On self-dual $\mathcal{N} = 4$ theories

MAX-PLANCK-GESELLSCHAFT

Iñaki García Etxebarria

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

with B. Heidenreich and D. Regalado, to appear

Classifying $\mathcal{N} = 4$ theories

Known $\mathcal{N} = 4$ theories in four dimensions are classified by a choice of gauge group G (with algebra $\mathfrak{g}$), and some discrete θ angles. [Aharony, Seiberg, Tachikawa '13]

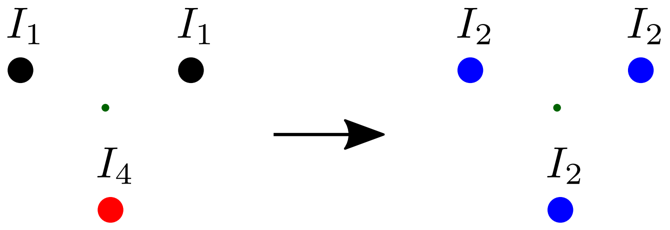
A prototypical example is $\mathfrak{su}(2) \to \{SU(2), SO(3)_\pm = (SU(2)/\mathbb{Z}_2)_\pm\}$. More generally, if $\widetilde{G}$ has center C, we can construct theories of the form $G_i = (\widetilde{G}/H)_i$ with $H \subseteq C$, and "i" some discrete data encoding the structure of extended operators not fully specified by the choice of G.

In the case of $\mathcal{N} = 4$ the structure of point operators is insensitive to the choice of theory, they only depend on $\mathfrak{g}$. One can detect the difference by studying the partition function on four-manifolds $\mathcal{M}_4$ with $H^2(\mathcal{M}_4, C) \neq 0$, or by studying the properties and correlators of extended operators.

A self-dual $\mathcal{N} = 4$ theory?

Recently, Argyres and Martone proposed the existence of a new $\mathcal{N} = 4$ theory with algebra $\mathfrak{su}(2)$. It has the property of mapping to itself under $SL(2, \mathbb{Z})$. This is interesting since it would allow us to construct new $\mathcal{N} = 3$ theories.

Evidence from the Seiberg-Witten curve, which for $\mathcal{N} = 4$ $\mathfrak{su}(2)$ has two formulations, related by a double covering map.



The geometry has a $\mathbb{Z}_3$ symmetry, but no known $\mathfrak{su}(2)$ theory maps to itself under $\mathbb{Z}_3$.

A wrong argument

Consider IIB string theory on $\mathbb{C}^2/\mathbb{Z}_N$, with vanishing B-field. At low energies this is described by the $(2,0)$ A_{N-1} theory. It is well known (this will be my basic assumption) that this theory on a T^2 flows to four dimensional $\mathcal{N}=4$ $\mathfrak{su}(N)$ theory. So we want to consider IIB string theory on $\mathbb{C}^2/\mathbb{Z}_N \times T^2$.

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Since string theory (with no background fields turned on) is invariant under large diffeomorphisms, the resulting 4d theory is invariant under large diffs on the T^2, and thus the 4d theory is self-dual.

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The problem is that IIB has a self-dual F_5, and (on $\mathbb{C}^2/\mathbb{Z}_N \times T^2 \times \mathcal{M}_4$) it is not consistent quantum mechanically to simply assume that this sector is invariant under large diffs.

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The rest of the talk will deal with the consequences of quantizing the self-dual form properly.

Quantization of self-dual forms

Self-dual forms are notoriously hard to understand quantum mechanically. The fact that $dB = \star dB$ implies that the kinetic term

$$\int_X dB \wedge \star dB \tag{1}$$

vanishes when B is an odd form, i.e. when $\dim(X) = 4k + 2$.

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The issue of quantizing self-dual forms appears in a number of contexts:

- Chiral theories in two dimensions. The chiral boson at radius k is one important example. [Witten '88], [Moore, Belov] It appears experimentally when describing the FQHE, for example.
- IIB string theory. [Witten '99]
- The $(2,0)$ A_{N-1} theory in 6d. [Witten '99]
- $AdS_5 \times S^5$ holography. (Not exactly a self-dual form, but equivalent structure.) [Witten '98]

Holography and global structure

What is the holographic interpretation of the possible variants for the $\mathfrak{su}(N)$ $\mathcal{N}=4$ theory in 4d?

We view the possible theories on the boundary as states in the Hilbert space of the bulk theory, taking the radial direction as "time". [Friedan, Shenker '87], [Verlinde '88], [Moore, Seiberg '88], [Witten '89], . . . , [Witten '98], . . . , [Belov, Moore], . . .

The partition function on each theory/state $|\psi\rangle$ is

$$Z_\psi(\tau) = \langle\psi|Z(\tau)\rangle \tag{2}$$

for some $|Z(\tau)\rangle$ to be described momentarily, the "partition vector".

Quantization of the bulk TQFT

(Following [Witten '98])

The reduction of IIB on S^5 gives an effective action

$$L_{\mathsf{CS}} = \frac{N}{2\pi i}\int_{X_5} B_2 \wedge dC_2\,. \tag{3}$$

In order to specify the boundary conditions, in addition to specifying the vevs of local gauge invariant operators, we need to specify

$$\alpha = \int_S B_2 \quad ; \quad \beta = \int_S C_2 \tag{4}$$

for any $S \subset \mathcal{M}_4$ near the boundary, $X_5 \approx \mathbb{R} \times \mathcal{M}_4$. The equations of motion are

$$dB_2 = dC_2 = 0 \tag{5}$$

and B_2, C_2 are canonically conjugate ($B = C = 0$ is disallowed!):

$$[B_{ij}(x), C_{kl}(y)] = -\frac{2\pi i}{N}\epsilon_{ijkl}\delta^4(x-y)\,. \tag{6}$$

Quantization of the bulk TQFT

(Following [Witten '98])

Define operators measuring the flux

$$\Phi_{\mathsf{RR}}(S) = \exp\left(i\int_S C_2\right) \quad ; \quad \Phi_{\mathsf{NS}}(T) = \exp\left(i\int_T B_2\right) . \qquad (7)$$

They do not commute:

$$\Phi_{\mathsf{RR}}(S)\Phi_{\mathsf{NS}}(T) = \Phi_{\mathsf{NS}}(T)\Phi_{\mathsf{RR}}(S)\exp\left(\frac{2\pi i}{N} S\cdot T\right) . \qquad (8)$$

The inequivalent operators are parameterized by classes in $H_2(\mathcal{M}_4, \mathbb{Z}_N)$, so the group of operators acting on the Hilbert space is the finite Heisenberg group W in

$$0 \to \mathbb{Z}_N \to W \to H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}} \times H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}} \to 0 . \qquad (9)$$

Quantization of the bulk TQFT

(Following [Witten '98])

Up to redefinitions W has a single representation. It can be constructed starting from a maximal isotropic subspace $\mathcal{I}$, i.e. a maximal commuting set of operators $\Phi(w)$.

Define $\mathbf{a} = (a_{\mathsf{NS}}, a_{\mathsf{RR}}) \in \mathfrak{H} \equiv H^2(\mathcal{M}_4, C)_{\mathsf{NS}} \times H^2(\mathcal{M}_4, C)_{\mathsf{RR}}$, and similarly $\mathbf{b}$. Introduce

$$\mathbf{a} \times \mathbf{b} = a_{\mathsf{NS}} \cdot b_{\mathsf{RR}} - a_{\mathsf{RR}} \cdot b_{\mathsf{NS}} \,. \tag{10}$$

Then $\mathcal{I}$ is a maximal subgroup of $\mathfrak{H}$ such that

$$\mathbf{a} \times \mathbf{b} = 0 \quad \forall a, b \in \mathcal{I} \,. \tag{11}$$

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(For the remainder of the talk:
polarization of $\mathfrak{H}$ $\equiv$ maximal isotropic subgroup of $\mathfrak{H}$.)

Quantization of the bulk TQFT

(Following [Witten '98])

For any such $\mathcal{I}$ there is a unique state $|\Omega_{\mathcal{I}}\rangle$ (up to normalization) such that

$$\Phi(w)\,|\Omega_{\mathcal{I}}\rangle = |\Omega_{\mathcal{I}}\rangle \quad \forall w \in \mathcal{I} \tag{12}$$

and a basis $\{|w\rangle_{\mathcal{I}}\}$ that diagonalizes $\Phi(w)$ for $w \in \mathcal{I}$:

$$\Phi(w)\,\left|w'\right\rangle_{\mathcal{I}} = \omega_N^{w\cdot w'}\,\left|w'\right\rangle_{\mathcal{I}} \qquad \text{for } w \in \mathcal{I} \tag{13}$$

$$\Phi(w)\,\left|w'\right\rangle_{\mathcal{I}} = \left|w'+w\right\rangle_{\mathcal{I}} \qquad \text{for } w \in \mathcal{J} \tag{14}$$

where $\omega_N = \exp(2\pi i/N)$, and $\mathfrak{H} = \mathcal{I} \oplus \mathcal{J}$.

We take the choice of duality frame where

$$|Z(\tau)\rangle = \sum_{w\in H^2(\mathcal{M}_4,\mathbb{Z}_N)} Z_w(\tau)\,|w\rangle_{\mathsf{NS}} \tag{15}$$

with $Z_w(\tau)$ the partition function of $\mathcal{N}=4$ $SU(N)/\mathbb{Z}_N$ in the sector with Stiefel-Whitney class w. [Witten '09] [Tachikawa '13]

An assumption

Not every $\mathcal{M}_4$ has a known/conventional holographic dual. For instance, K3 cannot be the boundary of any oriented manifold, since $p_1(K3) = -48 \neq 0$.

I will still assume that the previous prescription for constructing theories applies for the configurations I am interested in. Three and a half reasons:

- Whatever the dual is, it should asymptote to $\mathcal{M}_4 \times \mathbb{R}$.
- Right answer in 2d. (The same construction appears in condensed matter physics.)
- For the "gauge group" cases this prescription gives the same set of theories as [Aharony, Seiberg, Tachikawa '13].
- A not completely rigorous (yet) argument seems to give the same answer from the IIB orbifold viewpoint.

Reproducing the AST classification

The classification of [Aharony, Seiberg, Tachikawa '13] can be understood from this viewpoint [Tachikawa '13]:

Consider the polarization $\mathcal{I}_{T^2} \otimes H^2(\mathcal{M}_4, \mathbb{Z}_N)$, with $\mathcal{I}_{T^2}$ a polarization of $H^1(T^2, \mathbb{Z}_N)$. We have $H^1(T^2, \mathbb{Z}_N) = \mathbb{Z}_N^2$, so the conditions of maximality and Dirac quantization in AST map to maximality and isotropy of $\mathcal{I}_{T^2}$.

Examples:

- $\{(1,0),(2,0),\ldots,(N-1,0)\} \leftrightarrow \mathcal{I} = H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}}$

  $\mapsto \; SU(N)$
- $\{(0,1),(0,2),\ldots,(0,N-1)\} \leftrightarrow \mathcal{I} = H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}}$

  $\mapsto \; (SU(N)/\mathbb{Z}_N)_0$
- $\{a(N,0)+b(0,N)\} \leftrightarrow \mathcal{I} = NH^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}} + NH^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{NS}}$

  $\mapsto \; (SU(N^2)/\mathbb{Z}_N)_0$

Example: $(SU(N)/\mathbb{Z}_N)_0$ from six dimensions

Choose $\mathcal{I} = H^2(\mathcal{M}_4, \mathbb{Z}_N)_{\mathsf{RR}}$. We parameterize

$$|\Omega\rangle = \sum_w c_w \, |w\rangle_{\mathsf{NS}} \, . \tag{16}$$

Invariance under $\mathcal{I}$ then implies that for all v

$$\Phi(v)\, |\Omega\rangle = \sum_w c_w \Phi(v)\, |w\rangle_{\mathsf{NS}} = \sum_w c_w \, |v+w\rangle \stackrel{!}{=} |\Omega\rangle \tag{17}$$

which implies $c_w = c_{v+w}$ for all v. I.e. all c_w are equal, and

$$Z(\tau) = \langle \Omega | Z(\tau) \rangle = \sum_w Z_w(\tau) \, . \tag{18}$$

The $(0, 2)$ viewpoint

It is very natural to rephrase the previous discussion in terms of the 6d $(0, 2)$ A_{N-1} theory. [Witten '98] Holographically, the key term is

$$\mathcal{L} = N \int_{AdS_7} C_3 \wedge dC_3 + \dots \tag{19}$$

which implies that C_3 is the canonical momentum for itself:

$$[C_3, C_3] = \frac{i}{N} \tag{20}$$

so quantum mechanically $C_3 = 0$ is not a valid boundary condition.

The same arguments as before work basically unmodified. We end up with the requirement of choosing a maximal isotropic subgroup of $H^3(\mathcal{M}_6, \mathbb{Z}_N)$.

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When $\mathcal{M}_6 = \mathcal{M}_4 \times T^2$ there is a trivial map to the previous discussion, choosing one of the generators of $H^1(T^2, \mathbb{Z})$ as the "NS" direction and one as the "RR" direction, as usual.

Classifying $\mathcal{N} = 4$ theories, from the 6d perspective

We assume that the $(2,0)$ A_{N-1} theory compactified on T^2, in the limit of small T^2, gives rise to an effective $\mathcal{N} = 4$ 4d theory with gauge *algebra* $\mathfrak{su}(N)$. (What I will say generalizes to arbitrary Riemann surfaces.)

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The choice of gauge *group* for the 4d theory ($SU(N)$, $SU(N)/\mathbb{Z}_N$, $\ldots$) can be encoded in the choice of maximal isotropic subgroup in the six-dimensional theory (induced, conjecturally, from the choice in IIB).

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Do the theories in [Aharony, Seiberg, Tachikawa '13] generate all possible $\mathcal{I}$ in $H^3(\mathcal{M}_4 \times T^2, \mathbb{Z}_N)$?

A self-dual $\mathcal{N} = 4$ $\mathfrak{su}(2)$ theory?

Recently Argyres and Martone proposed the existence of a $\mathcal{N} = 4$ theory with algebra $\mathfrak{su}(2)$ which is completely invariant under $SL(2, \mathbb{Z})$, as part of their work on the classification of rank-1 $\mathcal{N} = 2$ SCFTs. [Argyres, Martone '16]

If it exists it implies the existence of new $\mathcal{N} = 2$ and $\mathcal{N} = 3$ theories via discrete gaugings of various subgroups of $SL(2, \mathbb{Z})$.

Such a theory should be associated with a polarization

$$\mathcal{I} = \mathcal{I}^{(2)}_{\mathsf{NS}} \oplus \mathcal{I}^{(2)}_{\mathsf{RR}} \subset H^2(\mathcal{M}, \mathbb{Z}_N)_{\mathsf{NS}} \times H^2(\mathcal{M}, \mathbb{Z}_N)_{\mathsf{RR}} \,, \tag{21}$$

with $\mathcal{I}^{(2)}$ a maximal isotropic subspace of $H^2(\mathcal{M}, \mathbb{Z}_2)$ with the standard product. (For full $SL(2, \mathbb{Z})$ duality when $N \in 2\mathbb{Z}$ we need a slightly stronger condition, to be explained momentarily.)

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Such a polarization exists
for any compact smooth orientable Spin fourfold (without torsion).

On the intersection form of four-manifolds

We take $\mathcal{M}_4$ to be compact, orientable, smooth and Spin (and having no torsion in $H^i(\mathcal{M}, \mathbb{Z})$, to make our lifes simpler).

A result of Donaldson [Donaldson '83] [Donaldson '87] shows that any such manifold has intersection form over $\mathbb{Z}$ given in some basis by

$$Q = (-\mathcal{C}(E_8))^{\oplus m} \oplus H^{\oplus n} \tag{22}$$

with $\mathcal{C}(E_8)$ the Cartan matrix of E_8 and

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{23}$$

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So the problem of existence of a $\mathcal{I}^{(2)}$ reduces to the existence of a polarization of $\mathcal{C}(E_8)$... Which obviously does not exist (in general) over $\mathbb{Z}$, since $\mathcal{C}(E_8)$ is definite positive!

Existence of $\mathcal{I}^{(2)}$

No obvious no-go in $\mathbb{Z}_2$: signature and positivity are ill-defined.

$$C(E_8) = \begin{pmatrix} 2 & 0 & -\mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -\mathbf{1} & 0 & 0 & 0 & 0 \\ -\mathbf{1} & 0 & 2 & -\mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & -\mathbf{1} & -\mathbf{1} & 2 & -\mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1} & 2 & -\mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{1} & 2 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{1} & 2 & -\mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{1} & 2 \end{pmatrix} . \quad (24)$$

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$$C(E_8)/2\mathbb{Z} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{pmatrix} . \qquad (25)$$

Existence of $\mathcal{I}^{(2)}$

An easy exercise shows that $S(C(E_8)/2\mathbb{Z})S^t = H^{\oplus 4}$, with

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

So, over $\mathbb{Z}_2$

$$Q = (-\mathcal{C}(E_8))^{\oplus m} \oplus H^{\oplus n} \to H^{\oplus(4m+n)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus(4m+n)} \tag{27}$$

as already remarked by [Vafa, Witten '94].

Partition function for $\mathcal{I} = \mathcal{I}^{(2)}_{\textbf{NS}} \oplus \mathcal{I}^{(2)}_{\textbf{RR}}$

$$|\Omega\rangle = \sum_{w \in H^2(\mathcal{M},\mathbb{Z}_2)} c_w \, |w\rangle \tag{28}$$

in the Stiefel-Whitney (NS) basis. For $u \in \mathcal{I}^{(2)}_{\text{NS}}$ we have

$$\Phi(u)\,|\Omega\rangle = \sum_{w \in H^2(\mathcal{M},\mathbb{Z}_2)} (-1)^{u\cdot w} c_w \, |w\rangle \overset{!}{=} |\Omega\rangle \tag{29}$$

so $c_w = 0$ if $w \notin \mathcal{I}^{(2)}$. Then, for $u \in \mathcal{I}^{(2)}_{\text{RR}}$

$$\Phi(u)\,|\Omega\rangle = \sum_{w \in \mathcal{I}^{(2)}} c_w \, |w+u\rangle \overset{!}{=} |\Omega\rangle \tag{30}$$

implies that $c_v = c_w$ for all $v, w \in \mathcal{I}^{(2)}$. So we have

$$\langle Z(\tau)|\Omega\rangle = \sum_{w \in \mathcal{I}^{(2)}} Z_w(\tau)\,. \tag{31}$$

Partition function on $K3$

As a quick check, we can compute explicitly the partition function on $K3$. [Vafa, Witten '94] We have that

$$Z_w(\tau) = \begin{cases} Z_0 = \frac{1}{4}G(q^2) + \frac{1}{2}\left[G(q^{1/2}) + G(-q^{1/2})\right] & \text{if } w = 0 \\ Z_e = \frac{1}{2}\left[G(q^{1/2}) + G(-q^{-1/2})\right] & \text{if } w \neq 0, \mathcal{P}(w) = 0 \\ Z_o = \frac{1}{2}\left[G(q^{1/2}) - G(-q^{-1/2})\right] & \text{if } w \neq 0, \mathcal{P}(w) = 2 \end{cases} \tag{32}$$

with $G(q) \equiv \eta(q)^{-24}$, $q = \exp(2\pi i\tau)$, and $\mathcal{P}(w)$ the Pontryagin square.

We find that if $\mathcal{P}(w) = 0$ for all $w \in \mathcal{I}^{(2)}$ (✓), the partition function

$$Z_{\mathcal{I}^{(2)}}(\tau) = Z_0(\tau) + (2^{11} - 1)Z_e(\tau) \tag{33}$$

is $SL(2, \mathbb{Z})$ invariant. (Or rather, appropriately covariant.)

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Uniqueness

In fact, this is the only combination of the form

$$Z_{(a,b,c)}(\tau) = aZ_0(\tau) + bZ_e(\tau) + cZ_o(\tau) \tag{34}$$

which is $SL(2, \mathbb{Z})$ invariant, up to overall normalization.

Extension to other primes

The construction works for any prime N. Consider

$$G[N] = \begin{pmatrix} \mathbb{I}_4 + N(2N+1)\mathbb{J}_4 & 2N^2\mathbb{J}_4 \\ 2N^2\mathbb{J}_4 & \mathbb{I}_4 + N(2N-1)\mathbb{J}_4 \end{pmatrix} . \qquad (35)$$

with $\mathbb{I}_4$ the 4×4 identity matrix, and $(\mathbb{J})_{ij} = 1$. It is easy to check that $\det(G[N]) = 1$, the matrix is positive definite, and that for $N \in 2\mathbb{Z}+1$ the associated bilinear form is even. So under a change of basis in $\mathbb{Z}$ this must be equivalent to $\mathcal{C}(E_8)$. On the other hand,

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Since, modulo a prime $a^2 + b^2 = -1$ always has a solution, we can introduce $\widetilde{e}_1 = ae_1 + be_2$, $\widetilde{e}_2 = -be_1 + ae_2$ to obtain

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$SL(2,\mathbb{Z})$ invariance of the partition function on $\mathcal{M}$

In general, the partition function $\sum_{w\in\mathcal{I}} Z_w(\tau)$ is $SL(2,\mathbb{Z})$ invariant for a polarization $\mathcal{I}$ with $\mathcal{P}(w)/2 = 0\ \forall w \in \mathcal{I}$: [Vafa, Witten '94]

$$Z_w(\tau+1) = \omega_N^{\mathcal{P}(w)/2} Z_w(\tau) \tag{38}$$

$$Z_w(-1/\tau) = N^{-b_2/2} \sum_{u\in H^2(\mathcal{M},\mathbb{Z}_N)} \omega_N^{u\cdot w} Z_u(\tau) \tag{39}$$

with $b_2 = \dim H^2(\mathcal{M},\mathbb{R})$ and $\omega_N = \exp(2\pi i/N)$. This holds, since

$$\sum_{w\in\mathcal{I}} \omega_N^{w\cdot u} = \begin{cases} 0 & \text{if } u \notin \mathcal{I} \\ |\mathcal{I}| = N^{b_2/2} & \text{if } u \in \mathcal{I} \end{cases} \tag{40}$$

so

$$\sum_{w\in\mathcal{I}} Z_w(-1/\tau) = N^{-b_2/2} \sum_{u\in H^2(\mathcal{M},\mathbb{Z}_N)} \left(\sum_{w\in\mathcal{I}} \omega_N^{u\cdot w}\right) Z_u(\tau) = \sum_{w\in\mathcal{I}} Z_w(\tau)\,. \tag{41}$$

Some properties of the resulting theories

The effective four dimensional properties that we obtain have a number of unfamiliar properties, inherited from the six dimensional parent theory:

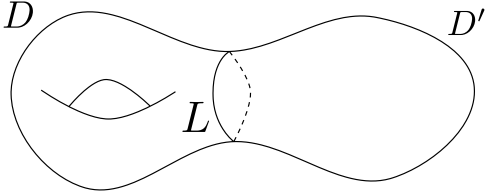
- Their partition function (or more generally, the theory itself) depends on a choice of maximal isotropic subgroup, which is generically not invariant under large diffs. If $\mathcal{M}_4$ has a T^2 factor this implies a failure of modular invariance. (That's OK as long we don't plan to couple to gravity.)
- There is no well-defined notion of gauge group, even in some specific duality frame. But there is a well-defined prescription for computing any observable, including the extended ones.
- There are no "genuine" line operators in these theories, making the line data in [Aharony, Seiberg, Tachikawa '13] $SL(2,\mathbb{Z})$ invariant in the trivial way.

Spectrum of extended operators

A basic distinction for line operators is whether a surface attached to the line needs to be specified for defining the line operator. If not, we say that the operator is *genuine*. [Kapustin,Seiberg '14]

If we have a gauge group G, Wilson lines in representations of G are genuine, and we then find the genuine magnetically charged lines by imposing mutual locality.

This can also be understood without reference to a gauge group (necessary if we want to start in 6d): consider two surface operators D, D' with boundary L. One finds that L is genuine if $S = D - D'$ cannot measure the fluxes in $\mathcal{I}$, for any choice of D, D'. More formally, when $v(S) = 0$ for all $v \in \mathcal{I}$.



Applications

Application 1: New $\mathcal{N}=3$ theories

We can use these ideas to construct some new $\mathcal{N}=3$ theories, as proposed by [Argyres, Martone '16], by gauging/taking quotients by appropriate subgroups of $SL(2,\mathbb{Z})\times SU(4)_R$.

The holographic dual will be rather like in [I.G.-E., Regalado '15], but with a clearer idea of the asymptotic behavior of the fields: In that paper we implicitly assumed "singleton" boundary conditions ($U(N)$ theories), while what we do here works without the need to include the singleton.

Application 2: Duality defects

Consider an elliptically fibered six-manifold $T^2 \to X \to \mathcal{M}_4$. If we make the T^2 small we should get $\mathcal{N} = 4$ theories with duality defects along the loci where the T^2 fiber degenerates.

For instance, we could have a $IV^* = E_6$ degeneration of the fiber over a divisor of $\mathcal{M}_4$. This leads to a $\mathbb{Z}_3 \subset SL(2,\mathbb{Z})$ duality around the divisor.

This makes no sense for the "gauge" $\mathfrak{su}(2)$ theories, because $\mathcal{I}$ is never $\mathbb{Z}_3$ invariant for these theories. Or more plainly, neither $SU(2)$ or $SO(3)$ maps to itself under $\mathbb{Z}_3$. But the choices of $\mathcal{I}$ described here will do the job, and allow such duality defects to exist.

Application 3: New 2d CFTs from reverse AGT

We are putting the $(2,0)$ theories on $\mathcal{M}_4 \times \Sigma$, and making Σ small to reach a 4d theory. We could alternatively make the $\mathcal{M}_4$ small, and go to a 2d theory on Σ.

The ordinary "gauge" $SU(2)$ theories are not $SL(2,\mathbb{Z})$ invariant, so while the 4d theory is modular invariant the 2d theory will not be.

Our choices of polarization are manifestly $SL(2,\mathbb{Z})$ invariant, so the 2d theory will be modular invariant.

Conclusions

It is illuminating to "expose" (an essential part of) the weirdness of the $(2,0)$ theory at weak coupling!

Quite naturally, we are led to consider $\mathcal{N}=4$ four dimensional theories without a good, well-defined notion of gauge group, and without modular invariance. We trade modular invariance for electromagnetic $SL(2,\mathbb{Z})$ invariance.

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Quite naturally, we are led to consider $\mathcal{N}=4$ four dimensional theories without a good, well-defined notion of gauge group, and without modular invariance. We trade modular invariance for electromagnetic $SL(2,\mathbb{Z})$ invariance.

But these are all "virtues" of the 6d theory already. So while more work remains to be done on the understanding of these theories, their existence seems to be forced on us from a careful study of the quantum properties of string theory, (condensed matter) and holography.

Open questions

- Existence of the isotropic polarization of $C(E_8)$ for all N.
- Classification of variants, other than the self-dual case.
- Incorporate torsion in $H^2(\mathcal{M}_4, \mathbb{Z})$.
- Other gauge algebras.

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- Prove that boundary conditions of IIB on $\mathbb{C}^2/\mathbb{Z}_N \times T^2 \times \mathcal{M}_4$ induce the different polarizations of the $(2,0)$ theory.
- Understand gluing axiom better for the $(2,0)$ theory: the anomaly theory has been recently worked out by [Monnier '16], [Monnier '17]. Despite being non-invertible, the anomaly theory obeys the gluing axioms. The $(2,0)$ theory is in some sense the "edge mode" of the anomaly theory, so it should be a sensible theory. (Same as the chiral boson in 2d at level $k > 1$.) But we would like to understand this point better.