Perfect Fluids, and new hydro

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First some fun

The relation between momentum and velocity

Classical Mechanics

$$H = \frac{p^2}{2m} \qquad \vec{p} = m\vec{v}$$

$$Boost: H \rightarrow H + \vec{v}_0 \cdot \vec{p} + c_0$$

Relativistic Mechanics

$$H = \sqrt{p^2c^2 + m^2c^4} \qquad \vec{p} = m\gamma\vec{v} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$Boost: \; H \rightarrow \gamma(H + \vec{v}_0 \cdot \vec{p})$$

Lifshitz mechanics

$$H = \lambda\left(\vec{p}^{\,2}\right)^{z/2} \qquad \left(z = 2:\ \lambda = \frac{1}{2m}\right)$$

$$\vec{p} = \left(\frac{1}{\lambda z}\right)^{\frac{1}{z-1}} \frac{1}{\left(\vec{v}^{\,2}\right)^{\frac{z-2}{2(z-1)}}}\vec{v}$$

No Boost Symmetry for $z \neq 1,2!$

Generically the relation between momentum and velocity can be complicated, and is determined from

$$\vec{v} = \frac{\partial H}{\partial \vec{p}}$$

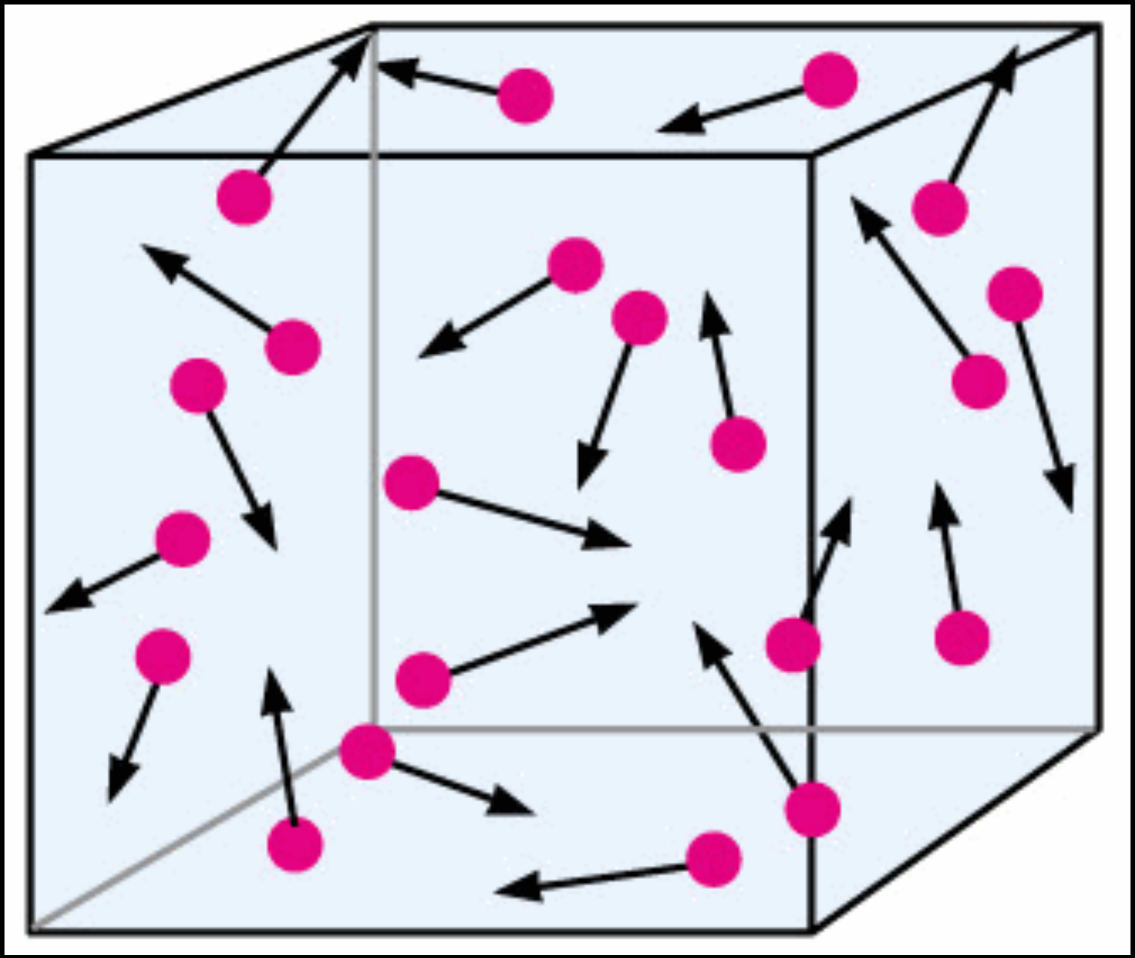
In the absence of boost symmetry we don't know much about this relation. Generically,

$$\vec{p} = f(v,\ldots)\vec{v} + \ldots$$

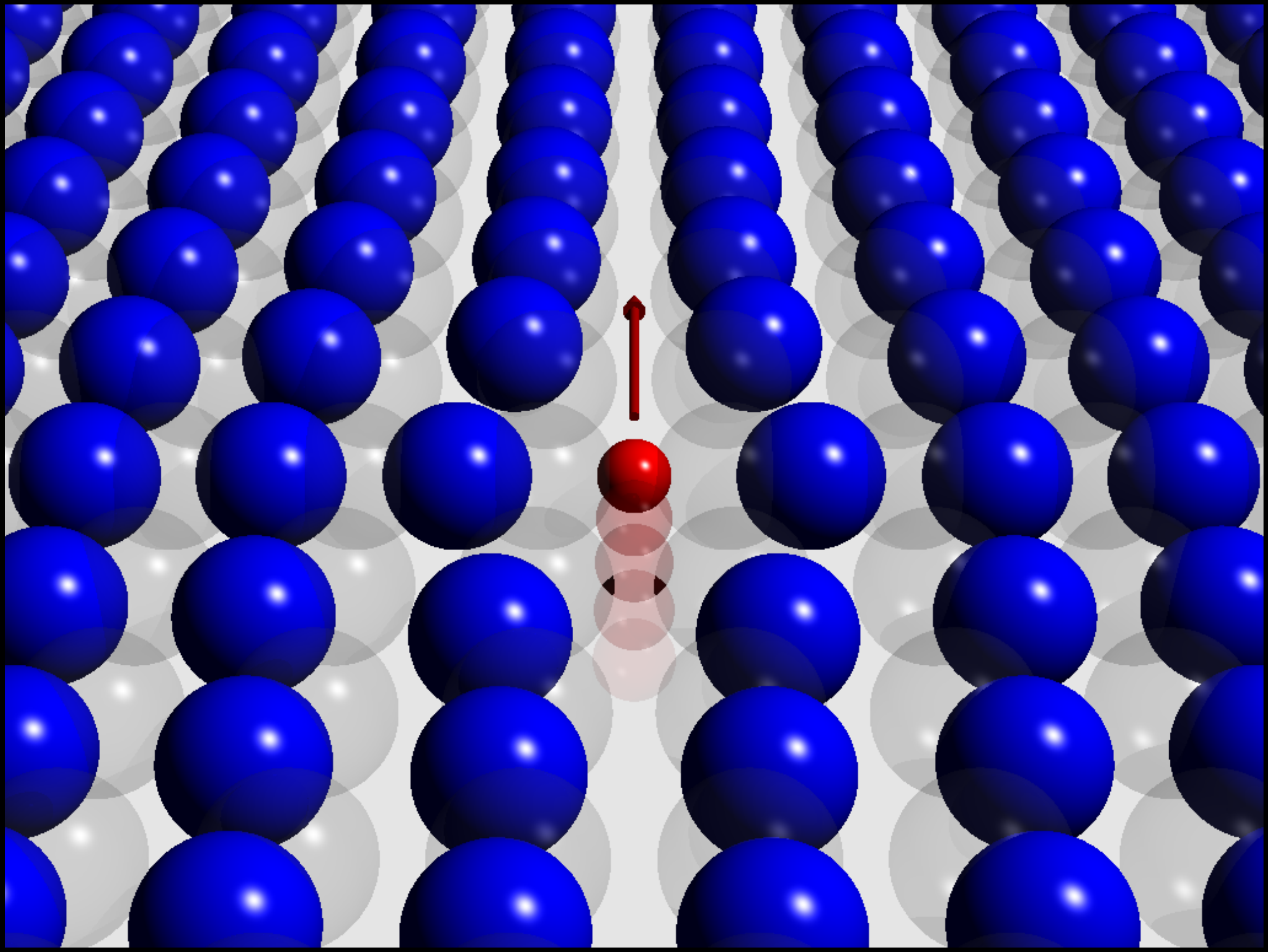
More serious now

What is the thermodynamics of a gas of such particles?

Isotropic, Homogeneous, but no boost symmetry!



Broken Boosts: Electrons in a lattice



Broken Boosts: Bird flocking

First thing is to look at thermodynamic equilibrium

Thermodynamics

$$Z = Tr\left(e^{-\beta(\hat{H} - \mu\hat{N} - \vec{v}\cdot\hat{\vec{P}})}\right)$$

$$\vec{v} = < \frac{\partial \hat{H}}{\partial \vec{P}} >$$

$$\Omega(T, V, \mu, \vec{v}) = -k_B T \log Z$$

$$d\Omega = -SdT - PdV - P_i dv^i - Nd\mu$$

Mass Density

Assuming translation and rotation symmetry, the average momentum (density) must be proportional to the velocity

$$\wp_i \equiv \frac{P_i}{V} = \rho v_i$$

New thermodynamic function ρ that must be computed! Dimension: mass density

Thermodynamic Relations

$$\varepsilon = Ts - P + \rho v^2 + \mu n$$

$$dP = s\,dT + n\,d\mu + \frac{1}{2}\rho\,dv^2$$

$$\rho = 2\left(\frac{\partial P}{\partial v^2}\right)_{T,\,\mu}$$

From this follows a new fluid dynamics. We assume translation and rotation symmetry, but no boost symmetry.

Boost Breaker

In the absence of Lorentz boost symmetry, the stress tensor is no longer symmetric:

$$T^{0i} \neq T^{i0}$$

This difference leads to a new fluid variable related to ρ.

Stress Tensor for a Perfect Fluid

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\varepsilon & \rho v_j \\ -(\varepsilon + P)v^i & P\delta^i_j + \rho v^i v_j \end{pmatrix}$$

U(1) current

$$J^{\mu} = (n, nv^i)$$

Entropy current

From the conservation of the stress-tensor and U(1) current, and the thermodynamic relations one can prove that

$$\partial_t s + \partial_i (s v^i) = 0$$

No entropy production, no dissipation, indeed perfect fluids.

Modified Euler Equation (1757)

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\partial})\vec{v} = -\frac{\vec{\partial} P}{\rho} - \frac{\vec{v}}{\rho}\left[\partial_t \rho + \partial_i (\rho v^i)\right]$$

For Galilei fluids, ρ=mn, and the extra term drops out because of particle number conservation!

Similarly for relativistic fluids, $\rho=(\varepsilon+P)/c^2$.

Modified speed of sound

For fluids at rest

$$c_s^2 = \frac{n}{\rho}\left(\frac{\partial P}{\partial n}\right)_{\frac{s}{n}}$$

Compare to, e.g. Landau-Lifshitz

$$c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_{\frac{s}{n}} \qquad \rho = mn$$

Scale Invariance & Equation of State

$$t \to \Lambda^z t \qquad \vec{x} \to \Lambda \vec{x}$$

$$z T^0_{\ 0} + T^i_{\ i} = 0$$

$$dP = z\varepsilon - \rho v^2$$

Example: Lifshitz ideal gas

Boltzmann gas of Lifshitz particles

$$H_1 = \lambda(\vec{p}^{\,2})^{\frac{z}{2}}$$

$$Z_1(T,V,\vec{v}) = \frac{V}{h^d}\int d^d\vec{p}\; e^{-\beta H_1 - \beta\vec{v}\cdot\vec{p}}$$

$$Z(N,T,V,\vec{v}) = \frac{1}{N!}\left[Z_1(T,V,\vec{v})\right]^N$$

$$\Omega(\mu,T,V,\vec{v}) = -k_B T\, e^{\beta\mu}\, Z_1(T,V,\vec{v})$$

Partition function

$$Z_1 = \frac{2V}{z}\left(\frac{\sqrt{\pi}}{h}\right)^d (\lambda\beta)^{-d/z} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\left(\frac{\beta v^2}{2}\right)^{2n}}{n!} \frac{\Gamma\left[\frac{d+2n}{z}\right]}{\Gamma\left[\frac{d+2n}{2}\right]} (\lambda\beta)^{-2n/z}$$

Thermodynamics (v=0)

- Equipartition $E=(d/z)\ Nk_BT$
- Ideal gas law $PV=Nk_BT$
- Heat capacity $C_V=(d/z)\ Nk_B$
- Adiabatic index $\gamma=1+d/z$
- z-Trace identity $\varepsilon=(d/z)P$

$$d\Omega = -SdT - PdV - P_i dv^i - Nd\mu$$

Mass density

$$\rho_0 = \frac{2}{zd}\left(\frac{\sqrt{\pi}}{h}\right)^d \frac{\Gamma\left[\dfrac{d+2}{z}\right]}{\Gamma\left[\dfrac{d}{2}\right]} \beta(\lambda\beta)^{-\left(\frac{d+2}{z}\right)} e^{\beta\mu}$$

Speed of sound $\omega = c_s k$

$$c_s^2 = (d+z)\frac{\Gamma\left(\frac{d}{z}\right)}{\Gamma\left(\frac{d+2}{z}\right)}(k_B T)^{2\left(\frac{z-1}{z}\right)}\lambda^{\frac{2}{z}}$$

$$z = 1 : c_s^2 = c^2 / d \quad ; \quad z = 2 : c_s^2 = \frac{d+2}{d}\frac{k_B T}{m}$$

Equation of state

$$\rho = \frac{d+z}{d}\frac{k_B T}{c_s^2} n = \frac{\gamma}{c_s^2} P$$

Quantum Bose Lifshitz gas

$$Z_{\vec{k}} = \sum_{n_{\vec{k}}} e^{-n_{\vec{k}}\beta(H_1+\vec{v}\cdot\vec{k}-\mu)}) = \left(1 - e^{-\beta(H_1+\vec{v}\cdot\vec{k}-\mu)}\right)^{-1}$$

$$Z = \prod_{\vec{k}} Z_{\vec{k}} \;\; ; \;\; \Omega = -k_B T \log Z \quad (\mu < 0)$$

Bose condensation (Yan ’00)

$$N \propto VT^{d/z} Li_{d/z}(e^{\beta\mu})$$

$$Bose\ condensation\ \ for\ d > z$$

Critical Temperature

$$k_B T_c = \lambda \left[\frac{N z h^d \Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2} V \zeta\left(\frac{d}{z}\right) \Gamma\left(\frac{d}{z}\right)} \right]^{\frac{z}{d}}$$

Speed of sound

$$T > T_c : c^2_{Bose} = c^2_{Boltzmann} \frac{Li_{[(d+2)/z]-1}[e^{\beta\mu}]}{Li_{(d/z)+1}[e^{\beta\mu}]}$$

$$T < T_c : c^2_{Bose} = c^2_{Boltzmann} \frac{\zeta\left(\frac{d}{z}+1\right)}{\zeta\left(\frac{d+2}{z}-1\right)}$$

$$2(z-1) < d$$

Back to the general theory. Hydrodynamic expansion beyond perfect fluid level: derivative expansion in velocities

Beyond perfect fluids

Hydrodynamic expansion: new hydro-coefficients. At first order in Landau frame

$$T^0_{\ 0} = -\varepsilon_0 - \delta\varepsilon$$

$$T^i_{\ 0} = -(\varepsilon_0 + P_0)\delta v^i$$

$$T^0_{\ j} = \rho_0 \delta v^j - \pi_0 \partial_t \delta v^j$$

$$T^i_{\ j} = (P_0 + \delta P)\delta^i_j - \zeta_0 \delta^i_j (\partial_k \delta v^k)$$

$$-\eta_0(\partial_j \delta v^i + \partial^i \delta v_j - \frac{2}{d}\delta^i_j \partial_k \delta v^k)$$

Shear modes

$$\omega_{shear} = -i\frac{\eta_0}{\rho_0}k^2$$

Sound modes

$$\omega_{sound} = c_s k - \frac{i}{2\rho_0}\left(\zeta_0 + \frac{2}{d}(d-1)\eta_0 + \pi_0 c_s^2\right)k^2$$

Enhancement of attenuation of sound

General Lifshitz fluids

$$\omega_s = c_s k - i\Gamma k^2$$

$$\Gamma = \frac{d-1}{d}\frac{\eta_0}{\rho_0} + \frac{1}{2}\frac{\pi_0}{\rho_0}c_s^2$$

$$\zeta_0 = 0$$

There will also be modifications of the Navier-Stokes equation (in preparation)

Outlook

New perfect fluids in absence of boost symmetry

New sound speed, Lifshitz gasses and hydro

A new hydrodynamic theory for isotropic and homogeneous fluids

Experimental consequences!