Higher secant varieties of classical projective varieties

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The classic Waring's problem (1770)

Lagrange

- every natural number is the sum of at most 4 squares.

Waring in Meditationes Algebricae

- every natural number is the sum of at most 9 cubes;
- every natural number is the sum of at most 19 forth powers.

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Waring proposed a generalization of Lagrange's four-square theorem

- every natural number is the sum of at most $g(d)$ d^{th} powers of natural numbers.
  So $g(2) = 4$, $g(3) = 9$, $g(4) = 19$.

The classic Waring's problem

Only in 1909 Hilbert proved that $g(d)$ exists for all d.

Note that:

g(3) = 9 but only 23 e 239 actually need 9 cubes, while 8 cubes suffice for all other natural numbers.
Even more is true, and only few natural numbers require 8 cubes and 7 cubes are enough for all large enough natural numbers.

The Waring Problem for Polynomials

Given a degree d form F in n variables, the

Waring Problem for Polynomials

asks for the least value of s for which there exist linear forms $L_1, ..., L_s$ such that

$$F = L_1^d + \cdots + L_s^d.$$

This value of s is called the Waring rank of F and will be denoted by $\mathrm{rk}(F)$, so

$$\mathrm{rk}(F) = \min\{s \mid F = L_1^d + \ldots + L_s^d\}.$$

Veronese Variety

A Veronese Variety is the image of the embedding

$$\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$$

$$N = \binom{n+d}{n} - 1$$

via the forms of degree d of the homogeneous coordinate ring

$$R = \mathbb{C}[x_0, ..., x_n].$$

Example

$$\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$(a, b, c) \longmapsto (a^2,\ ab,\ ac,\ b^2,\ bc,\ c^2)$$

Veronese Variety

A Veronese Variety is the image of the embedding

$$\mathbb{P}^n = \mathbb{P}(R_1) \longrightarrow \mathbb{P}^N = \mathbb{P}(R_d)$$

$$[L] \longmapsto [L^d]$$

Example

$$\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$[ax+by+cz] \longmapsto [a^2x^2+2abxy+2acxz+b^2y^2+2bcyz+c^2z^2].$$

Hence, if we use as a basis for R_d the monomial of degree d and order them lexicographically and write things with respect to coordinates, we get that the map is defined by

$$(a,b,c) \longmapsto (a^2,\ 2ab,\ 2ac,\ b^2,\ 2bc,\ c^2).$$

Veronese Variety

It is clear that the two methods of describing the Veronese embeddings are equivalent in characteristic zero.
The second method gives us a bridge between the

Waring problem for forms

and the

higher secant varieties of the Veronese variety.

Recall that

$$\text{the Waring rank of } F = \min\{s \mid F = L_1^d + \ldots + L_s^d\}.$$

Example

$$4xy = (x+y)^2 + (ix - iy)^2$$

$$(0,4,0) = (1,2,1) + (-1,2,-1)$$

HIGHER SECANT VARIETIES

Definition of higher secant variety

Let $\mathbb{X} \subset \mathbb{P}^N$ be a *reduced*, *irreducible*, *non-degenerate* projective variety, i.e., $\mathbb{X}$ arises as the set of zeros of a prime homogeneous ideal which does not contain any linear forms (i.e. $\mathbb{X}$ does not lie in any projective linear subspace of $\mathbb{P}^N$).

The s^{th} secant variety of $\mathbb{X}$, denoted by $\sigma_s(\mathbb{X})$, is the closure of the union of all linear spaces spanned by s linearly independent points of $\mathbb{X}$, that is

$$\sigma_s(\mathbb{X}) := \overline{\bigcup_{P_i \in \mathbb{X}} < P_1, \ldots, P_s >}$$

When $s = 2$ we refer to $\sigma_s(\mathbb{X})$ as the *secant line variety*, when $s = 3$ the *secant plane variety* and so on.

General forms

Example.

If F is a general form of degree d in $n+1$ variables, then the rank of F is the minimum s such that $\sigma_s(\mathbb{X})$ fiills the ambient space, where $\mathbb{X}$ is the Veronese variety $\nu_d(\mathbb{P}^n)$.

Problem

Important questions to answer about $\sigma_s(\mathbb{X})$.

- What is the minimal free resolution of the ideal of $\sigma_s(\mathbb{X})$?
- What is the degree of $\sigma_s(\mathbb{X})$?
- What are the equations defining $\sigma_s(\mathbb{X})$?

What is the dimension of $\sigma_s(\mathbb{X})$?

Dimension of Secant Variety

By a *parameter count* we get

$$\boxed{\exp \dim \sigma_s(\mathbb{X}) = \min\{s \dim \mathbb{X} + s - 1; N\}}$$

If $\sigma_s(\mathbb{X})$ does not have the expected dimension, we say that $\mathbb{X}$ is defective.

Note that, for

$$s(\dim \mathbb{X} + 1) \geq N + 1$$

we expect $\sigma_s(\mathbb{X})$ to fill $\mathbb{P}^N$, and it is not a coincidence that:

$\dim \mathbb{X} + 1$ is the degree of a double point on $\mathbb{X}$.

Motivation

Strong motivation for studying the secant varieties

- connections to questions in
  Representation Theory,
  Coding Theory,
  Algebraic Complexity Theory,
  Statistics,
  Phylogenetics ,
  Data Analysis,
  Electrical Engineering (Antenna Array Processing and Telecommunications),
  etc

Secant Varieties and Tensor decomposition

- Connection with the problem of how to minimally represent certain kinds of tensors as a sum of decomposable ones:

Rank of Tensors

Secant Varieties of Segre-Veronese Varieties and Tensor decomposition

Example:

The Segre-Veronese Variety is the image of the mapping

$$\nu_{(d_1,\dots,d_t)} : \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_t) \longrightarrow \mathbb{P}(S^{d_1}V_1 \otimes \ldots \otimes S^{d_t}V_t)$$

$$[v_1] \times \ldots \times [v_t] \longrightarrow [v_1^{\otimes d_1} \otimes \ldots \otimes v_t^{\otimes d_t}],$$

hence parameterizes partially symmetric tensors.

Segre Variety

A Segre Variety is the image of the embedding

$$\nu_{(1,\dots,1)} : \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_t} \hookrightarrow \mathbb{P}^N$$

$$N = \Pi(n_i + 1) - 1$$

via the forms of multi-degree $(1, \dots, 1)$ of the multi-graded homogeneous coordinate ring

$$\mathbb{C}[x_{0,1}, \dots, x_{n_1,1}; x_{0,2}, \dots, x_{n_2,2}; \dots; x_{0,t}, \dots, x_{n_t,t}].$$

Example

$$\nu_{(1,1,1)} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^7$$

$$(x_0, x_1) \times (y_0, y_1) \times (z_0, z_1) \longmapsto$$

$$(x_0y_0z_0,\ x_0y_0z_1,\ x_0y_1z_0,\ x_0y_1z_1,\ x_1y_0z_0,\ x_1y_0z_1,\ x_1y_1z_0,\ x_1y_1z_1).$$

Segre-Veronese Variety

A Segre-Veronese Variety is the image of the embedding

$$\nu_{(d_1,\dots,d_t)} : \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_t} \hookrightarrow \mathbb{P}^N$$

$$N = \Pi \binom{d_i+n_i}{n_i} - 1$$

via the forms of multi-degree $(d_1, \dots, d_t)$ of the multi-graded homogeneous coordinate ring

$$\mathbb{C}[x_{0,1}, \dots, x_{n_1,1}; x_{0,2}, \dots, x_{n_2,2}; \dots; x_{0,t}, \dots, x_{n_t,t}].$$

- For $t = 1$, $\mathbb{X}$ is a Veronese Variety.
- For $d_1 = \cdots = d_t = 1$, $\mathbb{X}$ is a Segre Variety.

Example

$$\nu_{(1,2)} : \mathbb{P}^2 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^8$$

$$(x_0, x_1, x_2) \times (y_0, y_1) \longmapsto$$
$$(x_0 y_0^2,\ x_0 y_0 y_1,\ x_0 y_1^2,\ x_1 y_0^2,\ x_1 y_0 y_1,\ x_1 y_1^2,\ x_2 y_0^2,\ x_2 y_0 y_1,\ x_2 y_1^2).$$

Varieties of λ-reducible forms

In 1954, Mammana introduced the varieties of reducible plane curves:
consider the space S_d of forms of degree d in 3 variables and let $\lambda = (d_1, ..., d_r)$ be a partition of d.

$$\mathbb{X}_{2,\lambda} \subset \mathbb{P}^N, \quad N = \binom{2+d}{d} - 1$$

is the variety parameterizing forms F which are the product of r forms F_i with $\deg F_i = d_i$, that is,

$$F = F_1 \cdots F_r \in S_d.$$

Varieties of λ-reducible forms

The varieties of reducible hypersurfaces are an obvious generalization:
consider the space S_d of forms of degree d in $n+1$ variables and let $\lambda = (d_1, ..., d_r)$ be a partition of d.

$$\mathbb{X}_{n,\lambda} \subset \mathbb{P}^N, \quad N = \binom{n+d}{d} - 1$$

is the variety parameterizing forms F which are the product of r forms F_i with $\deg F_i = d_i$

$$F = F_1 \cdots F_r \in S_d.$$

METHOD FOR SEGRE-VERONESE

Method for Segre-Veronese

The problem of determining the dimension of $\sigma_s(\mathbb{X})$ is related (via the theory of inverse systems or, equivalently, apolarity) to determining the Hilbert Function of a

0-dimensional scheme made by
the first infinitesimal neighbourhoods
of s generic points

(this is equivalent to what is classically known as Terracini's Lemma).

Step 1 - Terracini's Lemma

Let $P \in \sigma_s(\mathbb{X})$ be a generic point

$$P \in < P_1, ..., P_s >, \qquad (P_i \in \mathbb{X} \subset \mathbb{P}^N).$$

- Then by Terracini's Lemma :

$$T_P(\sigma_s(\mathbb{X})) =< T_{P_1}(\mathbb{X}), ..., T_{P_s}(\mathbb{X}) >$$

$$=< 2P_1|_{\mathbb{X}} + ... + 2P_s|_{\mathbb{X}} >$$

- hence

$$\begin{aligned} \dim \sigma_s(\mathbb{X}) &= \dim T_P(\sigma_s(\mathbb{X})) \\ &= \dim < 2P_1|_{\mathbb{X}} + ... + 2P_s|_{\mathbb{X}} > \\ &= N - \dim(I_{2P_1|_{\mathbb{X}}+.........+2P_s|_{\mathbb{X}}})_1 . \end{aligned}$$

Step 2 : Segre-Veronese map

Let

$$\nu_{(d_1,\dots,d_t)} : (\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_t}) \hookrightarrow \mathbb{X} \subset \mathbb{P}^N$$

$$Q_i \in \mathbb{P}^{n_1} \times \dots. \times \mathbb{P}^{n_t}; \qquad P_i \in \mathbb{X}$$

$$Q_i \leftrightarrow P_i$$

the scheme of double points

$$2P_1|_{\mathbb{X}} + \dots\dots\dots + 2P_s|_{\mathbb{X}} \subset \mathbb{X}$$

corresponds to the scheme

$$2Q_1 + \dots\dots + 2Q_s \subset \mathbb{P}^{n_1} \times \dots. \times \mathbb{P}^{n_t}.$$

Step 2 : Segre-Veronese map

and we have

$$\dim(I_{2P_1|_{\mathbb{X}}+\dots\dots\dots+2P_s|_{\mathbb{X}}})_1 = \dim(I_{2Q_1+\dots\dots+2Q_s})_{(d_1,\dots,d_t)}$$

hence

$$\begin{aligned} \dim\sigma_s(\mathbb{X}) &= N - \dim(I_{2P_1|_{\mathbb{X}}+\dots\dots\dots+2P_s|_{\mathbb{X}}})_1 \\ &= N - \dim(I_{2Q_1+\dots\dots+2Q_s})_{(d_1,\dots,d_t)} \\ &= HF\,(2Q_1+\dots\dots+2Q_s,(d_1,\dots,d_t)) - 1 \end{aligned}$$

Veronese Varieties

Thus, in case $\mathbb{X}$ is the Veronese Variety, we have

$$\begin{aligned} \dim \sigma_s(\mathbb{X}) &= = N - \dim(I_{2Q_1+......+2Q_s})_d \\ &= HF\,(2Q_1 + ... + 2Q_s, d) - 1 \end{aligned}$$

Veronese Varieties

The work by J. Alexander and A. Hirschowitz (1995) confirmed that apart from the quadratic Veronese varieties and a (few) well known exceptions, all the Veronese varieties $\mathbb{X} = \nu_d(\mathbb{P}^n)$ have higher secant varieties of the expected dimension. More precisely, $\sigma_s(\mathbb{X})$ is defective only in the following cases:

- $d = 2$ and $2 \leq s \leq n$;
- $n = 2$, $d = 4$, $s = 5$;
- $n = 3$, $d = 4$, $s = 9$;
- $n = 4$, $d = 3$, $s = 7$;
- $n = 4$, $d = 4$, $s = 14$.

Hence if $\mathbb{X}$ is a Veronese Variety the problem of finding

the dimension of $\sigma_s(\mathbb{X})$

is solved and the Waring Problem for a generic polynomial is solved also.

Veronese Varieties

Except the cases listed above, if

- F is a generic form of degree d in $n+1$ variables.

$$\mathrm{rk}(F) = \left\lceil \frac{\binom{d+n}{n}}{n+1} \right\rceil$$

(Alexander - Hirschowitz , 1995).

Example of a defective Segre variety

$\mathbb{X}$ = the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^5 \subset \mathbb{P}^{35}$

is defective for $s = 5$

$$\exp \dim \sigma_5(\mathbb{X}) = \min\{5 \cdot 8 + 4; N\} = \min\{44; 35\} = 35 = N$$

that is, we expect $\sigma_5(\mathbb{X}) = \mathbb{P}^N$, i.e., we expect that the ideal of 5 double points in degree $(1, 1, 1)$ is zero.

But there exist F_1 of degree $(1, 1, 0)$, and F_2 of degree $(0, 0, 1)$ in the ideal of 5 points, hence

$$F = F_1 \cdot F_2$$

is a form of degree $(1, 1, 1)$ in the ideal of 5 double points.

Step 3: the Multiprojective-Affine-Projective Method

- Let

$$T = k[x_{0,1}, \ldots, x_{n_1,1}; x_{0,2}, \ldots, x_{n_2,2}; \ldots; x_{0,t}, \ldots, x_{n_t,t}]$$

  be the multigraded homogeneus coordinate ring of $\mathbb{P}^{n_1} \times \ldots. \times \mathbb{P}^{n_t}$.

- Let

$$S = k[t, x_{1,1}, \ldots, x_{n_1,1}; x_{1,2}, \ldots, x_{n_2,2}; \ldots; x_{1,t}, \ldots, x_{n_t,t}]$$

  be the coordinate ring of $\mathbb{P}^{n_1+\ldots+n_t}$.

Step 3: the Multiprojective-Affine-Projective Method

$$\dim T_{(d_1,\dots,d_t)} = \dim(I_W)_{(d_1+\dots+d_t)}$$

where

$$T_{(d_1,\dots,d_t)}$$

is the vector space of the forms of multidegree $(d_1,\dots,d_t)$ of T;

$$W = (n_2+\dots+n_t)\Lambda_1 + (n_1+n_3+\dots+n_t)\Lambda_2 + \dots + (n_1+\dots+n_{t-1})\Lambda_t$$

and the

$$\Lambda_i \simeq \mathbb{P}^{n_i-1}$$

are generic linear spaces of $\mathbb{P}^{n_1+\dots+n_t}$.

(Examples: $\nu_{(2,1,1)}(\mathbb{P}^1 \times \mathbb{P}^5 \times \mathbb{P}^5)$, $\nu_{(2,1)}(\mathbb{P}^2 \times \mathbb{P}^1)$).

Step 3: the Multiprojective-Affine-Projective Method

Then

$$\dim(I_{2Q_1+\dots\dots+2Q_s})_{(d_1,\dots,d_t)} = \dim(I_{W+2P_1+\dots\dots+2P_s})_{d_1+\dots+d_t}$$

where

$$W + 2P_1 + \dots\dots + 2P_s \subset \mathbb{P}^{n_1+\dots+n_t}$$

$$P_1, \dots\dots, P_s \ \ \textit{are generic points}$$

$$W = (n_2 + \dots + n_t)\Lambda_1 + \dots\dots + (n_1 + \dots + n_{t-1})\Lambda_t$$

$$\Lambda_i \simeq \mathbb{P}^{n_i-1} \subset \mathbb{P}^{n_1+\dots+n_t} \ \ \textit{are generic linear spaces}$$

thus

$$\begin{aligned} \dim \sigma_s(\mathbb{X}) &= N - \dim(I_{2Q_1+\dots\dots+2Q_s})_{(d_1,\dots,d_t)} \\ &= N - \dim(I_{W+2P_1+\dots\dots+2P_s})_{d_1+\dots+d_t} \\ &= -1 + \textit{ number of conditions that } 2P_1 + \dots\dots + 2P_s \\ &\quad \textit{imposes to the forms of } (I_W)_{d_1+\dots+d_t} \end{aligned}$$

Notation

$\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_t}$
$(d_1, \ldots, d_t)$

instead of $\nu_{(d_1,\ldots,d_t)}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_t})$.

Example of a defective Segre-Veronese variety

$\mathbb{P}^1 \times \mathbb{P}^5 \times \mathbb{P}^5$
$(2,1,1)$

is defective for $7 \leq s \leq 11$

- $N = 3 \times 6 \times 6 - 1 = 107$; $\operatorname{expdim} \sigma_7(\mathbb{X}) = 7x12 - 1 = 83$;
- $W = 2\Lambda_1 + 3\Lambda_2 + 3\Lambda_3 \subset \mathbb{P}^{11}$ where $\Lambda_1 \simeq \mathbb{P}^0$, $\Lambda_2 \simeq \mathbb{P}^4$, $\Lambda_3 \simeq \mathbb{P}^4$, i.e., W is a subscheme of $\mathbb{P}^{11}$ formed by one double point and two triple linear spaces of dimension 4;
- there exists an r.n.c. C maximally intersecting the configuration $\Lambda_2 + \Lambda_3 +$ 8 points (– , Carlini (2009));
- $\deg C \cdot \deg F = 11 \cdot 4 = 44$, where $F \in (I_{W+2P_1+\ldots+2P_7})_4$;
- degree $(C \cap (W + 2P_1 + \ldots + 2P_7)) = 2 \times 2 \times 5 + 2 \times 8 = 46$.

Then we conclude that our variety is 7-defective.

Example of a defective Segre-Veronese variety

- expdim $\sigma_{11}(\mathbb{X}) = 107$; $N = 107$;
  hence we expect that $\sigma_{11}(\mathbb{X}) = \mathbb{P}^{107}$ or, equivalently, that there are not forms of degree 4 in $(I_{W+2P_1+\ldots+2P_{11}})$, where W is as above, that is, W is a subscheme of $\mathbb{P}^{11}$ formed by one double point and two triple linear spaces Λ_1 and Λ_2 of dimension 4.

But there is a quadric S_1 through $2\Lambda_1 + \Lambda_2 +$ the 12 points, and, analogously, there is a quadric S_2 through $\Lambda_1 + 2\Lambda_2 +$ the 12 points. (In fact the quadrics of $\mathbb{P}^{11}$ "are" 78, and the conditions given to the quadrics by $2\Lambda_1 + \Lambda_2 +$ 12 points are $50 + 15 + 12 = 77$). The form corresponding to the quartic $S_1 S_2$ is in $(I_{W+2P_1+\ldots+2P_{11}})_4$.

Example of a non defective Segre-Veronese variety

$\mathbb{P}^2 \times \mathbb{P}^1$
$(2,1)$

$$\exp\dim \sigma_s(\mathbb{X}) = \min\{4s-1; 11\}$$

hence we expect $\sigma_3(\mathbb{X})$ to fill $\mathbb{P}^N$ and $\dim \sigma_2(\mathbb{X}) = 7$.
$W \subset \mathbb{P}^3$ is formed by a line, say L, and one double point, say $2P_0$. Since

$$\dim(I_{L+2P_0+2P_1+2P_2})_3 = 4$$

$$\dim(I_{L+2P_0+2P_1+2P_2+2P_3})_3 = 0$$

we have

$$\dim \sigma_2(\mathbb{X}) = 11 - 4 = 7$$

$$\dim \sigma_3(\mathbb{X}) = 11.$$

Example of a defective Segre-Veronese variety

$\mathbb{P}^2 \times \mathbb{P}^5$

$(2,1)$

is defective iff $s = 5$

Let

$$\mathbb{X} = \nu_{(2,1)}(\mathbb{P}^2 \times \mathbb{P}^5) \subset \mathbb{P}^{35} \, ; \qquad \mathbb{Y} = \nu_{(1,1)}(\mathbb{P}^5 \times \mathbb{P}^5) \subset \mathbb{P}^{35}.$$

By a direct computation, we get $\sigma_4(\mathbb{X}) = \sigma_4(\mathbb{Y})$=31. Hence $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y})$ for $s \geq 4$.
Since

$$\dim \sigma_4(\mathbb{Y}) = 31 \qquad \exp\dim \sigma_4(\mathbb{X}) = 31$$

$$\dim \sigma_5(\mathbb{Y}) = 34, \qquad \dim \sigma_6(\mathbb{Y}) = 35$$

$$\exp\dim \sigma_5(\mathbb{X}) = \exp\dim \sigma_6(\mathbb{X}) = 35$$

we have that $\mathbb{X}$ is defective iff $s = 5$.

Step 4: Specialization and méthode d'Horace

Recall that:

let $Z \subset \mathbb{P}^n$ be a scheme corresponding to the ideal sheaf $\mathcal{I}_Z$ and let $H \subset \mathbb{P}^n$ be a hyperplane.
The trace of Z with respect to H is the schematic intersection $Tr_H(Z) = Z \cap H$.
The Residual $Res_H(Z)$ of Z with respect to H is defined by the ideal sheaf $\mathcal{I}_Z : \mathcal{O}_{\mathbb{P}^n}(-H)$.
Taking the global sections of the exact sequence

$$0 \to \mathcal{I}_{Res_H Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_H(Z)}(d) \to 0,$$

we obtain the so called Castelnuovo exact sequence

$$0 \to (I_{Res_H Z})_{d-1} \to (I_Z)_d \to (I_{Tr_H(Z)})_d$$

from which we get the following inequality

$$\dim(I_Z)_d \leq \dim(I_{Res_H Z})_{d-1} + \dim(I_{Tr_H(Z)})_d.$$

Step 4: Specialization and méthode d'Horace

- In order to compute $\dim(I_{W+2P_1+\ldots+2P_s})_{d_1+\ldots+d_t}$, specialize the scheme $Y = W + 2P_1 + \ldots + 2P_s$ by placing some of the P_i on a hyperplane. Let $\widetilde{Y}$ be the specialized scheme.
- In some cases, specialize Y in such a way that new linear spaces are in the base locus for the hypersurfaces defined by the forms of $(I_{\widetilde{Y}})_{d_1+\ldots+d_t}$.
- then use Castelnuovo Lemma

$$\boxed{\dim(I_{\widetilde{Y}})_d \leq \dim(I_{Res_H \widetilde{Y}})_{d-1} + \dim(I_{Tr_H \widetilde{Y}})_d}$$

(where H is a hyperplane)

- or Horace Differential Lemma

(example: $\sigma_4(\nu_{(2,1)}(\mathbb{P}^2 \times \mathbb{P}^1))$.

Example

$\mathbb{P}^1 \times .... \times \mathbb{P}^1$ (t times)

$(1, \ldots\ldots, 1)$

s even; $s = 2h$

we have to compute $\dim(I_X)_t$ where

$$X = (t-1)Q_1 + \ldots + (t-1)Q_t + 2P_1 + \ldots + 2P_{2h}.$$

Example:
for $t = 7$ and $s = 16$ we have to compute

$$\dim(I_{6Q_1+\ldots+6Q_7+2P_1+\ldots+2P_{16}})_7$$

(Note that $2^7 = 128 = 16 \cdot 8$))

- Specialize on a hyperplane H the points $Q_2, \dots, Q_7, P_1, \dots, P_8$.
- The 8 lines $L_1 = Q_1 P_9$, ..., $L_8 = Q_1 P_{16}$ are in the base locus of the hypersurfaces defined by the forms of $(I_X)_7$,
- hence

$$(I_X)_7 = (I_Y)_7$$

where

$$Y = X + L_1 + \dots + L_8.$$

- Now use Castelnuovo Lemma:

$$\dim(I_Y)_7 \leq \dim(I_{Res_H Y})_6 + \dim(I_{Tr_H Y})_7$$

and induction.

- Since the hypersurfaces defined by the forms of $(I_{Res_H Y})_6$ are cones with vertex in Q_1 we have

$$\dim(I_{Res_H Y})_6 = \dim(I_{5Q_2+\ldots+5Q_7+P_1+\cdots+P_8+2P_9+\ldots+2P_{16}})_6.$$

- Since $< Q_2, ..., Q_7 >$ is in the base locus of the hypersurfaces defined by the forms of $(I_{Tr_H Y})_7$, we get

$$\dim(I_{Tr_H Y})_7 = \dim(I_{5Q_2+\ldots+5Q_7+2P_1+\ldots+2P_8+8 \text{ simple points}})_6$$

- and now we are in

$\mathbb{P}^1 \times .... \times \mathbb{P}^1$ (6 times)

$(1, \ldots, 1)$

SECANT VARIETIES OF SEGRE VARIETIES WITH 2 FACTORS

Segre of two factors

$\mathbb{P}^m \times \mathbb{P}^n$

$(1, 1)$

everything is well known

The case for Segre varieties with two factors is very well understood since all of the theory is in terms of ranks of matrices and that is understood very well from both an algebraic and geometric standpoint.

Segre of two factors

ALL THE THEORY IS IN TERMS OF RANKS OF MATRICES

Example: $\mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^{11}$.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \times \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 \end{bmatrix}$$

The product of the two projective spaces ($\mathbb{P}^m \times \mathbb{P}^n$) can be identified with the $(m+1) \times (n+1)$ matrices of rank 1 and the general points on $\sigma_s(\mathbb{X})$ are the sums of s matrices of rank 1. (A matrix has rank $\leq s$ if and only if it is the sum of s matrices of rank 1.)

In the example

$$\dim \sigma_2(\mathbb{X}) = 11 - 2 = 9$$

By our method

Example: $\mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^{11}$.

$$\exp\dim \sigma_s(\mathbb{X}) = \min\{6s - 1; 11\}$$

hence

$$\exp\dim \sigma_2(\mathbb{X}) = 11$$

$$\dim \sigma_2(\mathbb{X}) = 11 - \dim(I_{W+2P_1+2P_2})_2$$

$W \subset \mathbb{P}^5$ is formed by a line and a plane. By projecting from the two points P_1 and P_2 we get

$$\dim(I_{W+2P_1+2P_2})_2 = \dim(I_{W'})_2$$

where $W' \subset \mathbb{P}^3$ is still formed by a line and a plane, hence

$$\dim \sigma_2(\mathbb{X}) = 11 - 2 = 9.$$

SECANT VARIETIES OF SEGRE-VERONESE VARIETIES WITH 2 FACTORS

Segre-Veronese of two factors

$\mathbb{P}^m \times \mathbb{P}^n$

(a, b)

there are only partial results

Segre-Veronese of two factors with $b = 1$

$\mathbb{P}^m \times \mathbb{P}^n$

$(a, 1)$

Segre-Veronese of two factors with $b = 1$

$\mathbb{P}^2 \times \mathbb{P}^1$	is defective for $s = 5$
$(3, 1)$	London (1890); Dionisi, Fontanari (2001); Carlini , Chipalkatti(2001)

$\mathbb{P}^2 \times \mathbb{P}^1$	is defective only for $a = 3$, $s = 5$ $(\delta = 1)$
$(a, 1)$	Dionisi, Fontanari (2001)

$\mathbb{P}^3 \times \mathbb{P}^2$	is defective for $s = 5$ $(\delta = 1)$
$(2, 1)$	Carlini, Chipalkatti(2001)

Segre-Veronese of two factors with $b = 1$

$\mathbb{P}^3 \times \mathbb{P}^4$	is defective for $s = 6$ $(\delta = 1)$
$(2, 1)$	Carlini, Chipalkatti(2001)

$\mathbb{P}^5 \times \mathbb{P}^2$	is defective for $s = 8$ $(\delta = 2)$
$(2, 1)$	Carlini, Chipalkatti(2001)

$\mathbb{P}^1 \times \mathbb{P}^n$	is never defective
$(a, 1)$	Chiantini, Ciliberto (2002)

$\mathbb{P}^m \times \mathbb{P}^n$	is never defective
$(n+1, 1)$	- , Geramita, Gimigliano (2005)

Segre-Veronese of two factors with $b = 1$

$\mathbb{P}^m \times \mathbb{P}^n$

$(a, 1)$

is not defective for $a \geq 3$ and $s \leq s_1$ or $s \geq s_2$, where:

$$s_1 \leq q(n+1) \leq \left\lfloor \frac{(n+1)\binom{m+a}{a}}{(m+n+1)} \right\rfloor, \quad (q \in \mathbb{N});$$

$$s_2 \geq t(n+1) \geq \left\lceil \frac{(n+1)\binom{m+a}{a}}{(m+n+1)} \right\rceil, \quad (t \in \mathbb{N}).$$

Bernardi, Carlini, - (2010)

Segre-Veronese of two factors with $b = 1$

$\mathbb{P}^{2k+1} \times \mathbb{P}^2$

$(2, 1)$

is defective for $s = 3k + 2$

Ottaviani (2008)

$\mathbb{P}^m \times \mathbb{P}^n$

$(2, 1)$

is not defective for $s \leq s_1$ and $n \leq m + 2$; or for $s \geq s_2$, where:

$s_1 = (n+1)\left\lfloor \frac{m}{2} \right\rfloor - \frac{(n-2)(n+1)}{2}$ if m is even;

$s_1 = (n+1)\left\lfloor \frac{m}{2} \right\rfloor - \frac{(n-3)(n+1)}{2}$ if m and n are odd ;

$s_1 = (n+1)\left\lfloor \frac{m}{2} \right\rfloor - \frac{(n-3)(n+1)+1}{2}$ if n is even and m is odd ;

$s_2 = (n+1)\left\lfloor \frac{m}{2} \right\rfloor + 1$ if m is even;

$s_2 = (n+1)\left\lfloor \frac{m}{2} \right\rfloor + 3$ otherwise.

Abo, Brambilla (2009)

Segre-Veronese of two factors with $b = 1$

$\mathbb{P}^m \times \mathbb{P}^m$	is never defective
$(2, 1)$	Abo (2010)

$\mathbb{P}^m \times \mathbb{P}^{m-1}$	is defective only for $(m, m-1) = (4, 3)$ and $s = 6$
$(2, 1)$	Abo (2010)

Conjectures

$\mathbb{P}^m \times \mathbb{P}^n$

$(2, 1)$

is never defective, except for

1) $n \geq \binom{m+2}{2} - m$ (i.e., it is unbalanced) and $\binom{m+2}{2} - m < s < \min\{n+1; \binom{m+2}{2}\}$;

2) $(m, n, s) = (2k+1, 2, 3k+2)$ con $k \geq 1$;

3) $(m, n, s) = (3, 4, 6)$.

Abo, Brambilla (2012)

Segre-Veronese of two factors with $n = 1$

$\mathbb{P}^m \times \mathbb{P}^1$

(a, b)

Segre-Veronese of two factors with $n = 1$

$\mathbb{P}^1 \times \mathbb{P}^1$	is defective only for $(a, b) = (2, 2d)$ and $s = 2d + 1$
(a, b)	- , Geramita, Gimigliano (2005)

$\mathbb{P}^2 \times \mathbb{P}^1$	is defective only for $(a, b) = (3, 1)$, $(a, b) = (2, 2d)$
(a, b)	Baur, Draisma (2007)

Segre-Veronese of two factors with $n = 1$

$\mathbb{P}^m \times \mathbb{P}^1$	is defective iff $d(m+1)+1 \leq s \leq d(m+1)+m$
$(2, 2d)$	Abrescia (2008)

$\mathbb{P}^m \times \mathbb{P}^1$	is never defective
$(2, 2d+1)$	Abrescia (2008)

$\mathbb{P}^m \times \mathbb{P}^1$	is defective only for $m = 2, b = 1$
$(3, b)$	Abrescia (2008)

Segre-Veronese of two factors with $n = 1$

$\mathbb{P}^m \times \mathbb{P}^1$

(a, b)

if $b \geq 3$, it is defective only for
$(a, b) = (2, 2d), \; d(m+1)+1 \leq s \leq d(m+1)+m$

Abo, Brambilla (2009)

$\mathbb{P}^m \times \mathbb{P}^1$

(a, b)

is defective only for
$m = 2, \quad (a, b) = (3, 1), \quad s = 5$ and for
$(a, b) = (2, 2d), \quad d(m+1)+1 \leq s \leq d(m+1)+m$

Ballico, Bernardi, - (2011)

Segre-Veronese of two factors

$\mathbb{P}^2 \times \mathbb{P}^2$	is defective for $s = 8$
$(2, 2)$	- , Geramita, Gimigliano (2005)

$\mathbb{P}^n \times \mathbb{P}^2$	is defectivefor $s = 3n + 2$
$(2, 2)$	- , Geramita, Gimigliano (2005) Bocci (2005)

$\mathbb{P}^3 \times \mathbb{P}^3$	is defective for $s = 15$
$(2, 2)$	- , Geramita, Gimigliano (2005)

$\mathbb{P}^3 \times \mathbb{P}^4$	is defective for $s = 19$
$(2, 2)$	Bocci (2005)

Conjecture

$\mathbb{P}^m \times \mathbb{P}^n$

(a, b)

is never defective, except for

1) $b = 1$, $m \geq 2$ and it is unbalanced ;
2) $(m, n) = (m, 1)$, $(a, b) = (2, 2d)$;
3) $(m, n) = (3, 4)$, $(a, b) = (2, 1)$;
4) $(m, n) = (m, 2)$, $(a, b) = (2, 2)$;
5) $(m, n) = (2k + 1, 2)$, $k \geq 1$, $(a, b) = (2, 1)$;
6) $(m, n) = (2, 1)$, $(a, b) = (3, 1)$;
7) $(m, n) = (2, 2)$, $(a, b) = (2, 2)$;
8) $(m, n) = (3, 3)$, $(a, b) = (2, 2)$;
9) $(m, n) = (3, 4)$, $(a, b) = (2, 2)$.

Abo, Brambilla (2011)

Conjecture

$\mathbb{P}^m \times \mathbb{P}^n$	for $(a, b) \geq (3, 3)$, it is never defective
(a, b)	Abo, Brambilla (2009)

SECANT VARIETIES OF SEGRE and SEGRE-VERONESE WITH MANY FACTORS

The unbalanced case

$\mathbb{P}^{n_1} \times .... \times \mathbb{P}^{n_t} \times \mathbb{P}^n$

$(1, ..., 1)$

is defective for
$N - \sum_{i=1}^{t} n_i + 1 < s \leq \min\{n; N\}$ where
$N = \prod_{i=1}^{t}(n_i + 1) - 1$

C. - Geramita - Gimigliano (2005)

$\mathbb{P}^{n_1} \times .... \times \mathbb{P}^{n_t} \times \mathbb{P}^n$

$(d_1, ..., d_t, 1)$

is defective for
$N - \sum_{i=1}^{t} n_i + 1 < s \leq \min\{n; N\}$, where
$N = \prod_{i=1}^{t}\binom{n_i+d_i}{d_i} - 1$

C. - Geramita - Gimigliano (2008)
Abo - Ottaviani - Peterson (2009)

Example (unbalanced)

$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^7$

$(1,1,1)$

is 4 and 5-defective

Let $\mathbb{X}$ denote this unbalanced Segre Variety. We have:
$\mathbb{X} \subset \mathbb{P}^{47}$, expdim $\sigma_4(\mathbb{X}) = 43$, exp dim $\sigma_5(\mathbb{X}) = 47$,

$$\dim(I_{2P_1+2P_2+2P_3+2P_4})_{(1,1,1)}) \geq (6-4)(8-4) = 8$$

$$\dim(I_{2P_1+2P_2+2P_3+2P_4+2P_5})_{(1,1,1)}) \geq (6-5)(8-5) = 3$$

hence $\dim \sigma_4(\mathbb{X}) \leq 47 - 8 = 39$, $\dim \sigma_5(\mathbb{X}) \leq 47 - 3 = 44$.

Segre - many copies of $\mathbb{P}^n$

$\mathbb{P}^n \times .... \times \mathbb{P}^n$ (t times)

$(1, ..., 1)$

has the expected dimension for $s = p(n+1)$

for $n = 1$, and some t,
Sloane(1982), Hill (1986), Roman (1992);
for $n = 1$, C. - Geramita - Gimigliano (2005);
for $n > 1$, Abo - Ottaviani - Peterson (2009).

Many copies of $\mathbb{P}^1$

$\mathbb{P}^1 \times .... \times \mathbb{P}^1$ (t times)

$(1, ..., 1)$

is never defective if $t \geq 5$

$t = 6$: Draisma (2008);
$t \geq 7$: C. - Geramita - Gimigliano (2011)

$\mathbb{P}^1 \times .... \times \mathbb{P}^1$ (t times)

$(d_1, ..., d_r)$

is never defective if $t \geq 5$

Laface - Postinghel (2013)

Conjecture for Segre Varieties (true for $s \leq 6$)

$\mathbb{P}^{n_1} \times .... \times \mathbb{P}^{n_t}$

$(1, ..., 1)$

is never defective, except

1) unbalanced;
2) $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$, for $s = 5$; $\delta_5 = 1$;
2) $\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n$, with n even,
   for $s = \frac{3n}{2} + 1$; $\delta_s = 1$;
3) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n$,
   for $s = 2n + 1$; $\delta_{2n+1} = 1$.

Abo - Ottaviani - Peterson (2009)

Example

$\mathbb{P}^1 \times .... \times \mathbb{P}^1$ (9 times)

$(1, ..., 1)$

$s = 51$

We have to compute $\dim(I_X)_9$
where

$$X = 8Q_1 + ... + 8Q_9 + 2P_1 + ... + 2P_{51} \subset \mathbb{P}^9$$

- Specialize on a hyperplane H the points $Q_2, ..., Q_9, P_1, ..., P_{25}$;
  specialize P_{27} on $< Q_1, Q_2, Q_3, P_{26} >= \Pi_1 \simeq \mathbb{P}^3$,
  specialize P_{29} on $< Q_1, Q_4, Q_5, P_{28} >= \Pi_2 \simeq \mathbb{P}^3$.
- Π_1 and Π_2 are in the base locus of the hypersurfaces defined by the forms of $(I_X)_9$.

- The 26 lines $L_i =< Q_1, P_i >$ $(i = 26, ..., 51)$ are in the base locus of the hypersurfaces defined by the forms of $(I_X)_9$.
- Let
$$Y = \Pi_1 + \Pi_2 + L_{26} + ... + L_{51} + X.$$
- Now use Castelnuovo Lemma:
$$\dim(I_Y)_9 \leq \dim(I_{Res_H Y})_8 + \dim(I_{Tr_H})_9.$$

METHOD FOR REDUCIBLE FORMS

Method for varieties of reducible forms

Recall that

$$\mathbb{X}_{n,\lambda} \subset \mathbb{P}^N, \quad N = \binom{n+d}{d} - 1$$

is the variety parameterizing forms F which are the product of r forms F_i with deg $F_i = d_i$, that is,

$$F = F_1 \cdots F_r.$$

Varieties of λ-reducible forms

Very little is known about the secant varieties of the varieties of reducible forms.

2011 - Arrondo -Bernardi

- $\lambda = (1, ..., 1)$ (Split hypersurfaces)

2012 - Shin

- Secant line variety to the varieties of split plane curves

2014 - Abo

- All the higher secant varieties of split plane curves

Varieties of λ-reducible forms

Conjecture (Arrondo, Bernardi)

- The higher secant varieties for split hypersurfaces always have the expected dimension

Step 1 - Terracini's Lemma

Let $Q \in \sigma_s(\mathbb{X})$ be a generic point

$$Q \in < P_1, ..., P_s >, \qquad (P_i \in \mathbb{X}_{n,\lambda} \subset \mathbb{P}^N).$$

Then by Terracini's Lemma :

$$T_Q(\sigma_s(\mathbb{X})) = < T_{P_1}(\mathbb{X}_{n,\lambda}), \dots, T_{P_s}(\mathbb{X}_{n,\lambda}) >$$

that is, the dimension of $\sigma_s(\mathbb{X}_{n,\lambda})$ is the dimension of the linear span of $T_{P_1}(\mathbb{X}_{n,\lambda}), \dots, T_{P_s}(\mathbb{X}_{n,\lambda})$.

Step 2- The tangent space to $\mathbb{X}_{n,\lambda}$ at a general point

Let $P = [F] \in \mathbb{X}_{n,\lambda}$ be a general point,

$$F = F_1 \cdots F_r$$

and let I_P be the following ideal

$$I_P = \left(\frac{F}{F_1}, \dots, \frac{F}{F_r} \right)$$

then

$$T_P = \mathbb{P}\left((I_P)_d \right).$$

Step 3- The dimension of $\sigma_s(\mathbb{X}_{n,\lambda})$

Let $P_1, ..., P_s$ be general points of $\mathbb{X}_{n,\lambda}$, and let

$$I = I_{P_1} + \cdots + I_{P_s}$$

then

$$\dim \sigma_s(\mathbb{X}_{n,\lambda}) = \dim\, (I)_d - 1.$$

Note that, if $s = 2$

$$\dim\, (I)_d = \dim(I_{P_1})_d + \dim(I_{P_2})_d - \dim(I_{P_1} \cap I_{P_2})_d.$$

Varieties of λ-reducible forms

2014 - - Geramita, Gimigliano, Shin

- All the higher secant line variety to the varieties of λ-reducible curves

2015 - - Geramita, Gimigliano, Harbourn, Migliore, Nagel, Shin

- Many higher secant line variety to the varieties of λ-reducible hypersurfaces

WHAT ABOUT THE WARING RANK OF A SPECIFIC FORM?

Answers

The problem of finding rk(F) is solved in few cases:

- F has degree 2.

$$\mathrm{rk}(F) = \mathrm{rk}(M)$$

where M is the symmetric matrix associated to F .

Answers

- F is a binary form, that is, $F \in \mathbb{C}[x, y]$.

  We have the Sylvester's algorithm (Sylvester 1886; Comas Seiguer 2001; Brachat, Comon, Mourrain,Tsigaridas 2009; Bernardi, Gimigliano, Idà (2011)).

  Let

  $$F^{\perp} = \{\partial \in \mathbb{C}[X, Y] \mid \partial F = 0\}.$$

  In this case $F^{\perp} = (f_1, f_2)$. If $\deg f_1 \leq \deg f_2$ we have

  $$\mathrm{rk}(F) = \begin{cases} \deg f_1 & \text{if } f_1 \text{ is square free} \\ \deg f_2 & \text{otherwise} \end{cases}$$

Answers

- If some algorithms work.

  (Iarrobino, Kanev; Landsberg, Teitler; Buczynska, Buczynski; Brachat, Comon, Mourrain, Tsigaridas; Bernardi, Gimigliano, Idà; Oeding, Ottaviani)

Answers

- $F \in \mathbb{C}[x, y, z]$ has degree 3, i.e. F represents a cubic curve.

  We have an explicit algorithm (Comon, Mourrain, Reznick; 1996).

  In particular we have that :

  the maximum rank for a ternary cubic is 5.

Maximum rank

A natural question arises:

What is the maximum rank

for a form of degree d in n variables?.

Answers

- The maximum rank for ternary quartics is 7.
- The maximum rank for ternary quintics is 10.

(De Paris, 2015)

Answers

- *F* is a monomial.

$$\mathrm{rk}(x_0^{a_0} \cdot \dots \cdot x_m^{a_m}) = (a_1 + 1) \cdot \dots \cdot (a_m + 1)$$

where $1 \leq a_0 \leq \ldots \leq a_m$

(Carlini, - , Geramita, 2011)
(Buczynska, Buczynski, Teitler, 2012)

Answers

- some reducible forms ($b \geq 2$):
  - $F = x_0^a(x_1^b + x_2^b)$

$$\mathrm{rk}(F) = \begin{cases} 2b & \text{if} \quad a+1 \leq b \\ 2(a+1) & \text{if} \quad a+1 \geq b \end{cases}$$

  - $F = x_0^a(x_0^b + x_1^b + x_2^b)$

$$\mathrm{rk}(F) = \begin{cases} 2b & \text{if} \quad a+1 \leq b \\ 2(a+1) & \text{if} \quad a+1 \geq b \end{cases}$$

(Carlini - Geramita, 2012)

Answers

- $F = x_0^a(x_1^b + \cdots + x_m^b)$

$$\operatorname{rk}(F) = (a+1)m \quad \text{if} \quad a+1 \geq b$$

- $F = x_0^a(x_0^b + x_1^b + \cdots + x_m^b)$

$$\operatorname{rk}(F) = (a+1)m \quad \text{if} \quad a+1 \geq b$$

(Carlini, - , Geramita, 2012)

Answers

- the Vandemonde determinant

$$V_n = \prod_{1\leq i<j\leq n}(x_i - x_j) \in \mathbb{C}[x_1, ..., x_n]$$

$$\mathrm{rk}(V_n) = (n+1)!$$

(Carlini, - , Chiantini, Geramita, Woo, 2015)

Answers

- $F = x_0^a G(x_1, ..., x_n)$

  If $G^\perp = (g_1, \dots, g_n)$ is a complete intersection and $\deg g_i \geq a+1$, then

$$\mathrm{rk}(F) = \Pi_1^n \deg g_i$$

(Carlini, - , Chiantini, Geramita, Woo, 2015)

METHOD TO FIND THE WARING RANK OF A SPECIFIC FORM

Method

- Apolarity Lemma

  Let

$$S = \mathbb{C}[x_0, ..., x_n], \quad T = \mathbb{C}[X_0, ..., X_n],$$

$$F \in S_d,$$

$$F^\perp = \{\partial \in T \mid \partial \circ F = 0\}.$$

Let $L_1, ..., L_r$ be pairwise linearly independent linear forms, with L_i corresponding to the point P_i, and $\mathbb{X} = \{P_1, ..., P_r\}$, then

$$F = a_1 L_1^d + \cdots + a_r L_r^d \quad \Leftrightarrow I_{\mathbb{X}} \subset F^\perp.$$

Method

- Let $\mathrm{rk}(F) = r$, let $\mathbb{X}$ be a set of points apolar to F.
  Let $t \in T$ be a linear form corresponding to the linear space Π, and let

  $$I_{\mathbb{Y}} = I_{\mathbb{X}} : (t), \quad (\text{ so } \mathbb{Y} = \mathbb{X} \setminus \Pi).$$

  Since t is a non-zero divisor in $T/I_{\mathbb{Y}}$, we have the following exact sequence

  $$0 \longrightarrow (T/I_{\mathbb{Y}})_{i-1} \xrightarrow{\cdot t} (T/I_{\mathbb{Y}})_i \longrightarrow (T/(I_{\mathbb{Y}} + (t)))_i \longrightarrow 0,$$

  for $t >> 0$, we get

  $$|\mathbb{Y}| = HF(T/I_{\mathbb{Y}}, t) = \sum_{i=0}^{t} HF(T/(I_{\mathbb{Y}} + (t)), i) = \ell(T/(I_{\mathbb{Y}} + (t))).$$

.

Method

Hence , since $I_{\mathbb{X}} \subset F^{\perp}$, we get

$$|\mathbb{X}| \geq |\mathbb{Y}| = \ell(T/(I_{\mathbb{Y}} + (t))) = \ell(T/(I_{\mathbb{X}} : (t) + (t)))$$

$$\geq \ell(T/F^{\perp} : (t) + (t)),$$

and so we have a lower bound for $\mathrm{rk}(F)$.

Example

$$F = x^2y^2z^3 \in \mathbb{C}[x, y, z]$$

Let $\Pi = \{X = 0\}$. Since

$$F^\perp : (X) = (X \circ F)^\perp = (2xy^2z^3)^\perp$$

$$= (X^2, Y^3, Z^4),$$

we have $F^\perp : (X) + (X) = (X, Y^3, Z^4)$ and so

$$\Delta HF(T/(F^\perp : (X) + (X))) = 1 \quad 2 \quad 3 \quad 3 \quad 2 \quad 1$$

Hence

$$\text{rk}(F) \geq 12.$$

Example

Now since

$$F^{\perp} = (x^2y^2z^3)^{\perp} = (X^3, Y^3, Z^4),$$

the ideal

$$(X^4 - Z^4,\ Y^4 - Z^4) \subset F^{\perp}$$

is the ideal of 12 distinct points.
It follows that

$$\mathrm{rk}(F) \leq 12.$$

Final observation

Now note that

$$F = x(y^2 + z^2) = xy^2 + xz^2.$$

$$\mathrm{rk}(xy^2) = 3\ ; \quad \mathrm{rk}(xz^2) = 3\ ;$$

$$\mathrm{rk}(F) = 4 < \mathrm{rk}(xy^2) + \mathrm{rk}(xz^2).$$

Example

But if we consider

$$F = xy^2 + zw^2 \in \mathbb{C}[x, y, z, w]$$

since

$$\text{rk}(xy^2) = 3 \; ; \quad \text{rk}(zw^2) = 3 \; ; \quad \text{rk}(F) = 6,$$

we have

$$\text{rk}(F) = \text{rk}(xy^2) + \text{rk}(zw^2).$$

Strassen's Additivity Conjecture

Let F and G be homogeneous polynomials in different sets of variables.

$$\mathrm{rk}(F + G) = \mathrm{rk}(F) + \mathrm{rk}(G)?$$

Thanks for your attention !