Comments on Topological Strings and Quantum Mechanics

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The general philosophy

Topological String (TS): framework for quantizing classical systems; can provide *analytic* solution to problems in Quantum Mechanics (QM)*

However: TS only defined perturbatively, many observables ill-defined (asymptotic series, divergences, …) due to missing *non-perturbative* terms

To solve QM problems analytically we need a complete theory of TS
$\Longrightarrow$ need to understand non-perturbative terms in TS; how?

*(QM problems: "relativistic", exponential dependence on momenta)

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$$\boxed{\begin{array}{c}\text{QM problem,}\\ \text{unknown solution}\end{array}} + \boxed{\begin{array}{c}\text{TS, unknown}\\ \text{non-pert. terms}\end{array}} \underset{\substack{\text{QM, TS}\\ \text{insight}}}{\Longrightarrow} \boxed{\begin{array}{c}\text{fix non-pert. TS terms,}\\ \text{solve QM problem}\end{array}}$$

Today: review what is known so far and study QM *eigenfunctions* via TS

[*](QM problems: "relativistic", exponential dependence on momenta)

Quantum Mechanics and Topological Strings

Quantum Mechanics from Topological Strings

Setup: TS on a local toric Calabi-Yau threefold X (in $X \times \mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_R$)

Toric Calabi-Yau X:	mirror symmetry	Mirror Calabi-Yau $\tilde{X}$:
identified by its toric diagram;	$\Longleftrightarrow$	identified by genus g_Σ mirror curve;
$g_\Sigma + r_\Sigma$ Kahler parameters t_l		g_Σ "true" moduli u_i, r_Σ masses m_j

Mirror curve Σ : $\boxed{W_X(e^x, e^p, \{u_i\}, \{m_j\}) = 0}$

Mainly focus on cases with 5d gauge theory interpretation (Σ : SW curve)

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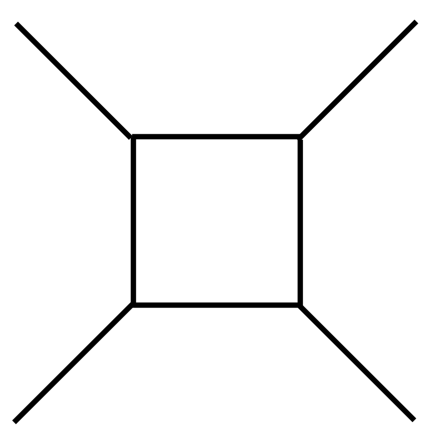
Mainly focus on cases with 5d gauge theory interpretation (Σ : SW curve)

Genus 1 example (5d pure $\mathcal{N}$ = 1 SU(2) theory / local F_0): main example



$$\Longleftrightarrow \qquad e^{-p} + m_0 e^p + e^{-x} + e^x - u = 0$$

How do we associate a Quantum Mechanical problem to X ?

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For genus 1, first rewrite the mirror curve Σ as

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Then, consider Weyl quantization of the mirror curve:

$$e^{ax+bp} \to e^{a\hat{x}+b\hat{p}}\,, \qquad [\hat{p}, \hat{x}] = -i\hbar$$

Obtain a self-adjoint operator on $L^2(\mathbb{R})$ with discrete energy spectrum $u = u^{(n)}$

$$\boxed{\widehat{O}_X(e^{\hat{x}}, e^{\hat{p}}, \{m_j\})\psi_n(x) = u^{(n)}\psi_n(x)}, \qquad n \in \mathbb{N}$$

and with inverse operator $\widehat{\rho}_X = \left[\widehat{O}_X\right]^{-1}$ of trace class

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Genus 1 example (pure SU(2) / local F_0): Quantum Mechanical problem

$$\left[e^{i\hbar\partial_x} + m_0 e^{-i\hbar\partial_x} + e^{-x} + e^x\right]\psi_n(x) = u^{(n)}\psi_n(x)$$

When the genus $g_\Sigma > 1$ we can instead define *two* different QM problems

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I) Quantum mirror curve as g_Σ finite-difference Schrodinger operators

First, pick up $u_\alpha \in \{u_i\}$ and rewrite the mirror curve Σ as

$$W_X(e^x, e^p, \{u_i\}, \{m_j\}) = 0 \quad \Longleftrightarrow \quad O_X^{(\alpha)}(e^x, e^p, \{u_i\}', \{m_j\}) = u_\alpha$$

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After Weyl quantization, obtain g_Σ self-adjoint operators on $L^2(\mathbb{R})$

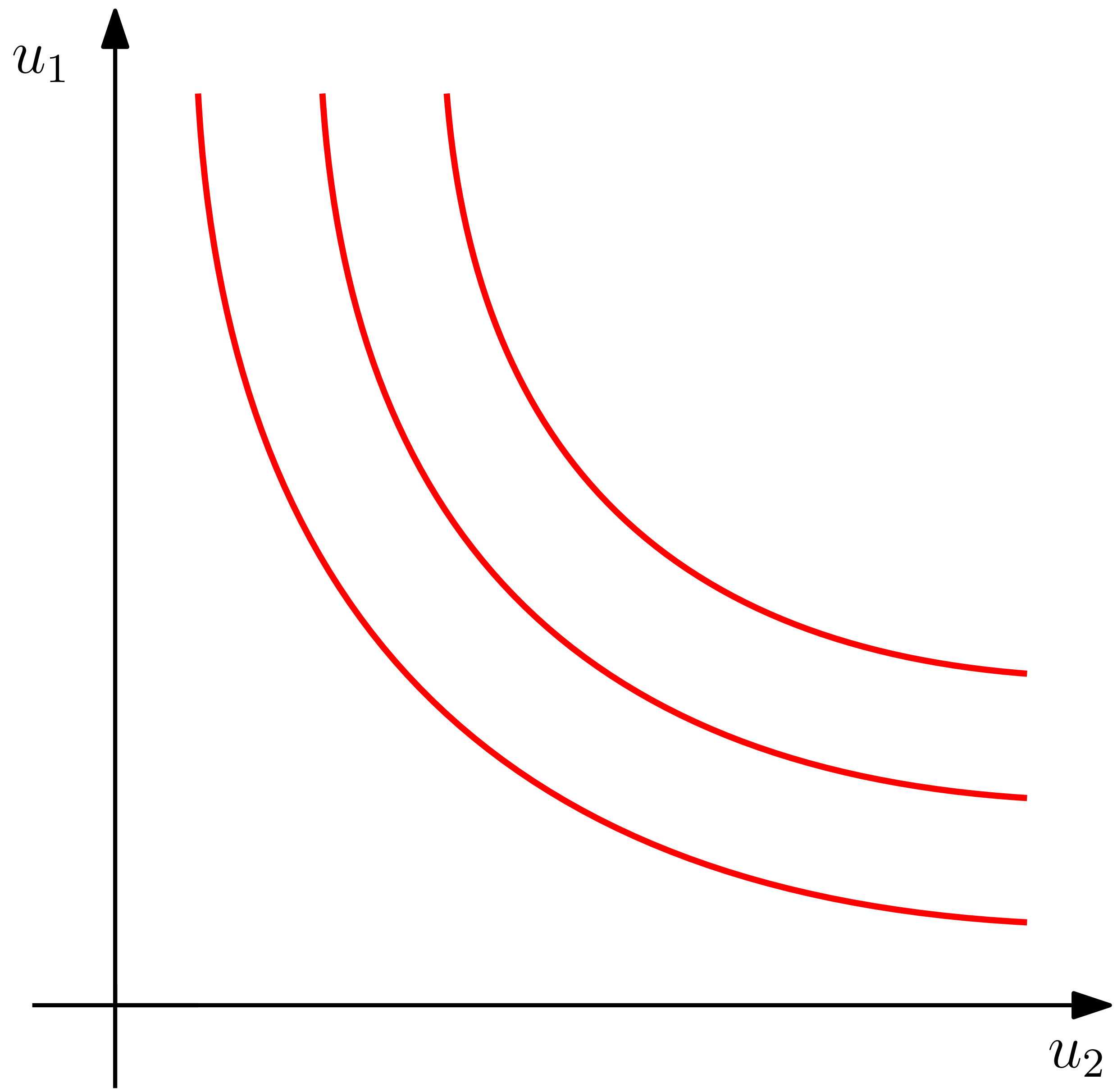
$$\boxed{\widehat{O}_X^{(\alpha)}(e^{\hat{x}}, e^{\hat{p}}, \{u_i\}', \{m_j\})\psi_n^{(\alpha)}(x) = u_\alpha^{(n)}\psi_n^{(\alpha)}(x)}, \qquad \alpha = 1, \ldots, g_\Sigma$$

with inverse operators $\widehat{\rho}_X^{(\alpha)} = \left[\widehat{O}_X^{(\alpha)}\right]^{-1}$ of trace class and discrete energy u_α

- Operators $\widehat{O}_X^{(\alpha)}$ and $\widehat{O}_X^{(\beta)}$ related by similarity transformations
- Solution $\widehat{O}_X^{(\alpha)}\psi_n^{(\alpha)}(x) = u_\alpha^{(n)}\psi_n^{(\alpha)}(x) \implies$ solution $\widehat{O}_X^{(\beta)}\psi_n^{(\beta)}(x) = u_\beta^{(n)}\psi_n^{(\beta)}(x)$
- Single quantization condition; discrete family of codimension-1 submanifolds in g_Σ-dimensional "energy" space $\{u_i\}$

A cartoon:

$$\widehat{O}_X^{(1)}(e^{\hat{x}}, e^{\hat{p}}, \{u_2\}, \{m_j\})\psi_n^{(1)}(x) = u_1^{(n)}\psi_n^{(1)}(x)$$
$$\widehat{O}_X^{(2)}(e^{\hat{x}}, e^{\hat{p}}, \{u_1\}, \{m_j\})\psi_n^{(2)}(x) = u_2^{(n)}\psi_n^{(2)}(x)$$

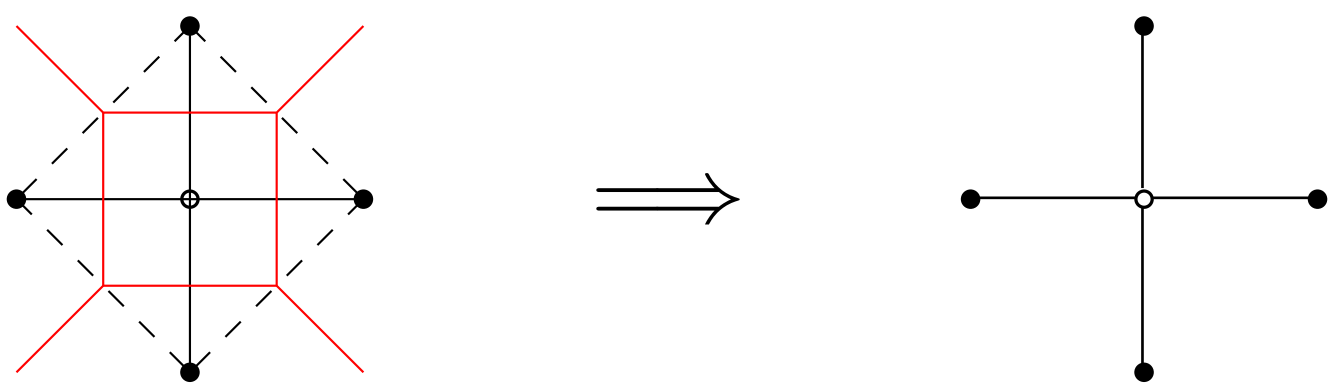


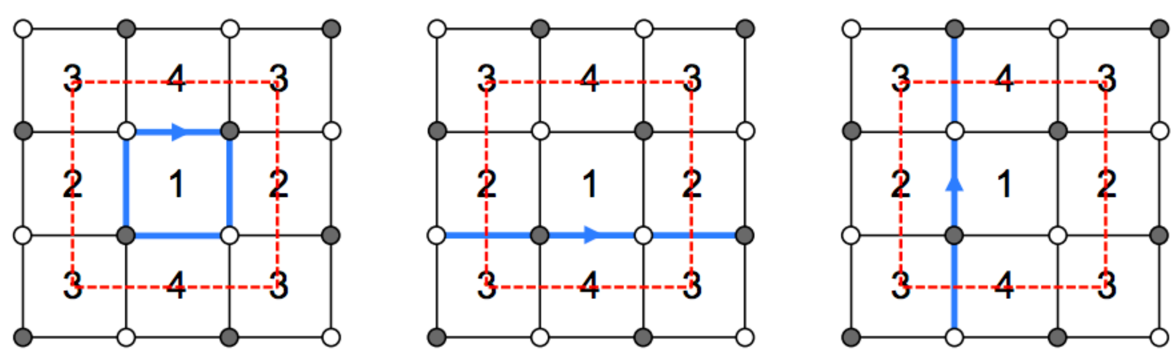
At fixed u_2, quantized energy $u_1 = u_1(u_2)$ (and viceversa)

II) Quantum mirror curve as Baxter equation (Separation of Variables)

To each X we can associate a classical (*cluster*) integrable system
with g_Σ Hamiltonians u_i + r_Σ Casimirs m_j [Goncharov-Kenyon '11]
[Franco-Hatsuda-Marino '15]

Toric diagram $X \Longleftrightarrow$ dimer model $\Longleftrightarrow$ *cluster* integrable system





Mirror / SW curve: spectral curve of the classical cluster integrable system

Quantum problem: $[\hat{p}_k, \hat{x}_l] = -i\hbar\delta_{k,l}$, $[\hat{u}_i, \hat{u}_j] = 0 \implies$

$$\boxed{\hat{u}_i \psi_{\vec{n}}(\vec{x}) = u_i^{(\vec{n})} \psi_{\vec{n}}(\vec{x})}$$

look for $L^2(\mathbb{R}^{g_\Sigma})$ -normalizable simultaneous eigenfunctions $\implies$
g_Σ quantization conditions, discrete set of points in energy space $\{u_i\}$

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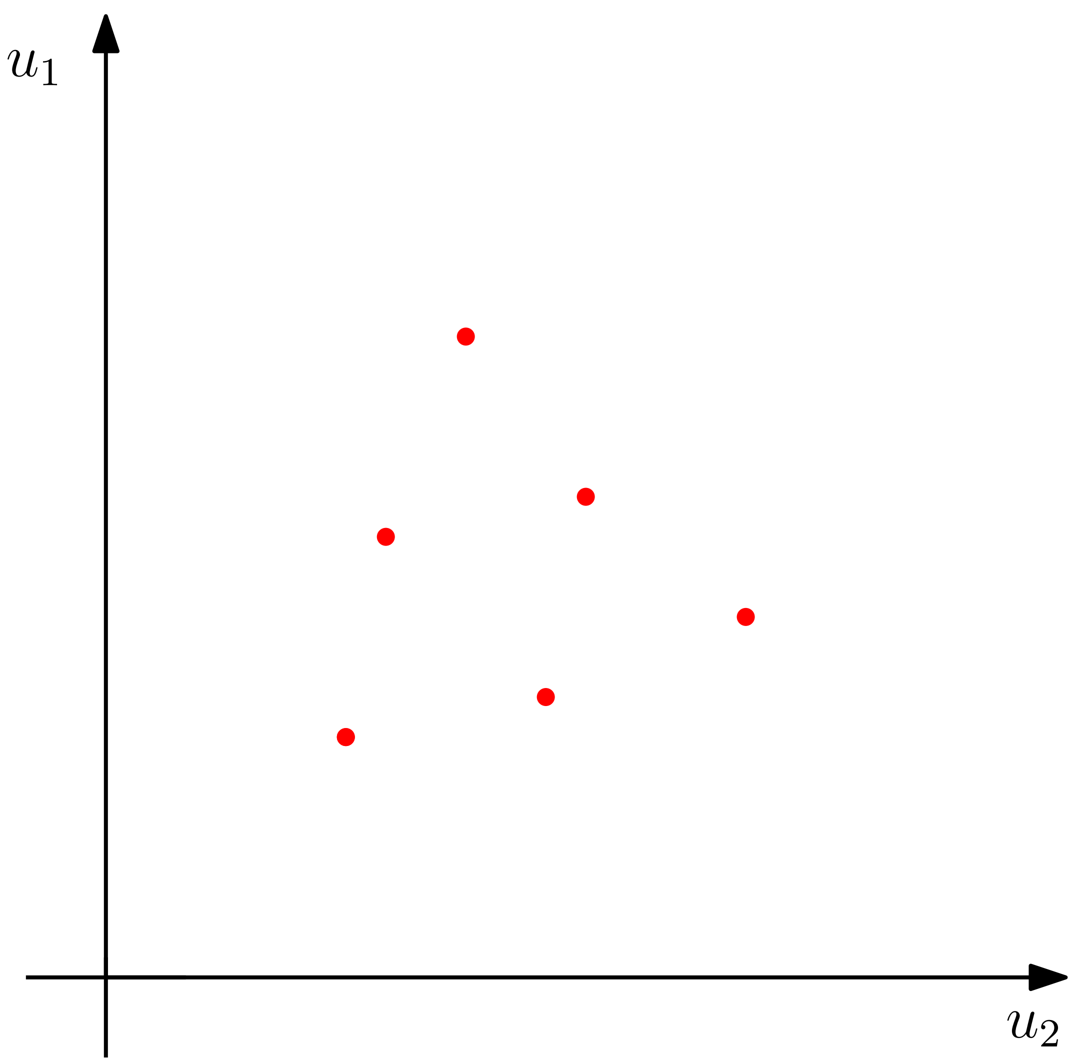

special case of QM problem I

Quantum problem:

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look for $L^2(\mathbb{R}^{g_\Sigma})$ -normalizable simultaneous eigenfunctions

'Shortcut': study quantum mirror curve / Baxter equation

auxiliary problem

$$\boxed{\widehat{W}_X(e^{\hat{x}}, e^{\hat{p}}, \{u_i\}, m_0) Q(x) = 0};$$

Requirements on $Q(x)$ (no $L^2(\mathbb{R})$ like in QM problem I):

- $\psi_{\vec{n}}(\vec{x})$ from $Q(x)$ via integral transform (Separation of Variables);
- Need $Q(x)$ *entire* + rapidly-decaying (such that $\psi_{\vec{n}}(\vec{x}) \in L^2(\mathbb{R}^{g_\Sigma})$)

Conditions on $Q(x)$ imply that all $\{u_i\}$ are quantized

How to *solve** these QM problems in terms of TS quantities of X?

*Solution: discrete energy levels + normalizable wave-functions; analytic solution unknown in QM

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- Problem I: Topological String / Spectral Theory correspondence (TS/ST)

[Grassi-Hatsuda-Marino '14][Codesido, Zakany, Moriyama, Okuyama, Kashaev, Gu, Klemm, Reuter,…]

Spectral determinant via *unrefined* TS free energy (+ *NS* non-pert. terms):

$$\Xi_X(u,\hbar) \equiv \det(1-u\widehat{\rho}_X) \equiv \prod_{n=0}^{\infty}(1-ue^{-E_n}) = \sum_{n\in\mathbb{Z}} e^{\boxed{J_X(\ln u+2\pi i n,\hbar)}}$$

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- Problem II: (revised) "Bethe/Gauge" correspondence

today's focus

[Nekrasov-Shatashvili '09][Huang, Wang, Zhang, Sun, Marino, Hatsuda, Franco, Kashani-Poor, A.S.,…]

Quantization conditions via *NS* limit of TS free energy (+ non-pert. terms):

$$a^{(n)}: \quad \partial_a \boxed{\mathcal{W}(a,\hbar)} = 2\pi n\,; \quad u^{(n)} = e^{E_n} = u(a^{(n)})$$

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Check with numerics; in both cases, *non-perturbative* terms play a key role

*Solution: discrete energy levels + normalizable wave-functions; analytic solution unknown in QM

Remarks on numerics (“experiment”)

QM problems often unsolved analytically, but can be studied numerically

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Given a QM problem $\boxed{\widehat{H}\psi(x) = E\psi(x)}$ defined on $L^2(\mathbb{R})$, diagonalize it in an orthonormal basis on $L^2(\mathbb{R})$ (in practice, harmonic oscillator basis)

$$\varphi_l(x) = (2^l l!)^{-\frac{1}{2}} (\pi\hbar)^{-\frac{1}{4}} e^{-\frac{x^2}{2\hbar}} H_l(\hbar^{-\frac{1}{2}} x) \qquad (m\omega = 1)$$

Consider a truncation of the infinite-dimensional matrix

$$\langle \varphi_{l_1} | \widehat{H} | \varphi_{l_2} \rangle$$

and compute its eigenvalues and eigenvectors; increasing the matrix size, these should converge to eigenvalues and eigenvectors of $\widehat{H}$

Analytical TS solution of QM problems always checked against numerics

Quantum Mechanical problem II:

Quantum Integrable Systems

QM problem II: (revised) “Bethe/Gauge” correspondence

Dictionary between 4d $\mathcal{N}$=2 / 5d $\mathcal{N}$=1 SUSY gauge theories and Quantum Integrable Systems: “Bethe/Gauge correspondence” [Nekrasov-Shatashvili ’09]

Bethe (QIS)		Gauge
Planck constant $\hbar$		NS limit $\epsilon_2 \to 0, \ \epsilon_1 \to i\hbar$
Spectral curve		Seiberg-Witten curve
Quantization conditions	$\Longleftrightarrow$	SUSY vacua equations
Eigenvalues		codim. 4 defects (Wilson)
Eigenfunctions ($\hat{u}_i$)		codim. 2 defects (quiver)
Eigenfunctions (Baxter)		codim. 2 defects (free chiral)

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Thought to involve gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$(4d) or $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_R$ (5d); however, correspondence *incomplete* in 5d (disagreement with numerics)

Need to revisit and reinterpret Bethe/Gauge correspondence in 5d

4d Bethe/Gauge example: N-particle closed Toda chain / 4d $\mathcal{N} = 2$ SU(N)

N-particle closed Toda chain Hamiltonians (QM solution known):

$$u_1 = \sum_{l=1}^{N} p_l \,, \qquad u_2 = \sum_{l<k}^{N} p_l p_k + \sum_{l=1}^{N} e^{x_l - x_{l+1}} \,, \qquad \ldots \qquad , \qquad u_N = \ldots$$

Quantization: $[\hat{p}_k, \hat{x}_l] = -i\hbar\delta_{k,l} \implies$ look for $L^2(\mathbb{R}^{N-1})$ eigenfunctions $\psi(\vec{x})$

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'Shortcut': study Baxter equation / quantum spectral (SW) curve; for N=2:

$$[e^{i\hbar\partial_x} + \tilde{m}_0 e^{-i\hbar\partial_x} + x^2]Q(x) = EQ(x)$$

rigorously proved

from SW curve $e^{-p} + Q_{4\mathrm{d}}e^p + x^2 = u$ (with $E \Leftrightarrow u, \ \tilde{m}_0 \Leftrightarrow Q_{4\mathrm{d}}$)

- Require $Q(x)$ entire + rapidly decaying $\implies$ quantized energy $E = E_n$
- Obtain $\psi(\vec{x})$ from $Q(x)$ via integral transform (Separation of Variables)
- Similar procedure for any N

Bethe/Gauge prescription on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ - quantization conditions + spectrum

[Nekrasov-Shatashvili '09]

- Define the Yang-Yang / twisted effective superpotential function

$$\mathcal{W}_{4\mathrm{d}}(a,\hbar,Q_{4\mathrm{d}}) = \lim_{\epsilon_2\to 0}\left[-\epsilon_2\epsilon_1 \log Z_{4\mathrm{d}}(a,\epsilon_1,\epsilon_2,Q_{4\mathrm{d}})\right]\Big|_{\epsilon_1=i\hbar}$$

- Discrete energy levels $E=E_n$ obtained at SUSY vacua as

$$u^{(n)} = E_n = Q_{4\mathrm{d}}\frac{d}{dQ_{4\mathrm{d}}}\mathcal{W}_{4\mathrm{d}}(a^{(n)},\hbar,Q_{4\mathrm{d}}) \quad \text{for} \quad a^{(n)} \,:\, \partial_a\mathcal{W}_{4\mathrm{d}}(a,\hbar,Q_{4\mathrm{d}}) = 2\pi\hbar n$$

codim. 4 defect: local observable

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codim. 4 defect: local observable

4d Bethe/Gauge correspondence seems to work well:

$\hbar=\sqrt{3}, Q_{4\mathrm{d}}=\frac{1}{\sqrt{2}}$	E_0	E_1	E_2
Bethe/Gauge	$3.44076329369006\ldots$	$7.25213834512333\ldots$	$11.60628188091683\ldots$
numerics	$3.44076329369006\ldots$	$7.25213834512333\ldots$	$11.60628188091683\ldots$

Bethe/Gauge prescription on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ - Baxter eigenfunction $Q(x)$

[Kozlowski-Teschner '10][Gaiotto-Kim '14][A.S. '17]

Couple 4d to two 2d $\mathcal{N}=(2,2)$ free chiral / anti-chiral on $\mathbb{R}^2_{\epsilon_1}$ (or $D^2_{\epsilon_1}$)

$$Q_n(x,\hbar,Q_{4\mathrm{d}}) = \underbrace{Q^{(\mathrm{ac})}_{\mathrm{NS}}(x,a^{(n)},\hbar,Q_{4\mathrm{d}})}_{\text{anti-chiral, NS limit}} - \xi^{(n)} \underbrace{Q^{(\mathrm{c})}_{\mathrm{NS}}(x,a^{(n)},\hbar,Q_{4\mathrm{d}})}_{\text{chiral, NS limit}}$$

<u>NOT</u> only chiral or only anti-chiral!

- $Q^{(\mathrm{c}/\mathrm{ac})}_{\mathrm{NS}}(x)$: 4d partition function with 2d defect (NS limit)
- Both fast-decay at $x\to\pm\infty$, but not entire (poles at $x\to\pm a$); requiring cancellation of both poles fixes $\xi^{(n)}$ (and $a^{(n)}$)

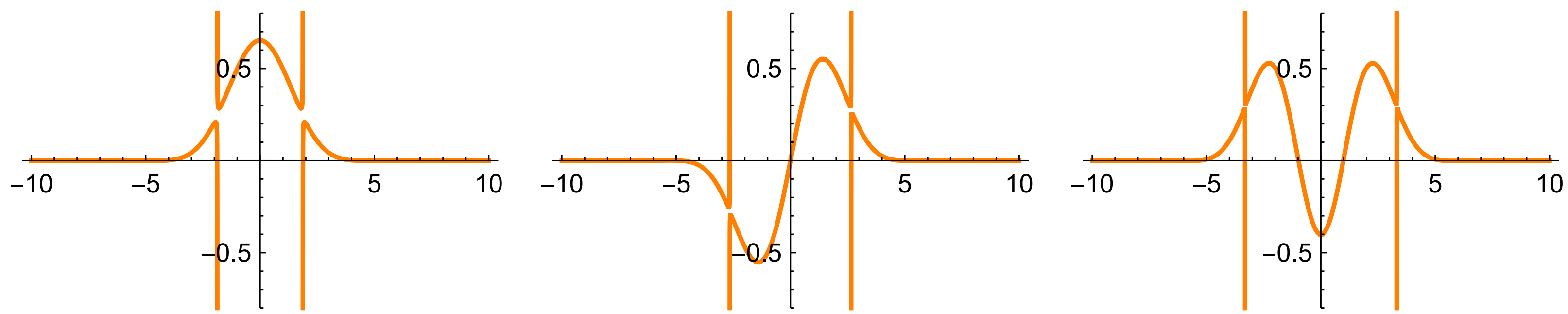
$$\operatorname*{Res}_{x=\pm a}[Q^{(\mathrm{ac})}_{\mathrm{NS}}(x,a)-\xi Q^{(\mathrm{c})}_{\mathrm{NS}}(x,a)]=0 \implies a=a^{(n)}$$

- $Q(x)$ *entire* / normalizable $\Longleftrightarrow$ SUSY vacua equations:

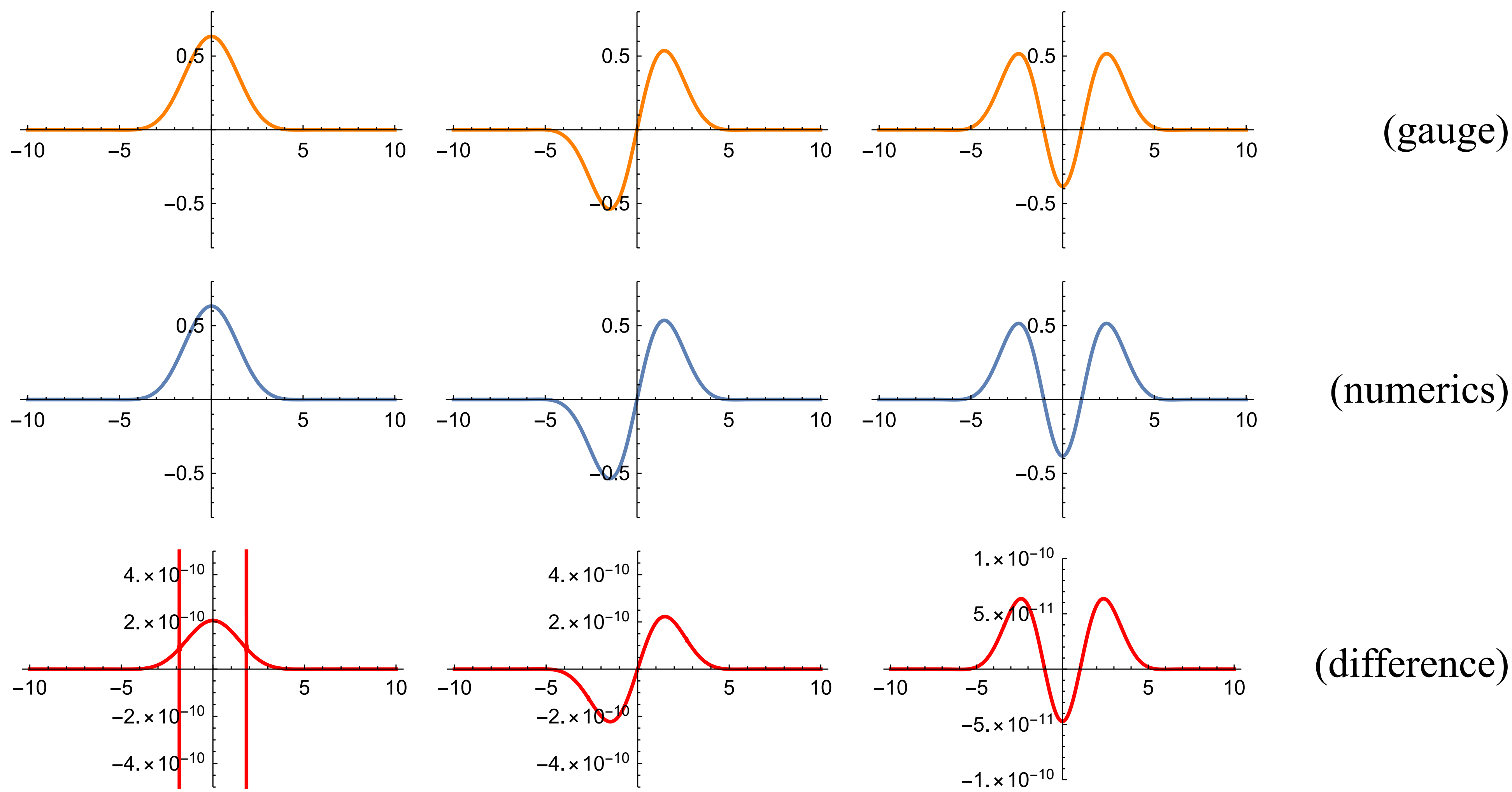
$$\operatorname*{Res}_{x=\pm a}[Q^{(\mathrm{ac})}_{\mathrm{NS}}(x,a)-\xi Q^{(\mathrm{c})}_{\mathrm{NS}}(x,a)]=0 \iff \partial_a\mathcal{W}_{4\mathrm{d}}(a)=2\pi\hbar n$$

Compare gauge theory result (4-instantons) with "experiment" (numerics)

Bethe/Gauge Baxter "eigenfunction" $Q(x)$ not entire for $a \approx a^{(n)}$:



On the other hand, both poles cancelled if $a = a^{(n)}$:



5d example: N-particle “relativistic” closed Toda chain / 5d $\mathcal{N} = 1$ SU(N)

N-particle “relativistic” closed Toda Hamiltonians (QM solution unknown):

$$u_1 = \sum_{l=1}^{N} [1 + e^{x_l - x_{l+1}}] e^{R p_l} \, , \quad \dots \quad , \quad u_N = \prod_{l=1}^{N} e^{R p_l}$$

Quantization: $[\hat{p}_k, \hat{x}_l] = -i\hbar \delta_{k,l} \implies$ look for $L^2(\mathbb{R}^{N-1})$ eigenfunctions $\psi(\vec{x})$

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$$[e^{i\hbar\partial_x} + m_0 e^{-i\hbar\partial_x} + e^{Rx} + e^{-Rx}] Q(x) = e^{E} Q(x)$$

NOT proved; naive guess

from SW curve (mirror curve local F_0) $e^{-p} + Q_{5\mathrm{d}} e^{p} + e^{Rx} + e^{-Rx} = u$

- Require $Q(x)$ entire + rapidly decaying $\implies$ quantized energy $e^{E} = e^{E_n}$
- Obtain $\psi(\vec{x})$ from $Q(x)$ via integral transform (Separation of Variables)
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(Putative) Bethe/Gauge prescription on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_R$ - quantized spectrum

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- Discrete energy levels $e^E = e^{E_n}$ obtained at SUSY vacua as

$$e^{E_n} = \langle W^{SU(2)}_{\square,\mathrm{NS}}(a^{(n)},\hbar,R,Q_{5\mathrm{d}})\rangle \quad \text{for} \quad a^{(n)} \,:\, \partial_a \mathcal{W}_{5\mathrm{d}}(a,\hbar,R,Q_{5\mathrm{d}}) = 2\pi n$$

codim. 4 defect: Wilson loop

(Putative) Bethe/Gauge prescription on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_R$ - quantized spectrum

[Nekrasov-Shatashvili '09]

- Define the Yang-Yang / twisted effective superpotential function

$$\mathcal{W}_{5\mathrm{d}}(a,\hbar,R,Q_{5\mathrm{d}}) = \lim_{\epsilon_2\to 0}\left[-\epsilon_2 \log Z_{5\mathrm{d}}(a,\epsilon_1,\epsilon_2,R,Q_{5\mathrm{d}})\right]\Big|_{\epsilon_1=i\hbar}$$

- Discrete energy levels $e^E = e^{E_n}$ obtained at SUSY vacua as

$$e^{E_n} = \langle W^{SU(2)}_{\square,\mathrm{NS}}(a^{(n)},\hbar,R,Q_{5\mathrm{d}})\rangle \quad \text{for} \quad a^{(n)} \,:\, \partial_a \mathcal{W}_{5\mathrm{d}}(a,\hbar,R,Q_{5\mathrm{d}}) = 2\pi n$$

codim. 4 defect: Wilson loop

However, 5d Bethe/Gauge correspondence on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_R$ ***inconsistent***:

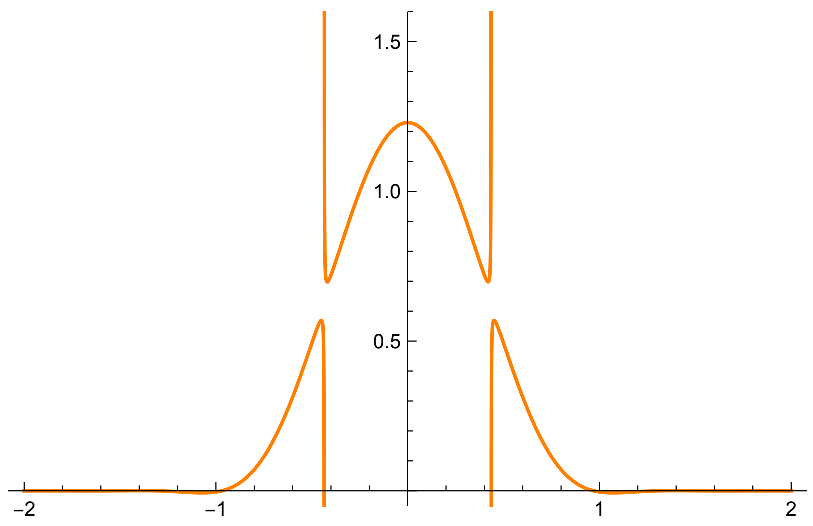
	$\hbar=\sqrt{2}\pi, Q_{5\mathrm{d}}=1$	$\hbar=\sqrt{2}\pi, Q_{5\mathrm{d}}=\sqrt{3}$	$\hbar=2\pi, Q_{5\mathrm{d}}=1$
Bethe/Gauge E_0	$2.4607618679\ldots$	$2.7531433944\ldots$	ill-defined
numerics E_0	$2.4605242719\ldots$	$2.7528481019\ldots$	$2.8818154299\ldots$

5d Bethe/Gauge prescription on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1_R$ inconsistent also for $Q(x)$

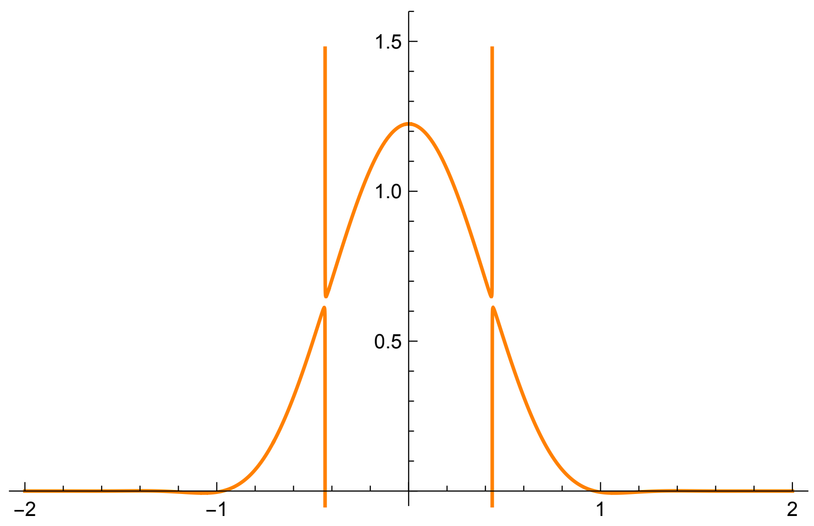
Couple 5d to 3d $\mathcal{N}=2$ chiral / anti-chiral on $\mathbb{R}^2_{\epsilon_1} \times S^1_R$ (or $D^2_{\epsilon_1} \times S^1_R$):

$$Q_n(x,\hbar,R,Q_{5\mathrm{d}}) = \underbrace{Q^{(\mathrm{ac})}_{\mathrm{NS}}(x,a^{(n)},\hbar,R,Q_{5\mathrm{d}})}_{\text{anti-chiral, NS limit}} - \xi^{(n)} \underbrace{Q^{(\mathrm{c})}_{\mathrm{NS}}(x,a^{(n)},\hbar,R,Q_{5\mathrm{d}})}_{\text{chiral, NS limit}}$$

Formally satisfies Baxter, fast-decaying at $x \to \pm\infty$, but never entire:



$\hbar = \sqrt{2}\pi, Q_{5\mathrm{d}} = \sqrt{3}, R = 1$ for $a \neq a^{(0)}$



$\hbar = \sqrt{2}\pi, Q_{5\mathrm{d}} = \sqrt{3}, R = 1$ for $a = a^{(0)}$

Moreover, ill-defined for $\hbar = 2\pi$ (and, more in general, for $\hbar \in 2\pi\mathbb{Q}$)

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Pole at $\hbar = 0$ (classical limit) but also dense set of poles at $\hbar \in 2\pi\mathbb{Q}$
(k-th instanton contribution $\sim Q_{\mathrm{5d}}^k (1-q^k)^{-1}$)

The same problem appears in the would-be Baxter eigenfunction $Q(x)$:

$$Q_{\mathrm{NS}}^{(\mathrm{c}),\mathrm{inst}}(x,a,\hbar) = 1 + \frac{Q_{\mathrm{5d}} q^2 X(\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}} - qX - q^2 X)}{(1-q)(1-q\mu)(1-q\mu^{-1})(1-qX\mu^{\frac{1}{2}})(1-qX\mu^{-\frac{1}{2}})} + O(Q_{\mathrm{5d}}^2) \qquad (X = e^{x})$$

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What happens? 5d Bethe/Gauge gives QM all-orders WKB expressions;
however, all-orders WKB may be corrected by QM instanton effects
(also related to quasi-constant ($\hbar$-periodic) ambiguity Toda eigenfunction)

This is *<u>NOT</u>* a problem of our flat-space 5d gauge theory / TS observables,
but of their correct interpretation in Quantum Mechanics

Can we fix this? Yes; Separation of Variable prescription: *modular duality*

[Kharchev,Lebedev,Tian-Shansky '01]

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Redefining all parameters, rewrite the "relativistic" Toda Hamiltonians as

$$u_1 = \sum_{l=1}^{N} [1 + e^{\frac{2\pi}{\omega_2}(x_l - x_{l+1})}] e^{\omega_1 p_l}\,, \quad \ldots \quad , \quad u_N = \prod_{l=1}^{N} e^{\omega_1 p_l}$$

Quantizing $[\hat{p}_k, \hat{x}_l] = -i\delta_{k,l}$, ambiguity $\psi(\vec{x})$ by $i\omega_1$-periodic functions

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Quantizing $[\hat{p}_k, \hat{x}_l] = -i\delta_{k,l}$, ambiguity $\psi(\vec{x})$ by $i\omega_1$-periodic functions; solved by considering modular dual "relativistic" Toda system ($\omega_1 \leftrightarrow \omega_2$)

$$\tilde{u}_1 = \sum_{l=1}^{N} [1 + e^{\frac{2\pi}{\omega_1}(x_l - x_{l+1})}] e^{\omega_2 p_l} \,, \quad \ldots \quad , \quad \tilde{u}_N = \prod_{l=1}^{N} e^{\omega_2 p_l}$$

and requiring $\psi(\vec{x})$ to be eigenfunctions of both sets of Hamiltonians $\hat{u}_i, \hat{\tilde{u}}_i$

$$\boxed{\hat{u}_i \psi_{\vec{n}}(\vec{x}) = e^{E^{(i)}_{\vec{n}}} \psi_{\vec{n}}(\vec{x})}, \quad \boxed{\hat{\tilde{u}}_i \psi_{\vec{n}}(\vec{x}) = e^{\tilde{E}^{(i)}_{\vec{n}}} \psi_{\vec{n}}(\vec{x})} \qquad ([\hat{u}_i, \hat{\tilde{u}}_j] = 0)$$

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Consequence: symmetry $\omega_1 \leftrightarrow \omega_2$, Baxter + *dual* Baxter equation, fixes $i\omega_1$ -periodic functions ambiguity, cancels poles at $\hbar \in 2\pi\mathbb{Q}$

Gauge theory realization of modular duality (*revised* Bethe/Gauge):* [Hatsuda '15] [A.S. '16, '17]

$$\mathbb{R}^2_{\epsilon_1} \times S^1_R \times \mathbb{R}^2 \ \text{(NS limit)} \Longrightarrow S^3_{\omega_1,\omega_2} \times \mathbb{R}^2$$

$S^3_{\omega_1,\omega_2}$ obtained from two copies of our previous $D^2_{\epsilon_1^{(l)}} \times S^1_{R^{(l)}}$, $l = 1, 2$:

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dual copy contains non-perturbative correction terms (in $\hbar \sim \omega_1/\omega_2$)

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non-trivial symmetry:
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"NS limit" of some 5d compact ($S^5_{\omega_1,\omega_2,\omega_3}$) / non-compact geometry?
("non-perturbative" completion of NS limit of TS [Lockhart-Vafa '12, Hatsuda '15])

*For TS geometries with 5d gauge theory interpretation

Revised Bethe/Gauge on $S^3_{\omega_1,\omega_2} \times \mathbb{R}^2$, 2-particle "relativistic" closed Toda:

- *Exact* quantization conditions (free from poles at $\hbar \in 2\pi\mathbb{Q}$) + energy:

$$a^{(n)} \; : \; \partial_a \Big[\underbrace{\mathcal{W}_{5\mathrm{d}}(a,\omega_1,\omega_2)}_{\text{exact WKB (old)}} + \underbrace{\mathcal{W}_{5\mathrm{d}}(a,\omega_2,\omega_1)}_{\text{non-pert. (new)}} \Big] = 2\pi n$$

[Wang-Zhang-Huang '15]
[Hatsuda-Marino '15]

$$e^{E_n} = \langle W^{SU(2)}_{\square,\mathrm{NS}}(a^{(n)},\omega_1,\omega_2)\rangle \qquad e^{\tilde{E}_n} = \langle W^{SU(2)}_{\square,\mathrm{NS}}(a^{(n)},\omega_2,\omega_1)\rangle$$

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- Need to find $Q(x)$ common solution to Baxter + dual Baxter equation:

$$[e^{i\omega_1\partial_x} + m_0 e^{-i\omega_1\partial_x} + e^{\frac{2\pi x}{\omega_2}} + e^{-\frac{2\pi x}{\omega_2}}]Q(x) = e^E Q(x)$$
$$[e^{i\omega_2\partial_x} + \tilde{m}_0 e^{-i\omega_2\partial_x} + e^{\frac{2\pi x}{\omega_1}} + e^{-\frac{2\pi x}{\omega_1}}]Q(x) = e^{\tilde{E}} Q(x)$$

- ***Exact*** solution to Baxter and dual Baxter (no poles at $\hbar \in 2\pi\mathbb{Q}$): [A.S. '17]

$$Q_n(x) = \underbrace{Q_{\rm NS}^{(\rm ac)}(x, a^{(n)}, \omega_1, \omega_2)}_{\rm WKB} \underbrace{Q_{\rm NS}^{(\rm ac)}(x, a^{(n)}, \omega_2, \omega_1)}_{\rm non\text{-}pert.} - \xi^{(n)} \underbrace{Q_{\rm NS}^{(\rm c)}(x, a^{(n)}, \omega_1, \omega_2)}_{\rm WKB} \underbrace{Q_{\rm NS}^{(\rm c)}(x, a^{(n)}, \omega_2, \omega_1)}_{\rm non\text{-}pert.}$$

- $Q(x)$ *entire* / normalizable $\Longleftrightarrow$ *exact* quantization conditions

$$\operatorname*{Res}_{x=\pm a}[Q(x,a)] = 0 \quad \Longleftrightarrow \quad \partial_a[\mathcal{W}_{\rm 5d}(a, \omega_1, \omega_2) + \mathcal{W}_{\rm 5d}(a, \omega_2, \omega_1)] = 2\pi n$$

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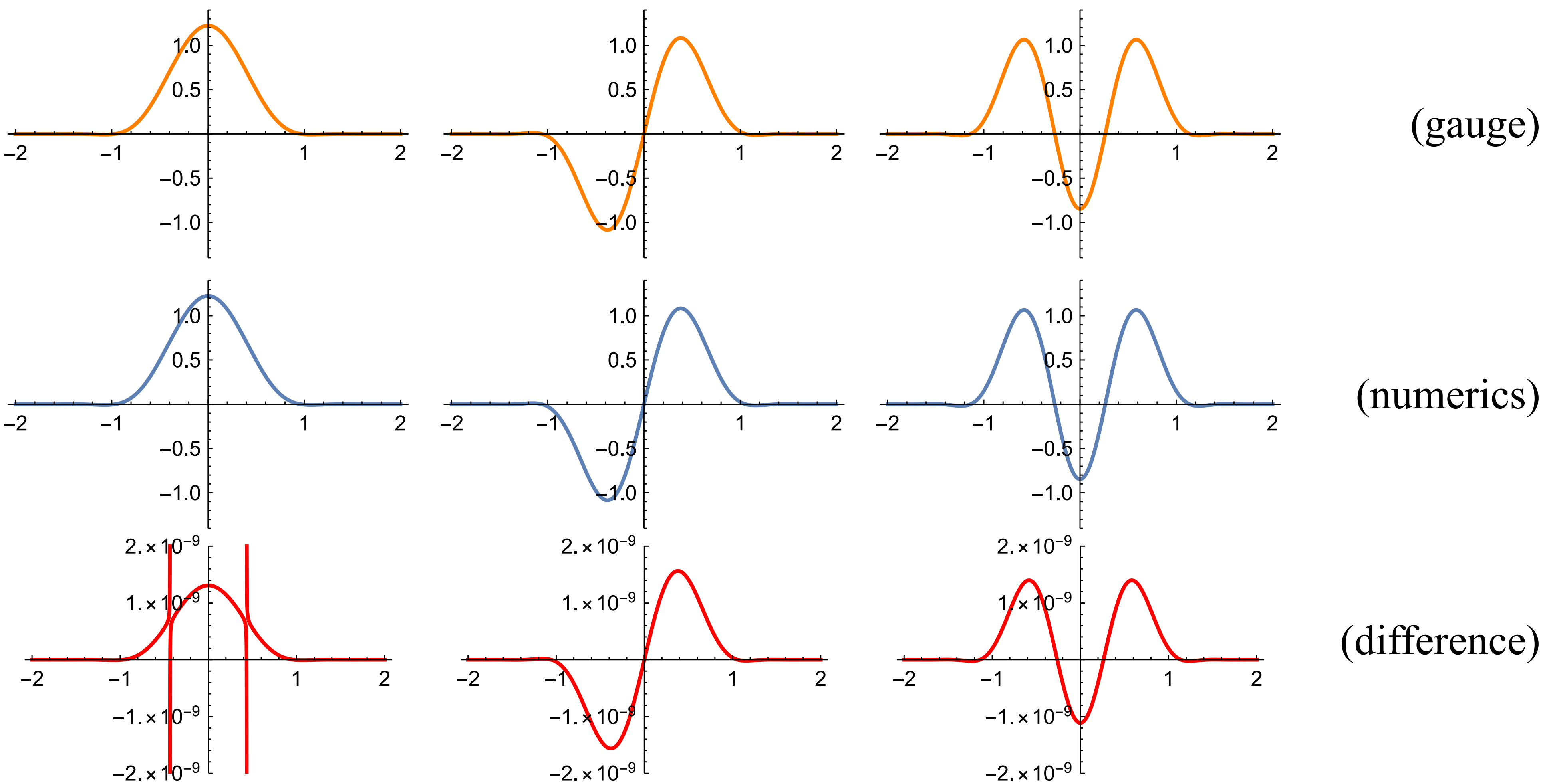

TS and QM problem II: summary

TS as universal framework for QM problems:

Topological Strings (NS) on X + non-pert. (NS) corrections	$\Longleftrightarrow$	Analytic solution QIS of X (Toda), all-orders WKB + QM instantons

TS, QM non-perturbative corrections from combining TS, QM techniques:

- QM: Separation of Variables, Baxter eq., modular dual, symmetry $\omega_1 \leftrightarrow \omega_2$
- TS: analytic formulae for energy, quantization conditions, eigenfunction

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Good match with numerics:

- Quantization conditions, energy spectrum: checked for many X
- Baxter eigenfunction: checked for $X = Y^{N,0}$, i.e. 5d $\mathcal{N} = 1$ SU(N)
  (NS limit of open TS only computable from gauge theory)

Gauge theory interpretation: <u>need</u> to promote $\mathbb{R}^2_{\epsilon_1} \times S^1_R \times \mathbb{R}^2 \Longrightarrow S^3_{\omega_1,\omega_2} \times \mathbb{R}^2$

same 4d limit!

Quantum Mechanical problem I:

Spectral Theory

QM problem I: TS/ST correspondence

QM problem II: very special case of QM problem I $(\alpha = 1, \ldots, g_\Sigma)$:

$$W_X(e^x, e^p, \{u_i\}, \{m_j\}) = 0 \quad \Longleftrightarrow \quad O_X^{(\alpha)}(e^x, e^p, \{u_i\}', \{m_j\}) = u_\alpha \qquad \text{(classical)}$$

$$\Longrightarrow \qquad \widehat{O}_X^{(\alpha)} \psi_n^{(\alpha)}(x) = u_\alpha^{(n)} \psi_n^{(\alpha)}(x) \qquad \text{(quantum)}$$

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TS solution to QM problem I: Top. String/Spectral Theory correspondence

[Grassi-Hatsuda-Marino '14][Codesido-Grassi-Marino '15]

Main idea: study Spectral Theory of $\widehat{\rho}_X^{(\alpha)} = \left[\widehat{O}_X^{(\alpha)}\right]^{-1}$ via TS quantities; provides a non-perturbative definition of (unrefined) TS free energy

Perturbative / non-perturbative (in g_s) TS effects $\underset{g_s \sim 1/\hbar}{\Longleftrightarrow}$ Non-perturbative / perturbative (in $\hbar$) QM effects

TS/ST correspondence much in the same spirit of AdS/CFT

Consider X of genus 1 (like 5d pure $\mathcal{N} = 1$ SU(2) theory / local F_0)

Main TS/ST conjecture: analytic TS expression for *spectral determinant*

$$\Xi_X(u,\hbar) \equiv \det(1 - u\widehat{\rho}_X) \equiv \prod_{n=0}^{\infty}(1 - ue^{-E_n}) = \sum_{n\in\mathbb{Z}} e^{J_X(\ln u(\mathbf{t})+2\pi i n,\hbar)}$$

with quantum mirror map $u = u(\mathbf{t},\hbar)$ and $\mathbf{t}$ Kahler parameters of X

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Grand potential decomposes as $J_X(\ln u,\hbar) = J_X^{\mathrm{WS}}(\ln u,\hbar) + J_X^{\mathrm{WKB}}(\ln u,\hbar)$:

- J_X^{WS} written in terms of unrefined (large volume) TS free energy:

$$J_X^{\mathrm{WS}}(\ln u(\mathbf{t}),\hbar) \qquad \Longleftrightarrow \qquad F_{\mathrm{TS},X}^{\mathrm{unref}}(\mathbf{t},g_s) \qquad \boxed{(g_s\sim 1/\hbar)}$$

- J_X^{WKB} written in terms of NS limit of (large volume) TS free energy:

$$J_X^{\mathrm{WKB}}(\ln u(\mathbf{t}),\hbar) \qquad \Longleftrightarrow \qquad F_{\mathrm{TS},X}^{\mathrm{NS}}(\mathbf{t},\hbar) \qquad (\epsilon_1=i\hbar,\epsilon_2=0)$$

WKB analysis of $\widehat{\rho}_X$ determines non-perturbative terms in unrefined TS; poles at $g_s=2\pi\mathbb{Q}$ of J_X^{WS} cancelled by poles $J_X^{\mathrm{WKB}} \Longrightarrow$ well-defined J_X

Spectrum of operator $\widehat{O}_X$ or $\widehat{\rho}_X$: look for zeroes of $\Xi_X(u, \hbar)$

$$\Xi_X(u(\mathbf{t}, \hbar), \hbar) = \sum_{n \in \mathbb{Z}} e^{J_X(\ln u(\mathbf{t}, \hbar) + 2\pi i n, \hbar)} = 0 \quad \Longrightarrow \quad \mathbf{t} = \mathbf{t}^{(n)}$$

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$$\widehat{O}_{F_0} = e^{-\hat{p}} + m_0 e^{\hat{p}} + e^{\hat{x}} + e^{-\hat{x}}$$

	$\hbar = \sqrt{2}\pi, m_0 = 1$	$\hbar = \sqrt{2}\pi, m_0 = \sqrt{3}$	$\hbar = 2\pi, m_0 = 1$
TS/ST E_0	$2.4605242719\ldots$	$2.7528481019\ldots$	$2.8818154299\ldots$
numerics E_0	$2.4605242719\ldots$	$2.7528481019\ldots$	$2.8818154299\ldots$

Remark: closed form for Ξ_X at $\hbar = 2\pi \implies$ *exact* *analytic* spectrum

Spectrum of operator $\widehat{O}_X$ or $\widehat{\rho}_X$: look for zeroes of $\Xi_X(u,\hbar)$

$$\Xi_X(u(\mathbf{t},\hbar),\hbar) = \sum_{n\in\mathbb{Z}} e^{J_X(\ln u(\mathbf{t},\hbar)+2\pi i n,\hbar)} = 0 \quad \Longrightarrow \quad \mathbf{t} = \mathbf{t}^{(n)}$$

Energies determined by the quantum mirror map:

$$e^{E_n} = u(\mathbf{t}^{(n)},\hbar)$$

Good match with numerics; 5d pure $\mathcal{N} = 1$ SU(2) theory / local F_0:

$$\widehat{O}_{F_0} = e^{-\hat{p}} + m_0 e^{\hat{p}} + e^{\hat{x}} + e^{-\hat{x}}$$

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Higher genus X: similar story, but only one "true" modulus u_α quantized

TS/ST conjecture - eigenfunction $\widehat{O}_X$: analytic TS expression [Marino-Zakany '16, '17]

analogue of QM II
chiral + antichiral

$$\psi(x,u,\hbar) = \sum_\sigma \psi_\sigma(x,u,\hbar)\,, \quad \psi_\sigma(x,u,\hbar) = \sum_{n\in\mathbb{Z}} e^{J_X^{(o)}(x,\ln u + 2\pi i n,\hbar)}$$

Sum over sheets σ of the mirror curve (single $\psi_\sigma(x)$ singular)

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Open grand potential $J_X^{(o)}(x,\ln u,\hbar) = J_X^{(o),\mathrm{WS}}(x,\ln u,\hbar) + J_X^{(o),\mathrm{WKB}}(x,\ln u,\hbar)$:

- $J_X^{(o),\mathrm{WS}}$ contains unrefined (large volume) open TS free energy:

$$J_X^{(o),\mathrm{WS}}(x,\ln u(\mathbf{t}),\hbar) \quad \Longleftrightarrow \quad F^{\mathrm{unref}}_{\mathrm{TS},X,\mathrm{open}}(x,\mathbf{t},g_s) \qquad \boxed{(g_s \sim 1/\hbar)}$$

- $J_X^{(o),\mathrm{WKB}}$ contains NS limit of (large volume) open TS free energy:

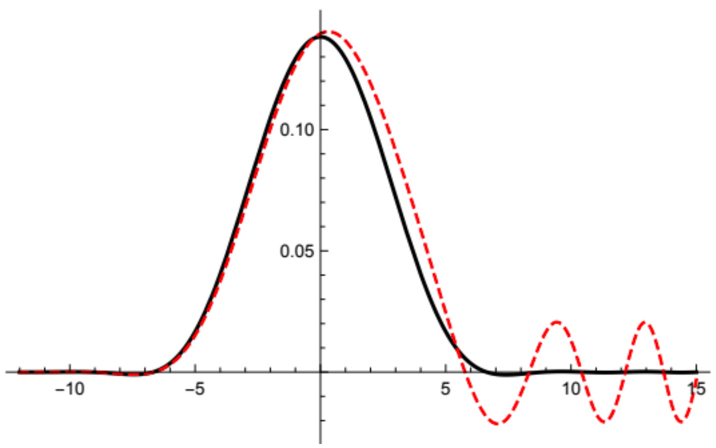
$$J_X^{(o),\mathrm{WKB}}(x,\ln u(\mathbf{t}),\hbar) \quad \Longleftrightarrow \quad F^{\mathrm{NS}}_{\mathrm{TS},X,\mathrm{open}}(x,\mathbf{t},\hbar) \qquad (\epsilon_1 = i\hbar, \epsilon_2 = 0)$$

QM WKB again determines non-perturbative terms in open unrefined TS; poles at $g_s = 2\pi\mathbb{Q}$ of $J_X^{(o),\mathrm{WS}}$ cancelled by poles $J_X^{(o),\mathrm{WKB}}$, well-defined $J_X^{(o)}$

Eigenfunction normalizability: $\psi(x,u,\hbar) \in L^2(\mathbb{R})$ only for $\mathbf{t} = \mathbf{t}^{(n)}$;

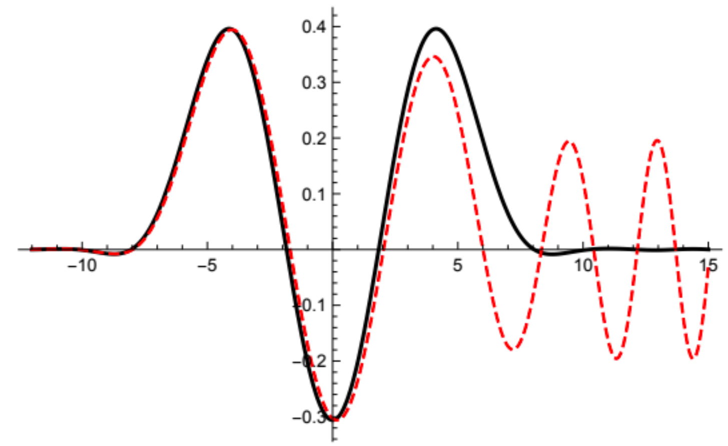
good decay at $x \to \infty \implies$ quantization condition $\Xi_X(u,\hbar) = 0$

[Marino-Zakany '16, '17]



$m_0 = 1, \hbar = 2\pi$

ground state



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second excited state

Closed form for $\psi(x)$ at $\hbar = 2\pi \implies$ <u>*exact*</u> <u>*analytic*</u> eigenfunction

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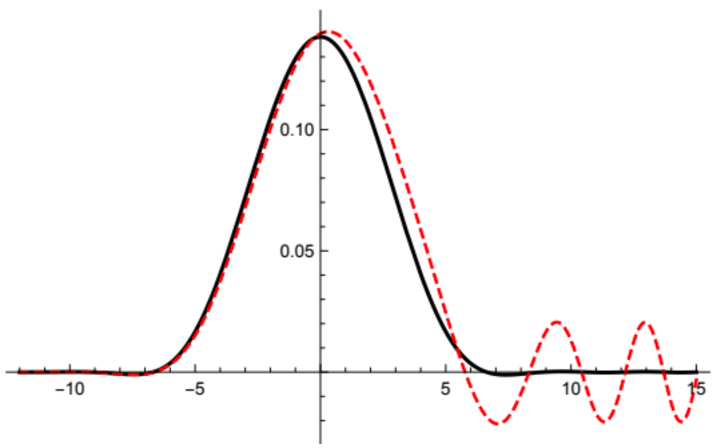

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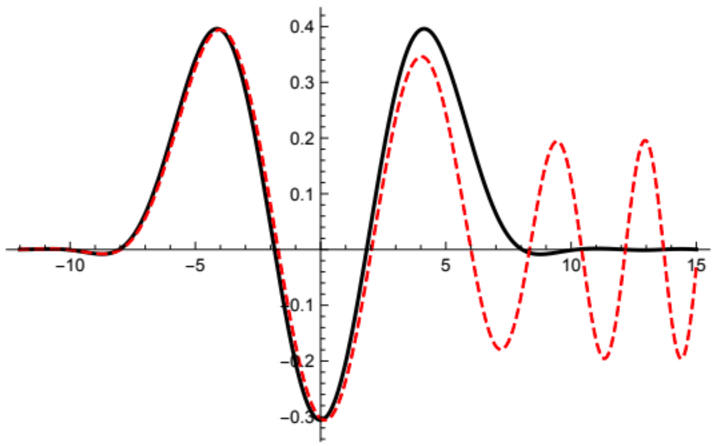

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Closed form for $\psi(x)$ at $\hbar = 2\pi \implies$ <u>*exact*</u> <u>*analytic*</u> eigenfunction

Higher genus X: similar story; at the very special points of QM problem II, <u>*enhanced*</u> <u>*decay*</u> at $x \to -\infty$ (to be better understood; symmetry $\omega_1 \leftrightarrow \omega_2$?)

TS and QM problem I: summary

TS as universal framework for QM problems:

Topol. String (unrefined) on X + non-pert. (NS) corrections	$\Longleftrightarrow$	Analytic Spectral Theory $\widehat{\rho}_X^{(\alpha)}/\widehat{O}_X^{(\alpha)}$, all-orders WKB + QM instantons

TS, QM non-perturbative corrections fixed à la AdS/CFT (TS/ST):

Perturbative / non-perturbative (in g_s) TS effects	$\underset{g_s \sim 1/\hbar}{\Longleftrightarrow}$	Non-perturbative / perturbative (in $\hbar$) QM effects

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Good match with numerics, *exact analytic expressions* at $\hbar = 2\pi$:

- Quantization condition, (single) energy spectrum: checked for many X
- Eigenfunction: fewer checks (NS limit open TS hard from top. vertex)

Gauge theory interpretation: very unclear ("unrefined" limit of $S^5_{\omega_1,\omega_2,\omega_3}$?)

Summary

Topological Strings and Quantum Mechanics: summary

Deep relation between TS and QM, useful for both:

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- TS computational tools $\Longrightarrow$ solve QM problems analytically

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Some of the points which must be clarified:

- Reduction QM problem I $\Longrightarrow$ QM problem II (QIS) at special points (emergence of symmetry $\omega_1 \leftrightarrow \omega_2$, non-trivial identities NS/unrefined)
- Application to other QM problems associated to X [Hatsuda's talk]
- Expand to $\hbar$ complex, other quantization slices ($x \in i\mathbb{R}$,...): no discrete spectrum, bands / gaps structure [Grassi-Marino '17]
- TS/ST non-pert. completion and resurgence: match! [Santamaria-Marino-Schiappa '16]
- TS/ST non-pert. completion and wall-crossing Riemann-Hilbert problem [Bridgeland '16, '17]

Thanks!

Remarks on TS/ST

Spectral determinant: *entire* function of u; Taylor expansion

$$\Xi_X(u,\hbar) \;=\; \det(1-u\widehat{\rho}_X) \;=\; 1+\sum_{N=1}^{\infty} Z_X(N,\hbar)(-u)^N$$

in terms of fermionic spectral traces

$$Z_X(N,\hbar) = \frac{1}{N!}\int \det[\rho_X(p_i,p_j)]d^N p$$

Non-perturbative definition of unrefined TS free energy (conifold frame):

$$F_X(\lambda,g_s) = \ln Z_X(N,\hbar)\,, \qquad g_s = \hbar^{-1},\;\lambda = N\hbar^{-1}$$

since Z_X well-defined for $N\in\mathbb{N}$, $\hbar\in\mathbb{R}_+$ (trace class) and in 't Hooft limit

$$N\to\infty,\quad \hbar\to\infty,\quad N\hbar^{-1}=\lambda \quad \text{finite}$$

$$F_X(\lambda,g_s) = \ln Z_X(N,\hbar) \sim \sum_{g\geq 0}\hbar^{2-2g}F_g(\lambda)$$

perturbative TS free energy (unrefined, conifold frame).

For general N, conjecture:

$$Z_X(N,\hbar) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{J_X(\ln u,\hbar) - N\ln u}\, d\ln u$$

Airy contour

$J_X(\ln u, \hbar)$ unrefined TS free energy (large volume) + non-perturbative terms

Fermionic spectral traces extended to *entire* functions in complex N plane