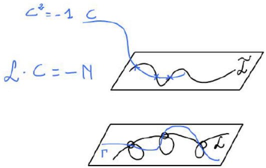
Example

$H(K[x_0, \ldots, x_n]/\wp_1^2 \cap \cdots \cap \wp_s^2, 2)$ is special if $s \leq n$. Actually, quadrics in $\mathbb{P}^n$ singular at s independent points $P_1, \ldots, P_a$ are cones with vertex the $\mathbb{P}^{s-1}$ spanned by $P_1, \ldots, P_s$. Therefore the system is empty as soon as $s \geq n+1$, whereas, if $s \leq n$ one easily computes:

$$\dim(\mathcal{L}) = exp \dim(\mathcal{L}) + \binom{s}{2}.$$

This corresponds to the fact that $\sigma_s(\nu_2(\mathbb{P}^n))$ are defective for all $s \leq n$.

More generally, one has special linear systems in the following situation.
Let $\mathcal{L}$ be a linear system on $\mathbb{P}^2$ which is not empty, let C be a (-1)-curve on $\widetilde{\mathbb{P}^2}$ corresponding to a curve Γ on $\mathbb{P}^2$, such that $\widetilde{\mathcal{L}} \cdot C = -N < 0$.



Then C (resp. Γ) splits off with multiplicity N as a fixed component from all curves of $\widetilde{\mathcal{L}}$ (resp. $\mathcal{L}$), and one has:

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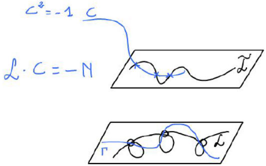
More generally, one has special linear systems in the following situation. Let $\mathcal{L}$ be a linear system on $\mathbb{P}^2$ which is not empty, let C be a (-1)-curve on $\widetilde{\mathbb{P}^2}$ corresponding to a curve Γ on $\mathbb{P}^2$, such that $\widetilde{\mathcal{L}} \cdot C = -N < 0$.



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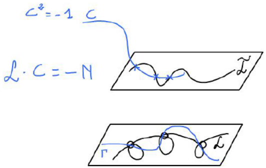

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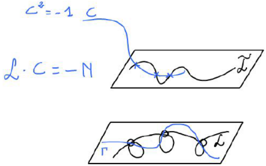

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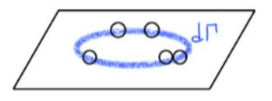
and therefore, if $N \geq 2$, then $\mathcal{L}$ is special.

Example

One immediately finds examples of special systems of this type by starting from the (-1)-curves of the previous example. For instance consider $\mathcal{L} := \mathcal{L}_{2,2d}(-\sum_{i=1}^{5} dP_i)$ which is not empty, consisting of the conic $\mathcal{L}_{2,2}(\sum_{i=1}^{d} P_i)$ counted d times, though it has virtual dimension $-\binom{d}{2}$.



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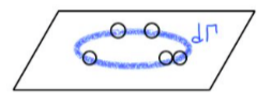

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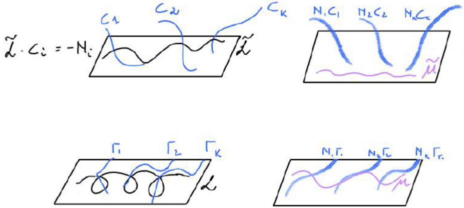
and $\widetilde{\mathcal{M}} \cdot C_i = 0$, for $i = 1, \ldots, k$. $\mathcal{L}$ is special if $N_i \geq 2$ for some i.



Furthermore $C_i \cdot C_j = \delta_{i,j}$, because the union of two (-1)-curves meeting moves, according to Riemann-Roch theorem, in a linear system of positive dimension on $\widetilde{\mathbb{P}}^2$, and therefore it cannot be fixed for $\widetilde{\mathcal{L}}$. The reducible curve $C := \sum_{i=1}^{k} C_i$ (resp. $\Gamma := \sum_{i=1}^{k} N_i \Gamma_i$) is called a (-1)-configuration on $\widetilde{\mathbb{P}}^2$ (resp. on $\mathbb{P}^2$).

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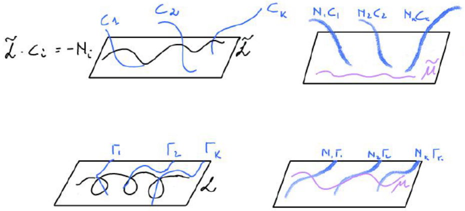

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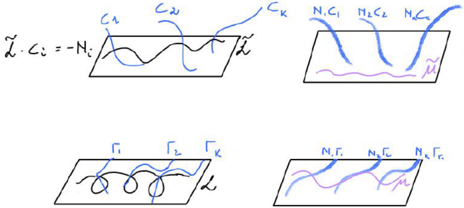

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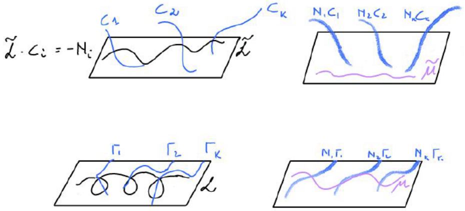

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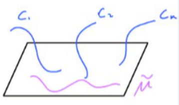
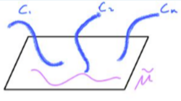
SHGH Conjecture

Definition

A linear system $\mathcal{L}$ on $\mathbb{P}^2$ is (-1)-reducible if $\widetilde{\mathcal{L}} = \sum_{i=1}^{k} N_i C_i + \widetilde{\mathcal{M}}$, where $C = \sum_{i=1}^{k} C_i$ is a (-1)-configuration, $\widetilde{\mathcal{M}} \cdot C_i = 0$, for all $i = 1, \dots, k$, and virtualdim$(\mathcal{M}) \geq 0$.



The system $\mathcal{L}$ is called (-1)-special if, in addition, there is an $i = 1, \dots, k$ such that $N_i > 1$.



Conjecture (Harbourne 1986, - Hirschowitz, 1989)

A linear system of plane curves $\mathcal{L}_{2,d}(-\sum_{i=1}^{s} m_i P_i)$ with general multiple base points is

special if and only if it is (-1)–special,

i.e. it contains some multiple rational curve of self-intersection -1 in the base locus.

NO SPECIAL SYSTEM HAS BEEN DISCOVERED EXCEPT (-1)-SPECIAL SYSTEMS.

SHGH Conjecture

One could hope tho address a weaker version of this problem. Nagata, in connection with his negative solution of the 14-th Hilbert problem, made such a conjecture,

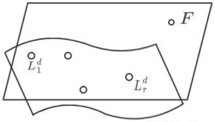
Conjecture (Nagata, 1960)

$\mathcal{L}_{2,d}(-\sum_{i=1}^{s} m_i P_i)$ is empty as soon as $s \geq 10$ and $d \leq \sqrt{s}$.

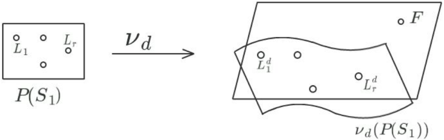
Nagata Conjecture is weaker than SHGH Conjecture yet still open for every non-square $s \geq 10$.

THIN PART

Rank and decomposition of the given element



- Find min $r \in \mathbb{N}$ s.t. $[F] \in r$-secant space to $X_{n,d}$
- Find the minimal length of a 0-dim'l smooth finite scheme $\Gamma \in \mathbb{P}(S_1)$ s.t. $T \in \langle \nu_d(\Gamma) \rangle$



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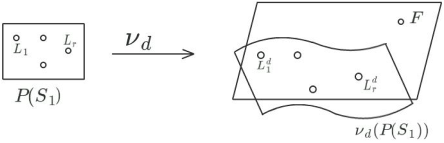


How to find the scheme Γ?

APOLARITY

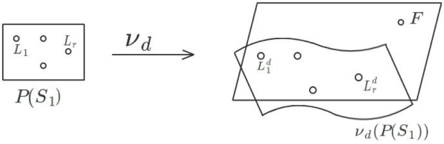
$$M_{d-k,k}(F): \quad T^kV \to S^{d-k}V$$

$$\frac{\partial^k}{\partial^{k_1}x_{i_1}\cdots\partial^{k_h}x_{i_h}} \mapsto \frac{\partial^k}{\partial^{k_1}x_{i_1}\cdots\partial^{k_h}x_{i_h}}F$$

$\ker(M_{d-k,k}) := (F^\perp)_k$ where $F^\perp = \{\varphi \in T \mid \varphi(F) = 0\}$ = Apolar ideal

Lemma (Apolarity Lemma)

$\Gamma \subset \mathbb{P}(S_1)$ *is apolar to* F *(i.e.* $I(\Gamma) \subset F^\perp \subset T$*) iff* $[F] \in \langle \nu_d(\Gamma) \rangle$



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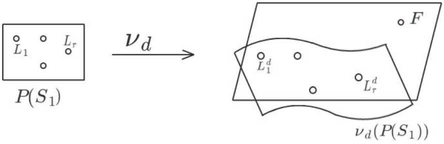

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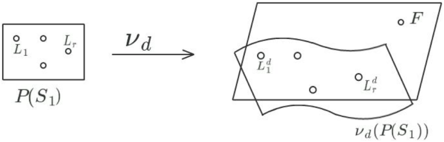

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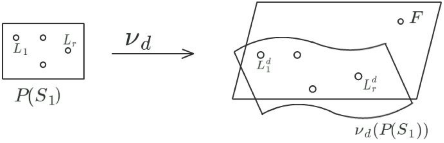

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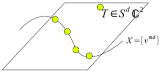

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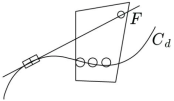

- $q \in \ker(M_{d-r,r}(F))$, find the roots of q, the min r s.t. the roots are distinct is the rank of F and the roots are the "dual" of the the L_i's.

– Sylvester XIX sec. –

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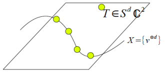

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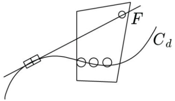

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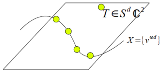

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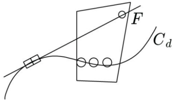

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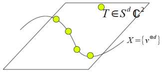

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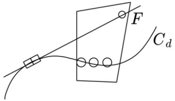

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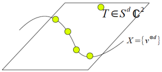

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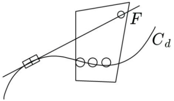

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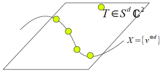
– Sylvester XIX sec. –

Rank and decomposition of the given element [Ex: 2 vars]

$F \in S^d\mathbb{C}^2 = \mathbb{C}[x_0, x_1]_d$



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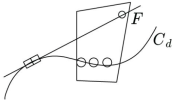

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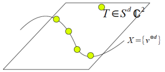

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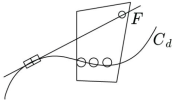

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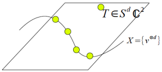

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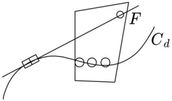

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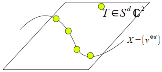

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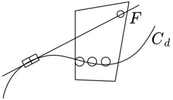

Is there any relation among the smooth and the non smooth scheme?

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Equations of secant varieties

- $I(\sigma_2(\nu_d(\mathbb{P}^n)))$ given by 3×3 minors of catalecticant. ([Pucci] Set Theoretical, [Parolin] Ideally theoretical)
  Analogously for secant lines of Segre-Veronese (Flattenings)([-] SV '07, [R] $\sigma_2(SV)$)
- Representation Theory[LM]:
  X_λ Homogeneous variety $\Rightarrow \exists$ highest weight vector v_λ whose orbit coincides with X_λ.

  Kostant ∼ 1960: $I(X_\lambda) = I(X_\lambda)_2 = V^*_{2\lambda} \subset S^2(S^d V^*)$

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- New results with this idea and combination of RT with generalization of Flattenings[LO].

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Sylvester'method (1886)

Theorem

$T(x_0, x_1) = \sum_{i=0}^{d} c_i \binom{d}{i} x_0^{d-i} x_1^i$ can be decomposed as

$$T = \sum_{k=1}^{r} \lambda_k (\alpha_k x_0 + \beta_k x_1)^d$$

iff there exists a polynomial q such that

$$\begin{bmatrix} c_0 & c_1 & \dots & c_r \\ c_1 & & & c_{r+1} \\ \vdots & & & \vdots \\ c_{d-r} & \dots & c_{d-1} & c_d \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_r \end{bmatrix} = 0$$

and of the form

$$q(x_0, x_1) := \mu \prod_{k=1}^{r} (\beta_k x_0 - \alpha_k x_1).$$

This is possible because

$$I(Z) \subset (f^{\perp}) = \ker(Cat) = (G_1, G_2)$$

where the scheme Z is the scheme defined by the zeros of q, G_1, G_2 are the two generators of the apolar ideal to f and " Cat " is the $(d - r + 1) \times (r + 1)$ Catalecticant matrix.

3. Sylvester's Algorithm, *[S.], [I.K.], [C.,S.]*

Sylvester Algorithm

1. Initialize $r = 0$;
2. Increment $r \leftarrow r + 1$;
3. If the rank of the matrix $M_{d-r,r}$ is maximum, then go to step (2);
4. Else compute a basis $\{l_1, \ldots, l_h\}$ of the right kernel of $M_{d-r,r}$;
5.
   - Take a vector q in the kernel, e.g. $q = \sum_i \mu_i l_i$;
   - Compute the roots of the associated polynomial $q(x_0, x_1) = \sum_{h=0}^{r} q_h x_0^h x_1^{r-h}$; Denote them by (α_j, β_j), where $|\alpha_j|^2 + |\beta_j|^2 = 1$;
   - If the roots are not distinct in $\mathbb{P}^1$, go to step (2);
   - Else if $q(x_0, x_1)$ admits r distinct roots then compute coefficients λ_j, $1 \leq j \leq r$, by solving the linear system below:

$$\begin{pmatrix} \alpha_1^d & \cdots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \cdots & \alpha_r^{d-1}\beta_r \\ \alpha_1^{d-2}\beta_1^2 & \cdots & \alpha_r^{d-2}\beta_r^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \cdots & \beta_r^d \end{pmatrix} \lambda = \begin{pmatrix} a_0 \\ 1/d a_1 \\ \binom{d}{2}^{-1} a_2 \\ \vdots \\ a_d \end{pmatrix};$$

6. $p(x_0, x_1) = \sum_{j=1}^{r} \lambda_j l_j(x_0, x_1)^d$, where $l_j(x_0, x_1) = (\alpha_j x_1 + \beta_j x_2)$.

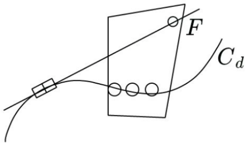
Improvement of Sylvester [–, Gimigliano, Idà]

Theorem

Let $C_d \subset \mathbb{P}^d$ be the rational normal curve of degree d, parameterizing decomposable symmetric tensors (i.e. $C_d = \{T \in \mathbb{P}(S^d\mathbb{C}^2) \mid rk(T) = 1\}$), i.e. projective classes of homogeneous polynomials in $\mathbb{C}[t_0, t_1]_d$ which are d-th powers of linear forms. Then:

$$\forall\, r,\ 2 \leq r \leq \left\lceil \frac{d+1}{2} \right\rceil : \qquad \sigma_r(C_d) \setminus \sigma_{r-1}(C_d) = \sigma_{r,r}(C_d) \cup \sigma_{r,d-r+2}(C_d)$$

where $\sigma_{r,r}(C_d)$ and $\sigma_{r,d-r+2}(C_d)$ are subsets of $\sigma_r(C_d)$ containing only elements of ranks r and $d - r + 2$ respectively.



The (Sylvester) Symmetric Rank Algorithm:
Input: The projective class T of a symmetric tensor $t \in S^d V$ with $\dim(V) = 2$
Output: $rk(t)$.

1. Initialize $r = 0$;
2. Increment $r \leftarrow r + 1$;
3. Compute $M_{d-r,r}(t)$'s $(r+1)$-minors; if they are not all equal to zero then go to step (2); else, $T \in \sigma_r(C_d)$ (notice that this happens for $r \leq \lceil \frac{d+1}{2} \rceil$); go to step (4).
4. Choose a solution $(\overline{u}_0, \ldots, \overline{u}_d)$ of the system $M_{d-r,r}(t) \cdot (u_0, \ldots, u_r)^t = 0$. If the polynomial $\overline{u}_0 t_0^d + \overline{u}_1 t_0^{d-1} t_1 + \cdots + \overline{u}_r t_1^r$ has distinct roots, then $rk(t) = r$, i.e. $T \in \sigma_{r,r}(C_d)$, otherwise $rk(t) = d - r + 2$, i.e. $T \in \sigma_{r,d-r+2}(C_d)$.

Generalization [–GI]

Theorem

Any $T \in \sigma_2(X_{n,d}) \subset \mathbb{P}(V)$, with $\dim(V) = n+1$, can only have symmetric rank equal to 1, 2 or d. More precisely:

$$\sigma_2(X_{n,d}) \setminus X_{n,d} = \sigma_{2,2}(X_{n,d}) \cup \sigma_{2,d}(X_{n,d}),$$

moreover $\sigma_{2,d}(X_{n,d}) = \tau(X_{n,d}) \setminus X_{n,d}$.

Since $r = 2$, every $Z \in Hilb_2(\mathbb{P}^n)$ is the complete intersection of a line and a quadric, so the structure of I_Z is well known: $I_Z = (l_1, \ldots, l_{n-1}, q)$, where $l_i \in R_1$, linearly independent, and $q \in R_2 - (l_1, \ldots, l_{n-1})_2$.
If $T \in \sigma_2(\nu_d(\mathbb{P}^n))$ we have two possibilities; either $rk(T) = 2$ (i.e. $T \in \sigma_2^0(\nu_2(\mathbb{P}^n)))$, or $rk(T) > 2$ i.e. T lies on a tangent line Π_Z to the Veronese, which is given by the image of a scheme Z of degree 2. We can view T in the projective linear space $H \cong \mathbb{P}^d$ in $\mathbb{P}(S^d V)$ generated by the rational normal curve $C_d \subset X_{n,d}$, which is the image of the line L defined by the ideal $(l_1, \ldots, l_{n-1})$ in $\mathbb{P}^n$ with $l_1, \ldots, l_{n-1} \in V^*$ (i.e. $L \subset \mathbb{P}^n$ is the unique line containing z); hence $rk(T) = d$.

Algorithm for the symmetric rank of an element of $\sigma_2(\mathbf{X_{n,d}})$
Input: The projective class T of a symmetric tensor $t \in S^d V$, with $\dim(V) = n+1$;
Output: $T \notin \sigma_2(X_{n,d})$, or $T \in \sigma_{2,2}(X_{n,d})$, or $T \in \sigma_{2,d}(X_{n,d})$, or $T \in X_{n,d}$.

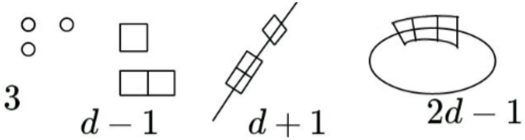
1. Consider the homogeneous polynomial associated to t and rewrite it with the minimum possible number of variables if this is 1 then $T \in X_{n,d}$; if it is > 2 then $T \notin \sigma_2(X_{n,d})$, otherwise T can be viewed as a point in $\mathbb{P}(S^d W) \cong \mathbb{P}^d \subset \mathbb{P}(S^d V)$, and $\dim(W) = 2$, so go to step 2.
2. Apply the algorithm in 2 vars to conclude.

Generalization [–GI]

This technique can be extend to $\sigma_3(X_{n,d})$ because a form there can always be written by using only 3 vars, and there always exist a length 3 apolar scheme of $\mathbb{P}^2$ (not necessary smooth).

Theorem

Let $d \geq 3$, $X_{n,d} \subset \mathbb{P}^(V)$. *Then:*
$\sigma_3(X_{n,3}) \setminus \sigma_2(X_{n,3}) = \sigma_{3,3}(X_{n,3}) \cup \sigma_{3,4}(X_{n,3}) \cup \sigma_{3,5}(X_{n,3})$, *while, for* $d \geq 4$*:*
$\sigma_3(X_{n,d}) \setminus \sigma_2(X_{n,d}) =$
$\sigma_{3,3}(X_{n,d}) \cup \sigma_{3,d-1}(X_{n,d}) \cup \sigma_{3,d+1}(X_{n,d}) \cup \sigma_{3,2d-1}(X_{n,d})$.



In [Ballico,–] we can do the analogous to classify the rank of $F \in \sigma_4(X_{n,d})$ by using the minimal apolar scheme.

We cannot use the same technique for classifying the ranks in the case of $\sigma_5(\nu_d(\mathbb{P}^n))$. In fact there is a famous contra-example due to W. Buczyńska, J. Buczyński that shows that in $\sigma_5(\nu_3(\mathbb{P}^4))$ there is at least a polynomial for which it doesn't exist any 0-dimensional scheme contained in $\nu_3(\mathbb{P}^4)$ whose span contains it.

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Generalization

Example (W. Buczyńska, J. Buczyński)

The following polynomial has border rank ≤ 5 but it doesn't exist any degree 5 zero-dimensional scheme contained in $\nu_3(\mathbb{P}^4)$ whose span contains it:

$$f = x_0^2 x_2 + 6x_1^2 x_3 - 3\,(x_0 + x_1)^2 x_4.$$

One can easily check that the following polynomial

$$f_\epsilon = (x_0 + \epsilon x_2)^3 + 6(x_1 + \epsilon x_3)^3 - 3(x_0 + x_1 + \epsilon x_4)^3 + 3(x_0 + 2\,x_1)^3 - (x_0 + 3x_1)^3$$

has rank 5 for $\epsilon > 0$, and that $\lim_{\epsilon \to 0} \frac{1}{3\epsilon} f_\epsilon = f$.
Therefore f has border rank at most 5.
Let us prove, by contradiction, that there is no saturated ideal $I \subset (f^\perp)$ of degree ≤ 5. Suppose on the contrary that I is such an ideal. Then $H_{R/I}(n) \geq H_{R/(f^\perp)}(n)$ for all $n \in \mathbb{N}$ and $H_{R/(f^\perp)} = [1, 5, 5, 1, 0, \ldots]$. As $H_{R/I}(n)$ is an increasing function of $n \in \mathbb{N}$ with $H_{R/(f^\perp)}(n) \leq H_{R/I}(n) \leq 5$, we deduce that $H_{R/I} = [1, 5, 5, 5, \ldots]$. This shows that $I_1 = \{0\}$ and that $I_2 = (f^\perp)_2$. As I is saturated, $I_2 : (x_0, \ldots, x_4) = I_1 = \{0\}$ since $H_{R/(f^\perp)}(1) = 5$. But an explicit computation of $((f^\perp)_2 : (x_0, \ldots, x_4))$ gives $\langle x_2, x_3, x_4 \rangle$. Contradiction: there is no saturated ideal of degree ≤ 5 such that $I \subset (f^\perp)$. The minimal zero-dimensional scheme contained in $\nu_3(\mathbb{P}^4)$ whose span contains f has deg. 6.

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Since the Apolarity Lemma is true for any number of variables, what can we say about a possible relation between $I(Z)$, $(f^{\perp})$ and $\ker(Cat)$?
Moreover, which is the best catalecticant matrix to use in order to control this relation?
Obviously, by definition, we have that $I(Z) \subset (f^{\perp})$, but in general the equality $(f^{\perp}) = \ker(Cat)$ is not true anymore, or better, it doesn't exist any catalecticant matrix for which this equality holds.
In particular this happens when $rk(f) \geq \max\{rkCat\}$ for the catalecticant matrix with maximum rank.
Therefore if we are in the case of $rk(f) \leq \max\{rkCat\}$ we can still use Catalecticant matrices to construct the apolar ideal $(f^{\perp})$ in order to find the linear forms for a decomposition of f (that is what is shown above by the algorithm of Iarrobino and Kanev), but out of that range Catalecticant matrices are not sufficient anymore.

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Generalization

A generalization of Sylvester's algorithm to any number of variables that uses this techniques is given by Iarrobino and Kanev only under the hypothesis $rk(f) = \max\{rkCat\}$.
[**Algorithm, Iarrobino and Kanev**]
Input: $f \in S^d V$, where $\dim V = n + 1$.

1. Construct the most square possible catalecticant $C_f^m = C_f$ with $m = \lceil \frac{d}{2} \rceil$.
2. Compute $\ker C_f$. If $rk(f) = rk(C_f)$ then continue, otherwise stop here.
3. Find the zero-set $Z' = \{[L_1], \ldots, [L_s]\}$ of the polynomials in $\ker C_f$.
4. Solve the linear system defined by $f = \sum_{i=1}^{s} c_i L_i^d$ in the unknowns c_i.

Output: Waring decomposition of f.

Generalization

Example

Compute a Waring decomposition of $f =$
$3x^4+12x^2y^2+2y^4-12x^2yz+12xy^2z-4y^3z+12x^2z^2-12xyz^2+6y^2z^2-4yz^3+2z^4$.
The most square Catalecticant matrix is the one associated to the map $K[\partial_x, \partial_y, \partial_z]_2 \to K[x, y, z]_2$ that is

$$M_{2,2} = \begin{pmatrix} 3 & 0 & 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & 0 & 1 & -1 \\ 0 & -1 & 2 & 1 & -1 & 0 \\ 2 & 0 & 1 & 2 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 \\ 2 & -1 & 0 & 1 & -1 & 2 \end{pmatrix}.$$

Now compute the Kernel of $M_{2,2}$ that is $K = \langle (1,0,0,-1,-1,-1), (0,1,0,-1,-2,0), (0,0,1,0,2,1) \rangle \subset K[\partial_x, \partial_y, \partial_z]_2$, hence the rank of $M_{2,2}$ is 3. Write for simplicity $X = \partial_x$, $Y = \partial_y$ and $Z = \partial_z$. Now $K = \langle X^2 - Y^2 - YZ - Z^2, XY - Y^2 - 2YZ, XZ + 2YZ + Z^2 \rangle$ and it is not difficult to see that the set of points $\{(1,1,0), (1,0,-1), (1,-1,1)\}$ vanishes on K. Hence we can take $L_1 = (1,1,0) = x + 1$, $L_2 = (1,0,-1) = x - z$ and $L_3 = (1,-1,1) = x - y + z$ and our original polynomial f turns out to be a linear combinations of those forms, in particular

Generalization

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Let the Henkel matrix is the matrix associated to the following map:

$$H_f : S^* \to S, \text{ such that } \partial \mapsto \partial(f).$$

Such an application is linear, where the entries of an associated matrix coincide with the entries of catalecticant matrices, but they are not all known.

Proposition

$Ker(H_f)$ is an ideal.

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Does $\ker(M_{d-r,r})$ define the apolar ideal?

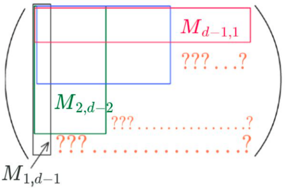
Hankel operators

$$\begin{array}{rcl} H_\Lambda : R & \to & R^* \\ p & \mapsto & p \cdot \Lambda \end{array}$$

Catalecticant Matrices

where $p \cdot \Lambda : q \mapsto \Lambda(p\,q)$.

$$\begin{array}{rcl} M_{d-k,k}(F): \quad T^k V & \to & S^{d-k} V \\ \partial^k & \mapsto & \partial^k F \end{array}$$



$rkH_\Lambda = r$ iff $\ker H_\Lambda = I_\Lambda$ is a radical ideal. $\Lambda = \sum_{i=1}^{r} \gamma_i \mathbf{1}_{z_i}$ with $\gamma_i > 0$ and z_i are distinct points of T_1.