# Lefschetz properties and moving beyond the SHGH Conjecture

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(Motivated by a paper of Di Gennaro - Ilardi - Vallès.)

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In this sense, the main topic of this talk shares this Lefschetz philosophy. There will be a direct connection at the end.

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Let $P \in \mathbb{P}^2$. What is the dimension of the linear system of plane curves of degree $j$ passing through $P$?

Answer. Regardless of the choice of $P$, the dimension is

$$\dim \mathcal{L}_j - 1 = \binom{j+2}{2} - 2.$$

That is, *$P$ imposes one independent condition* on $\mathcal{L}_j$.

Easy Question 2. Consider the complete linear system $\mathcal{L}_j$ of plane curves of degree $j$.

Let $\{P_1, \dots, P_d\} \subset \mathbb{P}^2$ be a set of points. How many conditions do $P_1, \dots, P_d$ impose on $\mathcal{L}_j$?

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Answer. If there aren't too many points, they impose independent conditions. More generally, they impose $\min\left\{\binom{j+2}{2}, d\right\}$ independent conditions.

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$mP$ is the scheme defined by the ideal $I_P^m$.

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Naive guess: Just like the case where $m_i = 1$ for all $i$ (mentioned above), if there is "room" then they should impose

$$\binom{m_1+1}{2} + \binom{m_2+1}{2} + \cdots + \binom{m_d+1}{2}$$

independent conditions.

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But consider $j = 4$. Since $\dim k[x, y, z]_4 = 15$, this means the "prediction" is that there is no curve of degree 4 double at all 5 points.

Is this true?

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Question. Staying in $\mathbb{P}^2$, what goes beyond this conjecture (as suggested in the title)?

Answer. Start with a linear system $\mathcal{L}$ that is not complete! Specifically,

$$\mathcal{L} = \left|[I_Z]_{j+1}\right|,$$

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- Are there connections between this and other interesting questions?
- Clearly this question is intractable as stated. What is the first non-trivial special case? Even $d = 1$ is interesting!

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Let $\mathcal{A}_f$ be the line arrangement in $\mathbb{P}^2$ defined by $f$. Note the lines of $\mathcal{A}_f$ are dual to a reduced set of $d$ distinct points $Z$.

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- $J = (f_x, f_y, f_z, f)$

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In this case it is not necessarily true that $f$ is in the ideal generated by its first partial derivatives, although it can happen.

Example. Let

$$f = xyz(x+y) = (x^2y + xy^2)z \text{ with } \mathrm{char}(K) = 2.$$

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Example. Let

$$f = xyz(x+y)(x+z) \text{ with } \mathrm{char}(K) = 5.$$

One can check that $f \notin J'$ so $J' \subsetneq J$.

Define the submodule

$$D(Z) \subset R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$$

to be the $K$-linear derivations $\delta$ such that $\delta(f) \in Rf$.

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Let $\mathcal{D}_Z$, $\widetilde{D(Z)}$ be the sheafifications of $D_0(Z)$ and $D(Z)$ resp. What can we say about $\mathcal{D}_Z$ and about $\widetilde{D(Z)}$?

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- The restriction of $\mathcal{D}_Z$ to a general line $\ell \cong \mathbb{P}^1$ splits as a direct sum
$$\mathcal{O}_{\mathbb{P}^1}(-a_Z) \oplus \mathcal{O}_{\mathbb{P}^1}(-b_Z)$$
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  The ordered pair $(a_Z, b_Z)$ is the splitting type of $\mathcal{D}_Z$ (or $Z$).

# Merging the two topics

Fix a set of points, $Z \subset \mathbb{P}^2$.

Let $\mathcal{L} = \left|[I_Z]_{j+1}\right|$. (Incomplete linear system.) Let $P \in \mathbb{P}^2$ be a general point.

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We expect that $jP$ will impose

$$\min\left\{\binom{j+1}{2}, \dim[I_Z]_{j+1}\right\}$$

independent conditions on $\mathcal{L}$.

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Note $Z$ might have unexpected curves in more than one degree.

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1. What properties of $Z$ force the existence of an unexpected curve? (Necessary and sufficient conditions.)

2. The definition of unexpected curves allowed for their existence in more than one degree. If there are unexpected curves at all, in what degrees do they exist?

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Recall that the splitting type of $\mathcal{D}_Z$ is $(a_Z, b_Z)$ with $a_Z \leq b_Z$ and $a_Z + b_Z = \deg Z - 1$.

The following is a (non-trivial) consequence of a result of Faenzi and Valles.

**Lemma.**

$$\dim[I_{Z+jP}]_{j+1} = \max\{0, j - a_Z + 1\} + \max\{0, j - b_Z + 1\}.$$

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# Main results

Definition. Let $Z$ be a reduced 0-dimensional subscheme of $\mathbb{P}^2$.

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Corollary *For the splitting type $(a_Z, b_Z)$ of $\mathcal{D}_Z$ we have $a_Z = m_Z$ and $b_Z = u_Z + 1$.*

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Is the converse true?

**Theorem.**

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Note

$$(b) \quad \Leftrightarrow \quad h^1(\mathcal{I}_Z(t_Z)) = 0$$

$\Leftrightarrow$ $Z$ imposes independent conditions on curves of degree $t_Z$.

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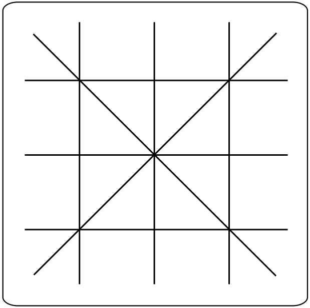
## Structure of unexpected curves

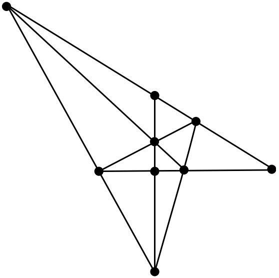
We give a careful description. Briefly, an unexpected curve consists of the union of

- an irreducible rational curve of some degree $e$ having a point of multiplicity $e - 1$ and
- certain lines.

# Some Examples/results

Example. [Di Gennaro, Ilardi and Vallès] (This motivated our paper!)





The points dual to the B-3 configuration admit an unexpected curve of degree 4.

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Consider the line configuration $\mathcal{A}_f$ given by the lines defined by

$$\begin{aligned} f \;=\;& xyz(x+y)(x-y)(2x+y)(2x-y)(x+z)(x-z) \\ & (y+z)(y-z)(x+2z)(x-2z)(y+2z)(y-2z) \\ & (x-y+z)(x-y-z)(x-y+2z)(x-y-2z). \end{aligned}$$

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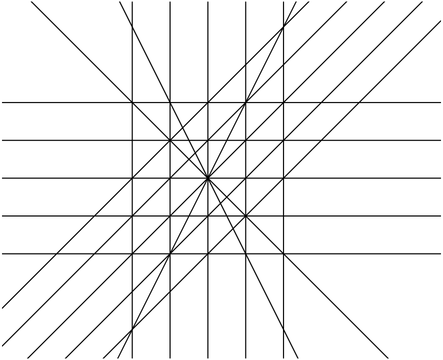
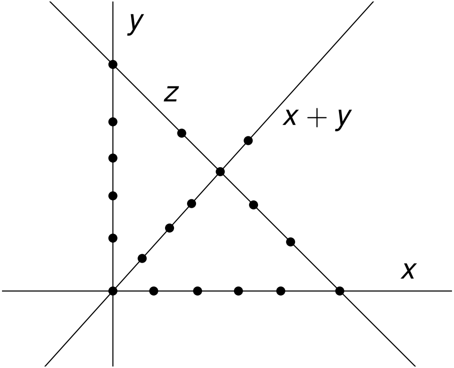
Note $d = 19$. Let $Z$ be the corresponding reduced scheme consisting of the 19 points that are dual to these lines.

The following figures show $\mathcal{A}_f$ and $Z$.




y
z
x + y
x

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Thus in our example there is an unexpected curve for each degree $j + 1$ with

$$8 + 1 \leq j + 1 < 9 + 1.$$

That is, 9 is the only degree in which $Z$ admits an unexpected curve. We have verified experimentally (using our criterion for irreducibility) that this curve is not irreducible.

Assume for convenience that $K$ has characteristic zero.

Definition. A line arrangement $\mathcal{A}_f$ in $\mathbb{P}^2$ is free if $\mathcal{D}_Z$ is free, i.e. if $J = J' = (f_x, f_y, f_z)$ is a saturated ideal.

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(This is far from talking about a general set of points.)

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Then $\dim[I_Z]_3 = 3$ and $2P$ should impose 3 conditions, so we expect there not to be a cubic containing $Z$ and singular at a general point $P = [\alpha, \beta, \gamma]$.

But in fact there is one. One can easily check that

$$f = \alpha^2 yz(y + z) + \beta^2 xz(x + z) + \gamma^2 xy(x + y)$$

defines a curve $C$ (reduced and irreducible in fact) which is singular at $P$, and hence $C$ is an unexpected curve of degree 3 for $Z$.

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where $k \geq 3$, $a_1, \ldots, a_k \geq 2$ and $L_1, \ldots, L_k$ linear forms in $K[x, y, z]$ (unlike my first talk, this time they are not necessarily general).

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But the above question about $\times L^2$ is meaningful.

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- $\mathcal{A}(f)$ *be a line arrangement in* $\mathbb{P}^2$, *where* $f = L_1 \cdots L_d$.
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*Then*

$$Z \text{ has an unexpected curve of degree } j+1$$

*if and only if*

$$\times L^2 : [R/I]_{j-1} \to [R/I]_{j+1} \text{ does not have maximal rank.}$$

Theorem. *Let*

- $\mathcal{A}(f)$ *be a line arrangement in* $\mathbb{P}^2$*, where* $f = L_1 \cdots L_d$*.*
- $Z$ *be the set of points in* $\mathbb{P}^2$ *dual to these lines.*
- $I = (L_1^{j+1}, \dots, L_d^{j+1})$*.*

*Then*

$Z$ *has an unexpected curve of degree* $j+1$

*if and only if*

$\times L^2 : [R/I]_{j-1} \to [R/I]_{j+1}$ *does not have maximal rank.*

There is one additional ingredient to prove this.

Theorem. (J. Emsalem and A. Iarrobino)

*Let $\wp_1, \ldots, \wp_m$ be the ideals of m distinct points in $\mathbb{P}^{n-1}$.*

Theorem. (J. Emsalem and A. Iarrobino)

*Let $\wp_1, \dots, \wp_m$ be the ideals of $m$ distinct points in $\mathbb{P}^{n-1}$.*

*Let $L_1, \dots, L_m$ be the dual linear forms.*

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*Choose positive integers $a_1, \ldots, a_m$.*

*Then for any integer $k \geq \max\{a_i\}$,*

$$\dim_K \left[R/(L_1^{a_1}, \ldots, L_m^{a_m})\right]_k = \dim_K \left[\wp_1^{k-a_1+1} \cap \cdots \cap \wp_m^{k-a_m+1}\right]_k.$$

**Theorem.** (J. Emsalem and A. Iarrobino)

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In particular, for a general point $P$ with defining ideal $\wp$ and dual linear form $L$, we have

$$\dim_K \left[R/(L_1^{j+1}, \ldots, L_d^{j+1}, L^2)\right]_{j+1} = \dim_K \left[\wp_1^1 \cap \cdots \cap \wp_n^1 \cap \wp^j\right]_{j+1} .$$

Thank you.