Singular loci of 3rd secant varieties of Veronese embeddings

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Background : tensors and tensor rank

The tensor product $V_1 \otimes \cdots \otimes V_d$ of d vector spaces $V_1, \cdots, V_d$ is a basic mathematical object which is fundamental in natural sciences and useful in many applications, including *Signal Processing, Phylogenetics, Quantum Information Theory and Complexity Theory*, etc. We call an element $t \in V_1 \otimes \cdots \otimes V_d$ a tensor.

We call a tensor t simple if $t = v_1 \otimes v_2 \otimes \cdots \otimes v_d$ with each $v_i \in V_i$.

e.g.) $t = e_1 \otimes e_2 + e_1 \otimes e_3 = e_1 \otimes (e_2 + e_3)$ ($\therefore$ simple)
$t = e_1 \otimes e_2 + e_2 \otimes e_1$ (Not simple)

The rank of a given tensor t, $R(t)$ is defined as the minimum number of simple tensors needed to express it as the sum.

e.g.) $R(e_1 \otimes e_2 + e_1 \otimes e_3) = 1$, $R(e_1 \otimes e_2 + e_2 \otimes e_1) = 2$

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Geometry of Tensors : Border rank

In $V_1 \otimes V_2$, the set $\{t \mid R(t) \leq k\}$ is Zariski-closed. But in case of 3-way tensors and of more factors, it does not hold any more.

Notion of Border rank : a tensor t has border rank r if $r = \min\{s \mid t = \lim_{\epsilon \to 0} t_\epsilon,\ R(t_\epsilon) = s\}$. Denote this by $\underline{R}(t)$.

Example

Let A, B, C be 3-dimensional vector spaces and a_i, b_j, c_k be basis elements for each vector space. Say

$$t = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 .$$

Note that $R(t) = 3$. But, $\underline{R}(t) = 2$, because $t = \lim_{\epsilon \to 0} t(\epsilon)$, where

$$t(\epsilon) = \frac{1}{\epsilon}\{(\epsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \epsilon a_2) \otimes (b_1 + \epsilon b_2) \otimes (c_1 + \epsilon c_2)$$

and $R(t(\epsilon)) = 2$.

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Geometry of Tensors : Higher secant variety

For tensor product $A \otimes B \otimes C$, there is the algebraic variety parametrizing decomposable (rank 1) tensors, which is Segre variety $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$. For instance, tensors of rank 2, like $a_0 \otimes b_0 \otimes c_0 + a_1 \otimes b_1 \otimes c_1$, lie in the line joining $a_0 \otimes b_0 \otimes c_0$ and $a_1 \otimes b_1 \otimes c_1$ on X. Tensors of rank k lie in the span of honest k points on the Segre variety.

The k-th secant variety of $X \subset \mathbb{P}W$, which is denoted by $\sigma_k(X)$, is defined by

$$\sigma_k(X) = \overline{\bigcup_{x_1 \cdots x_k \in X} \mathbb{P}\langle x_1 \cdots x_k \rangle} \subset \mathbb{P}W \tag{1}$$

where $\langle x_1 \cdots x_k \rangle \subset W$ denotes the linear span of the points $x_1 \cdots x_k$ and the overline denotes Zariski closure. It is the algebraic variety containing all tensors t with $\underline{R}(t) \leq k$.

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Geometry of parameter spaces of tensors

Similarly, Veronese variety can also be served as parameter space of symmetric tensors.

Let V be an $(n+1)$-dimensional complex vector space and consider a $t \in V \otimes V \otimes \cdots \otimes V$ (d-times). We call t symmetric tensor if t is invariant under permuting factors.

Let $W = S^d V$ be the d-th symmetric power of V. We can also think of W as the space of homogeneous polynomials of degree d in $n+1$ variables. The d-th Veronese embedding is the map

$$v_d : \mathbb{P}V \to \mathbb{P}W, \quad v_d([x]) = [x^d] .$$

In char 0, $W = S^d V$ can also be thought as the subspace of symmetric d-way tensors in $V^{\otimes d}$. Say $X = v_d(\mathbb{P}V)$.

Then,

$$X = v_d(\mathbb{P}V) \longleftrightarrow \{\text{rank one symmetric } d\text{-way tensors}\}$$

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Geometry of parameter spaces of tensors

Thus, it's natural to study geometry of higher secant variety of Veronese $\sigma_k(v_d\mathbb{P}V)$ (i.e. geometry of symmetric tensors of border rank at most k) for symmetric tensor problem.

Today, we consider singular loci of $\sigma_k(v_d\mathbb{P}V)$, $\mathrm{Sing}(\sigma_k(v_d\mathbb{P}V))$.

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Today, we consider singular loci of $\sigma_k(v_d\mathbb{P}V)$, $\mathrm{Sing}(\sigma_k(v_d\mathbb{P}V))$.

Singular loci of secant variety

Q: What is known for the singular locus of secant variety?

A: $\mathrm{Sing}(\sigma_{k+1}(X)) \supset \sigma_k(X)$ unless $\sigma_{k+1}(X)$ is linear (using Terracini lemma).

Terracini lemma For irreducible varieties $X, Y \subset \mathbb{P}V$ and for any $x \in X, y \in Y, z \in \langle x, y \rangle$, we have

$$T_z J(X,Y) \supset \langle T_x X, T_y Y \rangle$$

and "=" holds for general choices of x, y, z.

pf. Choose any $y \in \sigma_k(X)$. Then, for any $x \in X$

$$\begin{aligned} & T_y \sigma_{k+1}(X) \supset \langle T_y \sigma_k(X), T_x X \rangle \\ \Rightarrow\ & T_y \sigma_{k+1}(X) \supset \langle y, X \rangle \supset \langle X \rangle = \langle \sigma_{k+1}(X) \rangle \end{aligned}$$

Since $\sigma_{k+1}(X)$ is not linear, $\dim T_y \sigma_{k+1}(X) > \dim \sigma_{k+1}(X)$ for any $y \in \sigma_k(X)$.

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Singular loci of secant variety

Problem Let $V = \mathbb{C}^{n+1}$. Determine for which triple (k, d, n) it does hold that the singular locus

$$\mathrm{Sing}(\sigma_k(v_d(\mathbb{P}V))) = \sigma_{k-1}(v_d(\mathbb{P}V))$$

for every $k \geq 2, d \geq 2$ and $n \geq 1$ or describe $\mathrm{Sing}(\sigma_k(v_d(\mathbb{P}V)))$ if it is not the case.

Known results The following is known:

- First, it is classical that "=" is true for the binary case (i.e. $n = 1$)
- Also true for symmetric matrices (the case of quadratic forms (i.e. $d = 2$)
- Kanev proved that this holds for $k = 2$ and any d, n.

Thus, we only need to take care of the cases of $k \geq 3, d \geq 3$ and $n \geq 2$. From now on, $X = v_d(\mathbb{P}V)$.

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for every $k \geq 2, d \geq 2$ and $n \geq 1$ or describe $\mathrm{Sing}(\sigma_k(v_d(\mathbb{P}V)))$ if it is not the case.

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Degenerate forms and Non-degenerate forms

For any form $f \in S^dV$, following Landsberg-Teitler, we define the span of f to be $\langle f\rangle := \{\partial \in V^\vee | \partial(f) = 0\}^\perp$ in V.

So, f belongs to $S^d\langle f\rangle \subset S^dV$ and $\dim\langle f\rangle$ is the minimal number of variables in which we can express f as a homogeneous polynomial of degree d.

Note that $\dim\langle f\rangle = 1$ means $f \in v_d(\mathbb{P}V)$ by definition. We say a form $f \in \sigma_3(X) \setminus \sigma_2(X)$ to be degenerate if $\dim\langle f\rangle = 2$ and non-degenerate otherwise. Let's denote the locus of all degenerate forms in $\sigma_3(X) \setminus \sigma_2(X)$ by D.

In our case, by the equations from symmetric flattenings we know that for any $f \in \sigma_3(X) \setminus \sigma_2(X)$

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SL_{n+1}-orbits for $\sigma_3(X) \setminus \sigma_2(X)$ and their normal forms

Since there is a natural $\mathrm{SL}_{n+1}(\mathbb{C})$-group action on $\sigma_3(X)$, we may use the $\mathrm{SL}_{n+1}(\mathbb{C})$-orbits inside $\sigma_3(X)$ for our study of singularity.

First, suppose $f \in \sigma_3(X) \setminus \sigma_2(X)$ is non-degenerate (i.e. $\dim\langle f \rangle = 3$). There are 3 normal forms for such forms (by Landsberg-Teitler)

- (Fermat type) $x_0^d + x_1^d + x_2^d$
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- (Mixed type) $x_0^{d-2}x_1^2 + x_0^{d-1}x_2$

There are also degenerate forms D. So, we have roughly 4 types of forms we need to consider.

Note that for a general $f \in D$, we have a normal form $x_0^d + \alpha x_1^d + \beta(x_0 + x_1)^d$ for some nonzero $\alpha, \beta \in \mathbb{C}$.

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Reduction to the case $n = 2$ by fibration

For the locus of non-degenerate orbits in $\sigma_3(X) \setminus \sigma_2(X)$, we may consider a useful reduction method through the following arguments:

- For each $f \in \sigma_3(v_d(\mathbb{P}^n)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}^n)))$, $\exists$ a unique 3-dimensional subspace U such that $f \in \sigma_3(v_d(\mathbb{P}U))$.
- Thus, $\sigma_3(v_d(\mathbb{P}^n)) \setminus (D \cup \sigma_2(v_d(\mathbb{P}^n)))$ is smooth if $\sigma_3(v_d(\mathbb{P}^2)) \setminus \left(D \cup \sigma_2(v_d(\mathbb{P}^2))\right)$ is smooth for every $n \geq 2$ and $d \geq 3$, because the following map

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Case of $k = 3, d = 3, n = 2$: Aronhold hypersurface

For the first case $(k, d, n) = (3, 3, 2)$, note that D is empty when $d = 3$ (from equations of symmetric flattenings).

Classically, $\sigma_3(v_3(\mathbb{P}^2))$ in $\mathbb{P}^9$ is known as 'Aronhold hypersurface' and is defined by Pfaffian of the Young flattening

$$S_{(2,1)}(\mathbb{C}^3) \to S_{(3,2,1)}(\mathbb{C}^3) .$$

The singular locus of the Aronhold hypersurface $\sigma_3(v_3(\mathbb{P}^2))$ in $\mathbb{P}^9$ is equal to $\sigma_2(v_3(\mathbb{P}^2))$. It can be checked via several ways (e.g. using `Macaulay2`).

When $d = 3$, we also have an immediate corollary using fibration reduction:

Corollary ($d = 3$ case)

For every $n \geq 2$ and $d = 3$, $\sigma_3(v_3(\mathbb{P}^n)) \setminus \sigma_2(v_3(\mathbb{P}^n))$ is smooth.

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Symmetric flattening and Equations of $\sigma_3(v_d(\mathbb{P}^n))$

Consider the polynomial ring $S^\bullet V = \mathbb{C}[x_0, \ldots, x_n]$ (we call this ring S) and consider another polynomial ring $T = S^\bullet V^\vee = \mathbb{C}[y_0, \ldots, y_n]$, where $V^\vee$ is the *dual space* of V. Define the differential action of T on S as follows: for any $g \in T_{d-k}, f \in S_d$, we set

$$g \cdot f = g(\partial_0, \partial_1, \ldots, \partial_n) f \in S_k \ . \tag{2}$$

Let us take bases for S_k and T_{d-k} as

$$\mathbf{X}^I = \frac{1}{i_0! \cdots i_n!} x_0^{i_0} \cdots x_n^{i_n} \quad \text{and} \quad \mathbf{Y}^J = y_0^{j_0} \cdots y_n^{j_n} \ , \tag{3}$$

with $|I| = i_0 + \cdots + i_n = k$ and $|J| = j_0 + \cdots + j_n = d - k$. For a given $f = \sum_{|I|=d} a_I \cdot \mathbf{X}^I$ in S_d, we have a linear map

$$\phi_{d-k,k}(f) : T_{d-k} \to S_k, \quad g \mapsto g \cdot f$$

for any k with $1 \leq k \leq d-1$, which can be represented by the following $\binom{k+n}{n} \times \binom{d-k+n}{n}$-matrix.

Symmetric flattening and Equations of $\sigma_3(v_d(\mathbb{P}^n))$

We call this the *symmetric flattening* (or *catalecticant*) of f.

It is obvious that if f has rank 1, then any symmetric flattening $\phi_{d-k,k}(f)$ has rank 1. By subadditivity of matrix rank, we also know that rank $\phi_{d-k,k}(f) \leq r$ if $\underline{R}(f) \leq r$. Landsberg-Ottaviani showed

Proposition (Defining equations of $\sigma_3(v_d(\mathbb{P}^n))$)

Let X be the n-dimensional Veronese variety $v_d(\mathbb{P}V)$ in $\mathbb{P}^N$ with $N = \binom{n+d}{n} - 1$. For any (d, n) with $d \geq 4, n \geq 2$, $\sigma_3(X)$ is defined scheme-theoretically by the 4×4-minors of the two symmetric flattenings

$$\phi_{d-1,1}(F) \colon S^{d-1}V^{\vee} \to V \quad \text{and} \quad \phi_{d-\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor}(F) \colon S^{d-\lfloor \frac{d}{2} \rfloor}V^{\vee} \to S^{\lfloor \frac{d}{2} \rfloor}V \,,$$

where F is the form $\sum_{I \in \mathbb{N}^{n+1}} a_I \cdot \mathbf{X}^I$ of degree d as considering the coefficients a_I's indeterminate.

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Apolar ideal and Conormal space of $\sigma_3(X)$

For any given form $f \in S^d V$, we call $\partial \in T_t$ *apolar* to f if the differentiation $\partial(f)$ gives zero (i.e. $\partial \in \ker \phi_{t,d-t}(f)$). And we define the apolar ideal $f^\perp \subset T$ as $f^\perp := \{\partial \in T \mid \partial(f) = 0\}$. It is straightforward to see that $f^\perp$ is indeed an ideal of T. Moreover, it is well-known that the quotient ring $T_f := T/f^\perp$ is an *Artinian Gorenstein algebra with socle degree d.*

In our case, we have a nice description of the conormal space in terms of this apolar ideal as follows:

Proposition

Suppose any form f in $S^d V$ corresponds to a (closed) point of $\sigma_3(X) \setminus \sigma_2(X)$ and that $\operatorname{rank} \phi_{d-1,1}(f) = 3, \ \operatorname{rank} \phi_{d-\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor}(f) = 3.$ *Then, for any (d, n) with $d \geq 4, n \geq 2$ we have*

$$\hat{N}^\vee_f \sigma_3(X) = (f^\perp)_1 \cdot (f^\perp)_{d-1} + (f^\perp)_{\lfloor \frac{d}{2} \rfloor} \cdot (f^\perp)_{d-\lfloor \frac{d}{2} \rfloor}, \qquad (4)$$

where the sum is taken as a $\mathbb{C}$-subspace in $T_d = S^d V^\vee$.

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Suppose any form f in $S^d V$ corresponds to a (closed) point of $\sigma_3(X) \setminus \sigma_2(X)$ and that $\operatorname{rank} \phi_{d-1,1}(f) = 3, \ \operatorname{rank} \phi_{d-\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor}(f) = 3.$ *Then, for any (d,n) with $d \geq 4, n \geq 2$ we have*

$$\hat{N}_f^\vee \sigma_3(X) = (f^\perp)_1 \cdot (f^\perp)_{d-1} + (f^\perp)_{\lfloor \frac{d}{2} \rfloor} \cdot (f^\perp)_{d-\lfloor \frac{d}{2} \rfloor}, \qquad (4)$$

where the sum is taken as a $\mathbb{C}$-subspace in $T_d = S^d V^\vee$.

Apolar ideal and Conormal space of $\sigma_3(X)$

For any given form $f \in S^d V$, we call $\partial \in T_t$ *apolar* to f if the differentiation $\partial(f)$ gives zero (i.e. $\partial \in \ker \phi_{t,d-t}(f)$). And we define the apolar ideal $f^\perp \subset T$ as $f^\perp := \{\partial \in T \mid \partial(f) = 0\}$.It is straightforward to see that $f^\perp$ is indeed an ideal of T. Moreover, it is well-known that the quotient ring $T_f := T/f^\perp$ is an *Artinian Gorenstein algebra with socle degree* d.

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Idea for the proposition

As a simple case, consider a form

$$f \in S_k := \{g \in V \otimes W \mid \mathrm{rank}(g) \leq k\} \subseteq Hom(V^\vee, W)$$

Then, $T_f S_k = \{g \in V \otimes W \mid g(\ker f) \subseteq \mathrm{im}\, f\}$

pf.) Write $f = \sum_{i=1}^k v_i \otimes w_i$. Then, $\ker f = \langle v_1, \cdots, v_k \rangle^\perp$ and $\mathrm{im}\, f = \langle w_1, \cdots, w_k \rangle$. We know $T_f S_k = \sum_{i=1}^k v_i \otimes W + V \otimes w_i$. So, for any $g \in T_f S_k$, $g(\ker f) \subseteq \mathrm{im}\, f$. The other way by dimension count.

Cor.) $\hat{N}_f^\vee S_k = (T_f S_k)^\perp = (\ker f) \otimes (\mathrm{im}\, f)^\perp \subseteq V^\vee \otimes W^\vee$.

In the symmetric tensor case, similarly, we have

$$\hat{N}_f^\vee Z(M_4(\phi_{d-k,k}(F))) = (\ker \phi_{d-k,k}(f)) \otimes (\mathrm{im}\, \phi_{d-k,k}(f))^\perp .$$

Here, we have $(\ker \phi_{d-k,k}(f)) = (f)_{d-k}^\perp$ and

$$(\mathrm{im}\, \phi_{d-k,k}(f))^\perp = (\ker \phi_{d-k,k}(f)^T) = (\ker \phi_{k,d-k}(f)) = (f)_k^\perp .$$

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Computation conormal space I

We remark that for $n = 2$ case

$$\hat{N}^{\vee}_{f}\sigma_3(X) = (f^{\perp})_{\lfloor \frac{d}{2} \rfloor} \cdot (f^{\perp})_{d-\lfloor \frac{d}{2} \rfloor} \, . \tag{5}$$

First, consider 3 different normal forms for non-degenerate forms.

Case (i) It is well-known that this Fermat-type $f_1 = x_0^d + x_1^d + x_2^d$ becomes an almost transitive $\mathrm{SL}_3(\mathbb{C})$-orbit, thus, smooth here.
Case (ii) $f_2 = x_0^{d-1}x_1 + x_2^d$ (Unmixed-type). Say $s := \lfloor \frac{d}{2} \rfloor$. For $d \geq 4$, we have $2 \leq s \leq d - s \leq d - 2$. Since the summands of f_2 separate the variables (i.e. unmixed-type),

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Thus, if we denote the ideal (Q_1, Q_2, Q_3) by I, then $\dim \hat{N}^{\vee}_{f_2}\sigma_3(X)$ is equal to the value of Hilbert function $H(I^2, t)$ at $t = d$.

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But, it is easy to see that I^2 has a minimal free resolution as

$$0 \to T(-6) \to T(-5)^6 \to T(-4)^6 \to I^2 \to 0 ,$$

which shows the Hilbert function of I^2 can be computed as

$$H(I^2, d) = 6\binom{d-4+2}{2} - 6\binom{d-5+2}{2} + \binom{d-6+2}{2}$$
$$= \begin{cases} 0 & (d \leq 3) \\ \binom{d+2}{2} - 9 & (d \geq 4) \end{cases} .$$

This implies that $\dim \hat{N}^{\vee}_{f_2} \sigma_3(X) = \binom{d+2}{2} - 9$ for any $d \geq 4$, which means that our $\sigma_3(X)$ is smooth at f_2.

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$f_3 = x_0^{d-2}x_1^2 + x_0^{d-1}x_2$ (Mixed-type). In this case, we similarly use a computation of $\dim \hat{N}^{\vee}_{f_3}\sigma_3(X)$ via $(f_3^{\perp})_s \cdot (f_3^{\perp})_{d-s}$ to show the smoothness of f_3.

Let $Q_1 := y_0y_2 - \frac{d-1}{2}y_1^2 \in T_2$. We easily see that

$$f_3^{\perp} = \left(\{Q_1, Q_2 = y_1y_2, Q_3 = y_2^2\} \bigcup \{\text{other generators in degree} \geq d-1\}\right).$$

Let I be the ideal generated by three quadrics Q_1, Q_2, Q_3. By the same reasoning as (ii), we have

$$\dim \hat{N}^{\vee}_{f_3}\sigma_3(X) = \dim(f_3^{\perp})_s \cdot (f_3^{\perp})_{d-s} = H(I^2, d) = \begin{cases} 0 & (d \leq 3) \\ \binom{d+2}{2} - 9 & (d \geq 4) \end{cases},$$

because in this case I^2 also has the same minimal free resolution $0 \to T(-6) \to T(-5)^6 \to T(-4)^6 \to I^2 \to 0$. Hence, we obtain the smoothness of $\sigma_3(X)$ at f_3.

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Computation conormal space II

Now, time for degenerate forms D. When f_D is general, $f_D = x_0^d + \alpha x_1^d + \beta(x_0 + x_1)^d$ for some $\alpha, \beta \in \mathbb{C}^*$, so we have $F = y_0^2 y_1 - y_0 y_1^2$. Even for the case f_D being not general, we have $F = y_0^2 y_1$ up to change of coordinates, because the apolar ideal of this non-general f_D corresponds to the case with one multiple root on $\mathbb{P}^1$ (Comas-Seigurs, Landsberg-Teitler).
Therefore, we obtain that

$$f_D^\perp = \left(F = y_0^2 y_1 - y_0 y_1^2 \text{ or } y_0^2 y_1, G\right) \text{ for some polynomial } G \text{ of degree } (d-1)$$

and that $f_D^\perp$ *as an ideal in* $T = \mathbb{C}[y_0, y_1, \ldots, y_n]$ has its degree parts $(f_D^\perp)_{\lfloor \frac{d}{2} \rfloor}$ and $(f_D^\perp)_{d - \lfloor \frac{d}{2} \rfloor}$, both of which are generated by $F, y_2, \ldots, y_n$, since $d \geq 4$ so that $\lfloor \frac{d}{2} \rfloor, d - \lfloor \frac{d}{2} \rfloor < d - 1$.

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Computation conormal space II

i) $d = 4$ case (i.e. $\lfloor \frac{d}{2} \rfloor = 2$) : In this case, we have

$$\hat{N}^{\vee}_{f_D}\sigma_3(X) = (f_D^{\perp})_2 \cdot (f_D^{\perp})_2 = (y_2, \ldots, y_n)_2 \cdot (y_2, \ldots, y_n)_2 = (\{y_i y_j \mid 2 \leq i, j \leq n\})_4 .$$

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$$= \binom{4+n}{4} - 5 - 4(n-1) .$$

This shows us that $\sigma_3(X)$ is singular at f_D if and only if $n \geq 3$, because the expected codimension is $\binom{4+n}{4} - 3n - 3$.

ii) $d = 5$ case (i.e. $\lfloor \frac{d}{2} \rfloor = 2$) : Recall that F is $y_0^2 y_1 - y_0 y_1^2$ or $y_0^2 y_1$, the cubic generator of $f_D^{\perp}$. Then,

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$$\dim \hat{N}^{\vee}_{f_D}\sigma_3(X) = \dim T_5 - \dim \langle y_0^5, y_0^4 y_1, \cdots, y_1^5 \rangle$$

$$- \dim \left\langle \{y_0^4 \cdot \ell, y_0^3 y_1 \cdot \ell, y_0^2 y_1^2 \cdot \ell, y_0 y_1^3 \cdot \ell, y_1^4 \cdot \ell\} \setminus \{y_0 F \cdot \ell, y_1 F \cdot \ell \mid \ell \right.$$

$$= \binom{5+n}{5} - 6 - 3(n-1) = \text{expected codim}(\sigma_3(X), \mathbb{P}S^5 V) \,.$$

Computation conormal space II

iii) $d \geq 6$ case : Here, we have

$$\hat{N}^{\vee}_{f_D}\sigma_3(X) = (f_D^{\perp})_{\lfloor \frac{d}{2} \rfloor} \cdot (f_D^{\perp})_{d-\lfloor \frac{d}{2} \rfloor} = (F, y_2, \ldots, y_n)_{\lfloor \frac{d}{2} \rfloor} \cdot (F, y_2, \ldots, y_n)_{d-\lfloor \frac{d}{2} \rfloor} .$$

$$\begin{aligned}
&\dim \hat{N}^{\vee}_{f_D}\sigma_3(X) \\
&= \dim T_d - \dim \left\langle \{y_0^{d-1} \cdot \ell, y_0^{d-2} y_1 \cdot \ell, \ldots, y_1^{d-1} \cdot \ell\} \setminus \{y_0^{d-4} F \cdot \ell, \ldots, y_1^{d-4} F \cdot \ell \mid \ell \right. \\
&- \dim \left(\{y_0^d, y_0^{d-1} y_1, \cdots, y_1^d\} \setminus \{y_0^{d-6} \cdot F^2, y_0^{d-7} y_1 \cdot F^2, \ldots, y_1^{d-6} \cdot F^2\} \right) \\
&= \binom{d+n}{d} - \{d - (d-3)\}(n-1) - \{(d+1) - (d-5)\} \\
&= \binom{d+n}{d} - 3(n-1) - 6 = \text{expected codim}(\sigma_3(X), \mathbb{P}S^d V) ,
\end{aligned}$$

which implies that $\sigma_3(X)$ is also smooth at f_D.

Conclusion

Theorem (Singularity of $\sigma_3(v_d(\mathbb{P}^n))$)

Let X be the n-dimensional Veronese variety $v_d(\mathbb{P}V)$ in $\mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$. Then, the following holds that the singular locus

$$\mathrm{Sing}(\sigma_3(X)) = \sigma_2(X)$$

as a set for all (d, n) with $d \geq 3$ and $n \geq 2$ unless $d = 4$ and $n \geq 3$. In the exceptional case $d = 4$, for each $n \geq 3$ the singular locus $\mathrm{Sing}(\sigma_3(v_4(\mathbb{P}V)))$ is $D \cup \sigma_2(v_4(\mathbb{P}V))$, where D denotes the locus of all the degenerate forms f (i.e. $\dim\langle f\rangle = 2$) in $\sigma_3(v_4(\mathbb{P}V)) \setminus \sigma_2(v_4(\mathbb{P}V))$.

We can sum up all the relevant results into the following table:

$(\mathbf{k}, \mathbf{d}, \mathbf{n})$	$Sing\sigma_k(v_d(\mathbb{P}^n))$	**Comment**
$(\geq 2, \geq 2, 1)$	σ_{k-1}	Classical; case of binary forms
$(\geq 2, 2, \geq 1)$	σ_{k-1}	Symmetric matrice case
$(2, \geq 2, \geq 1)$	σ_1	Kanev
$(3, 3, 2)$	σ_2	Aronhold hypersurface
$(3, \geq 4, 2)$	σ_2	
$(3, 3, \geq 3)$	σ_2	
$(3, 4, \geq 3)$	$D \cup \sigma_2$	Only exceptional case ($d = 4$)
$(3, \geq 5, \geq 3)$	σ_2	

.

Figure: Singular locus of $\sigma_k(v_d(\mathbb{P}^n))$

.

As an immediate corollary of our Theorem, we obtain defining equations of the singular locus in our third secant of Veronese embedding $\sigma_3(X)$.

Corollary

Let X be the n-dimensional Veronese embedding as above. The singular locus of $\sigma_3(X)$ is cut out by 3×3-minors of the two symmetric flattenings $\phi_{d-1,1}$ and $\phi_{d-2,2}$ unless $d = 4$ and $n \geq 3$ case, in which the (set-theoretic) defining ideal of the locus is the intersection of the ideal generated by the previous 3×3-minors and the ideal generated by 3×3-minors of $\phi_{d-1,1}$ and 4×4-minors of $\phi_{d-\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor}$.

Compensating for Kwak's lecture

- By our theorem, for ND(2)-varieties,

$$\deg(X) \geq \binom{e+2}{2}$$

$$\deg(X) = \binom{e+2}{2} \Leftrightarrow I_X \text{ has ACM 3-linear resolution.}$$

- Note that $\deg(X) = \binom{e+1}{1} \Leftrightarrow I_X$ has ACM 2-linear resolution and del Pezzo-Bertini classification gave the geometric classification.
- **Problem** : What is a geometric classification/or characterization of 'Minimal degree varieties of the second kind'?

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Problem : Classfication of VMDs of 2nd kind

Example (Varieties having ACM 3-linear resolution)

(a) Hypercubic ($e = 1$);

(b) 3-minors of 4×4 generic symmetric matrix (i.e. $Sec(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);

(c) 3-minors of $3 \times (e+2)$ sufficiently generic matrices (e.g. $Sec(RNS)$).

- All the varieties with ACM 2-linear resolution are **determinantal**.
- All the examples above are also **determinantal**.
- **Question** : Anything else? Probably yes. But, for $n \geq 2$ there is **no known example** to give ACM 3-linear resolution outside the list.

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Thank you!