Little strings, Elliptic genera, and T-duality

Joonho Kim (KIAS)

Based on:

- 1709.????? with Kimyeong Lee, Jaemo Park.
- 1702.03166 "Little strings on D_n orbifolds" with Kimyeong Lee

Related works:

- 1503.07277 "Little strings and T-duality", J.Kim, S.Kim, K.Lee
- 1609.00310 "On Exceptional Instanton Strings", Del Zotto et al.
- 1701.00765 "Refined BPS invariants of 6d SCFTs … ", Gu et al.

Instanton and SW solution

- In 4d N=2 theory, the Seiberg-Witten prepotential determines an exact low energy effective action in the Coulomb branch. [Seiberg,Witten]

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\,\text{Im}\Big[\int d^4\theta \frac{dF}{dA}\bar{A} + \int d^2\theta \frac{1}{2}\frac{d^2F}{dA^2}W_\alpha W^\alpha\Big]$$

  - It receives non-perturbative instanton contributions.

- It can be systematically obtained via microscopic instanton counting.

[Nekrasov] [Nekrasov,Okounkov]

$$Z(a,\mathfrak{n},z) = Z_0(a,z)\cdot\left(1+\sum_{k=1}^{\infty} Z_k(a,z)\mathfrak{n}^k\right) \xrightarrow{\epsilon_{1,2}\to 0} \exp\left(-\frac{F(a,\mathfrak{n},z)}{\epsilon_1\epsilon_2}\right)$$

  - SUSY partition function on Euclidean R^4 deformed by $\epsilon_1,\ \epsilon_2$
  - Coefficient Z_k : integration over the k-instanton moduli space.

R^4 X T^2 partition function

- This talk: 6d SUSY partition function on Ω-deformed R^4 X T^2

$$Z(a,q,\mathfrak{n},z) = \mathcal{I}_0(a,q,z)\cdot\left(1+\sum_{k=1}^{\infty}\mathcal{I}_k(a,q,z)\,\mathfrak{n}^k\right)$$

  - 6d Yang-Mills instantons = BPS *strings* wrapped on T^2
  - Coefficient I_k captures the BPS spectrum made of k winding modes and an infinite tower of left-moving momentum modes.

$$\mathcal{I}_k(a,q,z) = \mathrm{Tr}_{\mathcal{H}_k}\left[(-1)^F q^{H_L}\bar{q}^{H_R}\,e^{2\pi i\epsilon_+(J_r+J_R)}e^{2\pi i\epsilon_- J_l}e^{2\pi i a\cdot G}e^{2\pi i z\cdot F}\right]$$

- Existence of BPS strings is a characteristic of 6d SCFTs and LSTs, even without gauge symmetry. Different types of strings can exist.

$$Z = \mathcal{I}_0\cdot\left(1+\sum\nolimits_{\{k_i\}}\mathfrak{n}_1^{k_1}\cdots\mathfrak{n}_N^{k_N}\cdot\mathcal{I}_{\{k_i\}}\right)$$

R^4 X T^2 partition function

$$Z = \mathcal{I}_0 \cdot \left(1 + \sum_{\{k_i\}} \mathfrak{n}_1^{k_1} \cdots \mathfrak{n}_N^{k_N} \cdot \mathcal{I}_{\{k_i\}}\right)$$

- Tensor branch observable: all BPS strings acquire non-zero tension.

$$\mathfrak{n}_i \sim \exp\left(-2\pi R \cdot \langle \Phi^i \rangle\right) \neq 1$$

circumference string tension

- The factor I_0 is the partition function of pure momentum sector, decoupled from stringy excitations.

- A specialty of *6d* partition function: each coefficient I_k behaves as a weak Jacobi form of weight 0 and index i(z). [Huang,Katz,Klemm]

$$\mathcal{I}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \varepsilon(a,b,c,d) \exp\left(\frac{-\pi i c \cdot \mathfrak{i}(z)}{c\tau+d}\right) \mathcal{I}(\tau, z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$$

$q = e^{2\pi i \tau}$ phase factor index polynomial

Plan

- Introduction

- Properties of 6d string elliptic genera

  Modular property, Casimir energy, Pole structure,
  Ring of Weyl invariant Jacobi forms, Known examples.

- LST partition function and T-duality

  T-duality of LSTs, Bootstrapping the LST partition function.

- Conclusion

Elliptic genus

- 6d BPS strings are generally described by 2d CFT with (0,4) SUSY.
  - The k-string coefficient I_k should be the elliptic genus of (0,4) SCFT.

$$\mathcal{I}_k(\tau, z) = \mathrm{Tr}_{\mathrm{RR}}\left[(-1)^F e^{2\pi i(\tau H_L - \bar{\tau} H_R)}\, e^{2\pi i z\cdot F}\right]$$

- complex structure: $\tau = \frac{\beta}{2\pi}(\mu + i) \qquad q = e^{2\pi i\tau}$
- z collectively denotes all background U(1) gauge fields including ϵ_1, ϵ_2

- Reducing the temporal circle, the effective 1d CS action is determined by 2d anomalies under large diffomorphism & global symmetry.

[Di Pietro, Komargodski][Golkar, Sethi][S.Kim, Nahmgoong]

$$\frac{\pi i}{6}\frac{(c_R - c_L \pmod{12})}{\beta}\int_{S^1_x} a + \sum_z \frac{i\beta\, n_z}{4\pi}\int_{S^1_x} A_z\,\Phi_z$$

High temp. free energy is obtained by inserting background field values. $a = \frac{\mu\, dx}{1+\mu^2},\ A_z = -\frac{2\pi z}{\beta}\cdot\frac{\mu\, dx}{1+\mu^2},\ \Phi_z = \frac{2\pi z}{\beta}$

Modular property & Casimir energy

- Taking the limit on the S-transformation property of elliptic genus:

$$\lim_{\tau\to i0} \mathcal{I}\left(-\tfrac{1}{\tau}, \tfrac{z}{\tau}\right) = \lim_{\tau\to i0} \left[\varepsilon \exp\left(-\frac{\pi i}{\tau}\cdot \mathfrak{i}(z)\right) \mathcal{I}(\tau, z)\right]$$

the Casimir energy and the index can be identified.

$$\mathfrak{i}(z) = -\sum_i n_z \cdot z^2, \quad E_0 = -\frac{c_R - c_L}{12} \pmod 1$$

coefficients appearing in the anomaly polynomial of 6d strings [Benini-Eager-Hori-Tachikawa]

- The elliptic genus of 6d BPS strings takes the form of

$$\mathcal{I}(\tau, z) = \eta(\tau)^{24E_0} \cdot \mathcal{N}(\tau, z)/\mathcal{D}(\tau, z)$$

- weight ($+12\ E_0$) and index 0
- q-expansion starts from q^{E_0}

[Del Zotto, Lockhart]

- weight ($-12\ E_0$) and index $\mathfrak{i}(z)$
- q-expansion starts from q^0

$(q \equiv e^{2\pi i \tau})$

Pole structure

- Elliptic genus of 6d string is expected to have various poles, related to the bosonic zero modes lifted by background chemical potentials.

- Zero modes for the center-of-mass motion along $R^4 \subset R^4 \times T^2$
  - Divergence (re)appears if we turn off the regulators $\epsilon_1, \epsilon_2 \longrightarrow 0$
  - Degree of divergence: the 6d *single-particle* spectrum index should have a simple pole at $\epsilon_1 = 0$ and $\epsilon_2 = 0$ [Gopakumar, Vafa]

$$f_{\text{single}} = \text{PE}^{-1}[Z] \xrightarrow{\epsilon_1,\epsilon_2\to 0} -\frac{F_0}{\epsilon_1\epsilon_2} \quad \text{with} \quad \text{PE}[f(\tau,z)] \equiv \exp\Big(\sum_{n=1}^{\infty}\frac{f(n\tau,nz)}{n}\Big)$$

  - Denominator of k string elliptic genus: [Del Zotto, Lockhart] [Gu, Huang, Kashani-Poor, Klemm]

$$\tilde{\mathcal{D}}_{\text{com}}(k) = \prod_{n=1}^{k}\frac{\theta_1\left(n\epsilon_+ \pm n\epsilon_-\right)}{\eta^6}$$

Pole structure

- Zero modes of the *reduced* instanton moduli space
  - Since the same zero modes must exist regardless of dimension, we examine various known 5d instanton partition functions.
  - For SU(2) k instantons, the simple poles are located at $n\epsilon_1 + m\epsilon_2 \pm \alpha(a) = 0$ for positive integers (n,m) such that nm ≤ k.

$$\tilde{\mathcal{D}}^{SU(2)}_{\text{reduced}}(\{\alpha\}, k) = \prod_{\substack{nm \leq k \\ n,m>0}} \frac{\theta_1(n\epsilon_1 + m\epsilon_2 \pm \alpha(a))}{\eta^6}$$

α: SU(2) root

  - Generally for simple, non-Abelian G, k instanton solutions can be constructed by embedding SU(2) BPST instantons into G.

[Bernard, Christ, Guth, Weinberg]

Pole structure

- Zero modes of the *reduced* instanton moduli space
  - For simply-laced G, all roots equally embed SU(2) instantons.

$$\tilde{\mathcal{D}}^{G}_{\text{reduced}}(\Delta, k) = \prod_{\alpha\in\Delta} \tilde{\mathcal{D}}^{SU(2)}_{\text{reduced}}(\{\alpha\}, k)$$

[Del Zotto, Lockhart]

  - For non-simply-laced G, a short-root embedding solution carries a *doubled* or *tripled* instanton charge of SU(2) solution's.

$$\tilde{\mathcal{D}}^{G}_{\text{reduced}}(\Delta, k) = \prod_{\alpha_l\in\Delta_l} \tilde{\mathcal{D}}^{SU(2)}_{\text{reduced}}(\{\alpha\}, k) \cdot \prod_{\alpha_s\in\Delta_s} \tilde{\mathcal{D}}^{SU(2)}_{\text{reduced}}(\{\alpha_s\}, \lfloor k/c \rfloor)$$

    - c=2 for B_n, C_n, F_4
    - c=3 for G_2

  match with various 5d partition functions.

- Summary: $\mathcal{I}(\tau, z) = \eta(\tau)^{24E_0} \cdot \mathcal{N}(\tau, z)/\mathcal{D}(\tau, z)$

$$= \tilde{\mathcal{D}}_{\text{com}}(k) \cdot \tilde{\mathcal{D}}^{G}_{\text{reduced}}(\Delta, k)$$

Ring of Jacobi forms

- Numerator of the 6d string elliptic genus
  - a weak Jacobi form of certain weight & index
  - invariant under the Weyl group of the global symmetry.

- It is in the *finitely-generated* ring of Weyl invariant Jacobi forms. [Wirthmuller]

$SU(2)_L$	$SU(2)_R$	F (6d flavor)	G (6d gauge)
$\mathfrak{L}_{2,1} = \theta_1(\pm\epsilon_-)/\eta^6$ $\mathfrak{L}_{0,1} = 4\sum_{a=2}^{4}\frac{\theta_a(\epsilon_-)^2}{\theta_a(0)^2}$	$\mathfrak{R}_{2,1} = \theta_1(\pm\epsilon_+)/\eta^6$ $\mathfrak{R}_{0,1} = 4\sum_{a=2}^{4}\frac{\theta_a(\epsilon_+)^2}{\theta_a(0)^2}$	|F| + 1 generators $\mathfrak{f}_{w,m}$	|G| + 1 generators $\mathfrak{g}_{w,m}$

for explicit expressions of generators: [Bertola], [Sakai]

- This determines the numerator up to a finite number of coefficients, which can be fixed through comparison with initially given BPS data.

[Huang, Katz, Klemm] [Del Zotto, Lockhart]

Examples

- Let us reproduce 1 string elliptic genera in various 6d SCFTs.
- M-string: $\mathcal{N} = -\frac{1}{12}\mathfrak{R}_0\mathfrak{f}_{2,1} + \frac{1}{12}\mathfrak{R}_2\mathfrak{f}_{0,1}$ [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa]
- SU(2) with 4 flavor: [Haghighat, Kozcaz, Lockhart, Vafa]

$$\begin{aligned}\mathcal{N} = \frac{1}{2^{11}3^5}\Big(&54E_4^3\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^4 - 36E_4^2\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^2\mathfrak{R}_0^2 + 9E_4^2\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{R}_0 - 48E_4^2\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{R}_0 - 6E_4^2\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^4\\
&- 9E_4^2\mathfrak{f}_{2,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^4 - 32E_4E_6\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{R}_0 - 16E_4E_6\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^4 - 2E_4\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_0^4 + 3E_4\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{R}_0^3\\
&- 16E_4\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2\mathfrak{R}_0^3 - 12E_4\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^2\mathfrak{R}_0^2 - 108E_4\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^2\mathfrak{R}_0^2 - 3E_4\mathfrak{f}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^3\mathfrak{R}_0 + 72E_4\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{R}\\
&+ 36E_4\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^4 - 64E_6^2\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^4 - 16E_6\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{R}_0^3 + 12E_6\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^2\mathfrak{R}_0^2 - 48E_6\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{R}_0^2\\
&- 16E_6\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^3\mathfrak{R}_0 - 96E_6\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{R}_0 - 8E_6\mathfrak{f}_{2,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{R}_0 - 4E_6\mathfrak{f}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^4 + 96E_6\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^4\\
&+ 2\mathfrak{f}_{4,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_0^4 + 12\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_0^4 + \mathfrak{f}_{2,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_0^4 - \mathfrak{f}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2\mathfrak{R}_0^3 + 24\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2\mathfrak{R}_0^3 - 36\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_2^2\mathfrak{R}_0^2\Big).\end{aligned}$$

- Sp(1) with 10 flavor: [JK, S.Kim, K.Lee]

$$\begin{aligned}\mathcal{N} = \frac{1}{2^{17}3^9}\big(&-3^3\mathfrak{f}_{10,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_2^3E_4^6 + 3^2\mathfrak{f}_{10,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^2E_4^5 + 3^2\mathfrak{f}_{10,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_2\mathfrak{R}_0^2E_4^5 + 3^3\mathfrak{f}_{10,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^2\mathfrak{g}_{0,1}\mathfrak{R}_0E_4^5\\
&+ 3\mathfrak{f}_{10,1}\mathfrak{g}_{2,1}^2\mathfrak{g}_{0,1}\mathfrak{R}_0^3E_4^4 + 3^2\mathfrak{f}_{10,1}\mathfrak{g}_{2,1}\mathfrak{R}_2\mathfrak{g}_{0,1}^2\mathfrak{R}_0^2E_4^4 - 2^63^3\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}E_4^4 + 2^23^1\mathfrak{f}_{10,1}E_6\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}E_4^4\\
&+ 3^1\mathfrak{f}_{10,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^3\mathfrak{R}_0E_4^4 + 2^63^3\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0E_4^4 + 2^23^1\mathfrak{f}_{10,1}E_6\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0E_4^4 + 59^1\mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{2,1}^3\mathfrak{R}_2^3E_4^3\\
&- 2^63^2\mathfrak{f}_{4,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^3E_4^3 + 2^1\mathfrak{f}_{10,1}E_6\mathfrak{R}_2^3\mathfrak{g}_{0,1}^3E_4^3 + 2^63^2\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_0^3E_4^3 + 2^1\mathfrak{f}_{10,1}E_6\mathfrak{g}_{2,1}^3\mathfrak{R}_0^3E_4^3 - \mathfrak{f}_{10,1}\mathfrak{g}_{0,1}^3\mathfrak{R}_0^3E_4^3\\
&- 2^43^3\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^2E_4^3 + 2^43^3\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_2\mathfrak{R}_0^2E_4^3 + 2^63^3\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{g}_{0,1}\mathfrak{R}_0^2E_4^3 + 2^13^2\mathfrak{f}_{10,1}E_6\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{g}_{0,1}\mathfrak{R}_0^2E_4^3\\
&- 2^53^4\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}E_4^3 + 2^53^4\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0E_4^3 - 2^63^3\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^2\mathfrak{R}_0E_4^3 + 2^13^2\mathfrak{f}_{10,1}E_6\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^2\mathfrak{R}_0E_4^3\\
&+ 2^63^3\mathfrak{f}_{0,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^3E_4^2 - 2^63^3\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^3\mathfrak{R}_0^3E_4^2 - 2^63^2\mathfrak{f}_{4,1}\mathfrak{g}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_0^3E_4^2 + 2^43^2\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}^2\mathfrak{g}_{0,1}\mathfrak{R}_0^3E_4^2\\
&- 3^2\mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^2E_4^2 - 2^63^27^1\mathfrak{f}_{4,1}E_6\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^2E_4^2 + 2^63^2\mathfrak{f}_{4,1}\mathfrak{R}_2\mathfrak{g}_{0,1}^3\mathfrak{R}_0^2E_4^2 - 3^2\mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{2,1}^3\mathfrak{R}_2\mathfrak{R}_0^2E_4^2\\
&+ 2^63^27^1\mathfrak{f}_{4,1}E_6\mathfrak{g}_{2,1}^3\mathfrak{R}_2\mathfrak{R}_0^2E_4^2 - 2^63^4\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{g}_{0,1}\mathfrak{R}_0^2E_4^2 - 2^43^25^1\mathfrak{f}_{2,1}E_6\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}E_4^2 - 2^43^2\mathfrak{f}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^3\mathfrak{R}_0E_4^2\\
&+ 2^43^25^1\mathfrak{f}_{2,1}E_6\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0E_4^2 + 2^63^4\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^2\mathfrak{R}_0E_4^2 - 3^3\mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{2,1}^2\mathfrak{R}_2^2\mathfrak{g}_{0,1}\mathfrak{R}_0E_4^2 - 2^53^2\mathfrak{f}_{2,1}E_6\mathfrak{R}_2^3\mathfrak{g}_{0,1}^3E_4\\
&+ 2^53^2\mathfrak{f}_{2,1}E_6\mathfrak{g}_{2,1}^3\mathfrak{R}_0^3E_4 + 2^53^3\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_0^3E_4 - 3^1\mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{2,1}^2\mathfrak{g}_{0,1}\mathfrak{R}_0^3E_4 + 2^63^2\mathfrak{f}_{4,1}E_6\mathfrak{g}_{2,1}^2\mathfrak{g}_{0,1}\mathfrak{R}_0^3E_4\\
&+ 2^53^37^1\mathfrak{f}_{0,1}E_6\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^2E_4 - 2^53^3\mathfrak{f}_{0,1}\mathfrak{R}_2\mathfrak{g}_{0,1}^3\mathfrak{R}_0^2E_4 - 3^2\mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{2,1}\mathfrak{R}_2\mathfrak{g}_{0,1}^2\mathfrak{R}_0^2E_4 - 2^53^37^1\mathfrak{f}_{0,1}E_6\mathfrak{g}_{2,1}^3\mathfrak{R}_2\mathfrak{R}_0^2E_4\\
&+ 2^53^3\mathfrak{f}_{2,1}E_6\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{g}_{0,1}\mathfrak{R}_0^2E_4 - 2^23^1\mathfrak{f}_{10,1}E_6^3\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}E_4 - 2^73^2\mathfrak{f}_{4,1}E_6^2\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}E_4 - 3^1\mathfrak{f}_{10,1}E_6^2\mathfrak{R}_2^2\mathfrak{g}_{0,1}^3\mathfrak{R}_0E_4\\
&- 2^63^2\mathfrak{f}_{4,1}E_6\mathfrak{R}_2^2\mathfrak{g}_{0,1}^3\mathfrak{R}_0E_4 - 2^23^1\mathfrak{f}_{10,1}E_6^3\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0E_4 + 2^73^2\mathfrak{f}_{4,1}E_6^2\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0E_4 - 2^53^3\mathfrak{f}_{2,1}E_6\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^2\mathfrak{R}_0E_4\\
&- 2^5\mathfrak{f}_{10,1}E_6^4\mathfrak{g}_{2,1}^3\mathfrak{R}_2^3 - 2^1\mathfrak{f}_{10,1}E_6^3\mathfrak{R}_2^3\mathfrak{g}_{0,1}^3 - 2^63^2\mathfrak{f}_{4,1}E_6^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}^3 - 2^1\mathfrak{f}_{10,1}E_6^3\mathfrak{g}_{2,1}^3\mathfrak{R}_0^3 + 2^63^2\mathfrak{f}_{4,1}E_6^2\mathfrak{g}_{2,1}^3\mathfrak{R}_0^3\\
&+ \mathfrak{f}_{10,1}E_6^2\mathfrak{g}_{0,1}^3\mathfrak{R}_0^3 - 2^43^2\mathfrak{f}_{2,1}E_6\mathfrak{g}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{R}_0^3 - 2^53^3\mathfrak{f}_{0,1}E_6\mathfrak{g}_{2,1}^2\mathfrak{g}_{0,1}\mathfrak{R}_0^3 - 2^63^2\mathfrak{f}_{2,1}E_6^2\mathfrak{g}_{2,1}\mathfrak{R}_2^3\mathfrak{g}_{0,1}^2 + 2^43^2\mathfrak{f}_{2,1}E_6\mathfrak{R}_2\mathfrak{g}_{0,1}^3\mathfrak{R}\\
&+ 2^63^2\mathfrak{f}_{2,1}E_6^2\mathfrak{g}_{2,1}^3\mathfrak{R}_2\mathfrak{R}_0^2 - 2^13^2\mathfrak{f}_{10,1}E_6^3\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{g}_{0,1}\mathfrak{R}_0^2 + 2^63^3\mathfrak{f}_{4,1}E_6^2\mathfrak{g}_{2,1}^2\mathfrak{R}_2\mathfrak{g}_{0,1}\mathfrak{R}_0^2 + 2^83^3\mathfrak{f}_{0,1}E_6^2\mathfrak{g}_{2,1}^2\mathfrak{R}_2^3\mathfrak{g}_{0,1}\\
&+ 2^53^3\mathfrak{f}_{0,1}E_6\mathfrak{R}_2^2\mathfrak{g}_{0,1}^3\mathfrak{R}_0 - 2^83^3\mathfrak{f}_{0,1}E_6^2\mathfrak{g}_{2,1}^3\mathfrak{R}_2^2\mathfrak{R}_0 - 2^13^2\mathfrak{f}_{10,1}E_6^3\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^2\mathfrak{R}_0 - 2^63^3\mathfrak{f}_{4,1}E_6^2\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{g}_{0,1}^2\mathfrak{R}_0\big).\end{aligned}$$

- pure SU(3): [H.-C. Kim, S.Kim, J.Park]

$$\begin{aligned}\mathcal{N} = \frac{1}{2^{12}3^4}\big(&-24E_4^3\tilde{\mathfrak{R}}_2^3\mathfrak{g}_{2,1}\mathfrak{g}_{3,1}^2 + 24E_4^2\tilde{\mathfrak{R}}_2\tilde{\mathfrak{R}}_0^2\mathfrak{g}_{2,1}\mathfrak{g}_{3,1}^2 + 3E_4^2\tilde{\mathfrak{R}}_2^2\tilde{\mathfrak{R}}_0\mathfrak{g}_{2,1}^3 - 288E_4^2\tilde{\mathfrak{R}}_2^2\tilde{\mathfrak{R}}_0\mathfrak{g}_{0,1}\mathfrak{g}_{3,1}^2\\
&+ 36E_4^2\tilde{\mathfrak{R}}_2^3\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}^2 + 24E_4E_6\tilde{\mathfrak{R}}_2^2\tilde{\mathfrak{R}}_0\mathfrak{g}_{2,1}\mathfrak{g}_{3,1}^2 - 96E_4E_6\tilde{\mathfrak{R}}_2^3\mathfrak{g}_{0,1}\mathfrak{g}_{3,1}^2 + E_4\tilde{\mathfrak{R}}_0^3\mathfrak{g}_{2,1}^3 - 96E_4\tilde{\mathfrak{R}}_0^3\mathfrak{g}_{0,1}\mathfrak{g}_{3,1}^2\\
&- 36E_4\tilde{\mathfrak{R}}_2\tilde{\mathfrak{R}}_0^2\mathfrak{g}_{0,1}\mathfrak{g}_{2,1}^2 - 432E_4\tilde{\mathfrak{R}}_2^2\tilde{\mathfrak{R}}_0\mathfrak{g}_{0,1}^2\mathfrak{g}_{2,1} + 1728E_4\tilde{\mathfrak{R}}_2^3\mathfrak{g}_{0,1}^3 + 32E_6^2\tilde{\mathfrak{R}}_2^3\mathfrak{g}_{2,1}\mathfrak{g}_{3,1}^2 + 8E_6\tilde{\mathfrak{R}}_0^3\mathfrak{g}_{2,1}\mathfrak{g}_{3,1}^2\\
&+ 4E_6\tilde{\mathfrak{R}}_2\tilde{\mathfrak{R}}_0^2\mathfrak{g}_{2,1}^3 - 288E_6\tilde{\mathfrak{R}}_2\tilde{\mathfrak{R}}_0^2\mathfrak{g}_{0,1}\mathfrak{g}_{3,1}^2 - 576E_6\tilde{\mathfrak{R}}_2^3\mathfrak{g}_{0,1}^2\mathfrak{g}_{2,1} - 144\tilde{\mathfrak{R}}_0^3\mathfrak{g}_{0,1}^2\mathfrak{g}_{2,1} - 1728\tilde{\mathfrak{R}}_2\tilde{\mathfrak{R}}_0^2\mathfrak{g}_{0,1}^3\big).\end{aligned}$$

- G_2 with 1 flavor: [JK, H.-C. Kim, S.Kim, K.-H. Lee, J. Park, to appear]

$$\begin{aligned}\mathcal{N} = \frac{1}{2^83^2}\big(&-96\mathfrak{g}_{6,2}\mathfrak{g}_{0,1}E_6\mathfrak{f}_{2,1} - 96\mathfrak{g}_{6,2}\mathfrak{g}_{0,1}E_4\mathfrak{f}_{0,1} + 8\mathfrak{g}_{2,1}\mathfrak{g}_{6,2}E_4^2\mathfrak{f}_{2,1} - 4\mathfrak{g}_{2,1}^3E_6\mathfrak{f}_{2,1}\\
&+ \mathfrak{g}_{2,1}^3E_4\mathfrak{f}_{0,1} + 8\mathfrak{g}_{2,1}\mathfrak{g}_{6,2}E_6\mathfrak{f}_{0,1} - 5184\mathfrak{g}_{0,1}^3\mathfrak{f}_{2,1} - 144\mathfrak{g}_{2,1}\mathfrak{g}_{0,1}^2\mathfrak{f}_{0,1} + 84\mathfrak{g}_{2,1}^2\mathfrak{g}_{0,1}E_4\mathfrak{f}_{2,1}\big)\end{aligned}$$

Plan

- Introduction

- Properties of 6d string elliptic genera

  Modular property, Casimir energy, Pole structure,
  Ring of Weyl invariant Jacobi forms, Known examples.

- LST partition function and T-duality

  T-duality of LSTs, Bootstrapping the LST partition function.

- Conclusion

Little string theory

- UV-complete, non-gravitational string theory in 6 dimensions. [Seiberg '97]

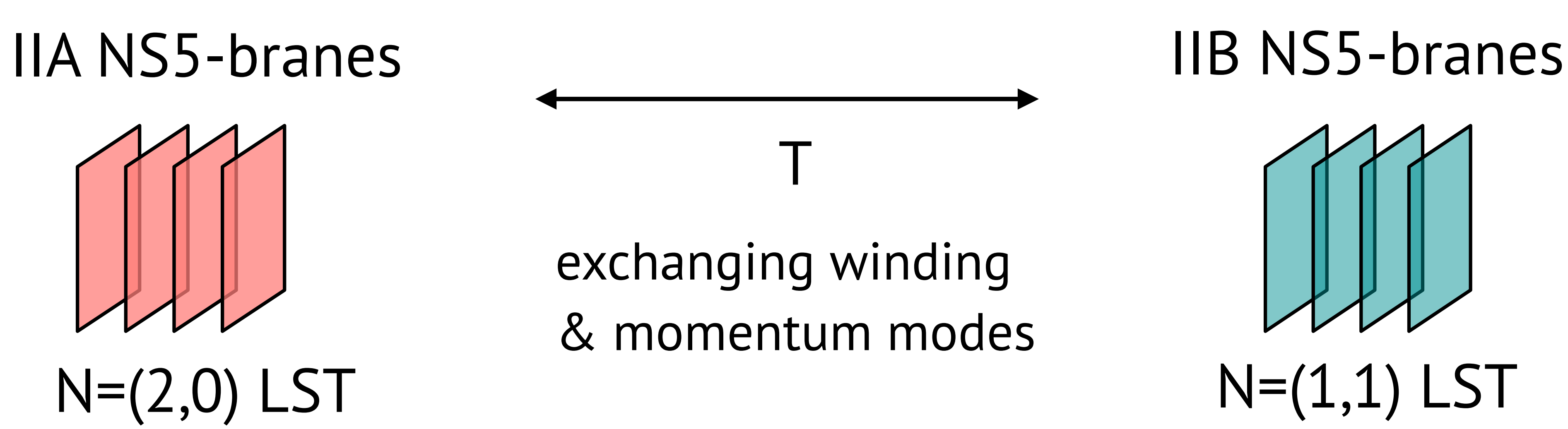
  1) It depends on a mass scale M_{st}, determining the *string tension*.
  2) Its density of states grows *exponentially*. [Aharony et al.'98]

  $$\rho(E) = E^{\alpha}\, e^{\beta E}$$

  3) It exhibits *T-duality* upon toroidal compactification.

- Example: type II NS5-branes in the limit $g_s \longrightarrow 0, \;\; \alpha' = (\text{finite})$

  [Dijkgraaf et al '96] [Seiberg '97]



T-duality of LSTs

- T-duality: equivalence between two circle compactified LSTs.
- BPS partition functions for dual LSTs are expected to agree, with a proper identification of winding and momentum fugacity variables.

[J.Kim, S.Kim, K.Lee] [Hohenegger et al.] [JK, K.Lee]

- Example:

[J.Kim, S.Kim, K.Lee]

Modes	SU(n) type (2,0) LST	SU(n) type (1,1) LST
winding	n *fractional* strings wrapping the circle $\mathfrak{n}_1,\, \mathfrak{n}_2,\, \cdots,\, \mathfrak{n}_n$	a *fully* wrapping string $\mathfrak{n}'$
momentum	a *full* circle momentum q	n *fractional* modes by SU(n) holonomy $q'_1,\, q'_2,\, \cdots,\, q'_n$

$$Z^{(2,0)}\big(\{\mathfrak{n}_i\}_{i=1}^n,\, q,\, z\big) = Z^{(1,1)}\big(\mathfrak{n}',\, \{q'_i\}_{i=1}^n,\, z\big) \quad \text{if} \quad \mathfrak{n}' = q,\ q'_i = \mathfrak{n}_i$$

Bootstrapping 6d partition function

- T-duality relation provides a sufficient amount of initial BPS data to construct the full LST partition function.

- Example: (2,0) and (1,1) LSTs of SU(2) type

$$Z^{(2,0)} = \mathcal{I}_0 \cdot \Big(1 + \sum_{k_1,k_2} \mathfrak{n}_1^{k_1}\mathfrak{n}_2^{k_2} \cdot \mathcal{I}_{\{k_1,k_2\}}\Big) \qquad Z^{(1,1)} = \mathcal{I}_0' \cdot \Big(1 + \sum_{k} \mathfrak{n}'^k \cdot \mathcal{I}_k'\Big)$$

(1) Pure momentum sector index can be obtained by counting the single trace operators. $t = e^{2\pi i \epsilon_+},\ u = e^{2\pi i \epsilon_-},\ v = e^{2\pi i m}$

SU(2) x SU(2) = $SO(4)_L$ SU(2) ⊂ $SO(4)_R$

$$\mathcal{I}_0 = \mathrm{PE}\Big[\frac{2\,t\left(v+v^{-1}-u-u^{-1}\right)}{(1-tu)(1-tu^{-1})}\frac{q}{1-q}\Big]$$

$$\mathcal{I}_0' = \mathrm{PE}\Big[\frac{t\left(v+v^{-1}-t-t^{-1}\right)}{(1-tu)(1-tu^{-1})}\frac{q_1'+q_2'+2q_1'q_2'}{1-q_1'q_2'}\Big]$$

expand in q_1', q_2'
read the coefficients at q_1', q_2', $q_1'q_2'$

Bootstrapping 6d partition function

(2) Using the BPS coefficients from I_0', one can determine

$$\mathcal{I}_{\{1,0\}} = \mathcal{I}_{\{0,1\}} = \frac{\eta^6(\frac{1}{12}\mathfrak{R}_0\mathfrak{f}_{2,1} - \frac{1}{12}\mathfrak{R}_2\mathfrak{f}_{0,1})}{\theta_1(\epsilon_+ \pm \epsilon_-)} = \frac{\theta_1(m \pm \epsilon_+)}{\theta_1(\epsilon_+ \pm \epsilon_-)}$$

$$\mathcal{I}_{\{1,1\}} = \frac{\eta^{12}\left(\frac{1}{12}\mathfrak{f}_{2,1}\mathfrak{L}_0 - \frac{1}{12}\mathfrak{f}_{0,1}\mathfrak{L}_2\right)^2}{\theta_1(\epsilon_+ \pm \epsilon_-)^2} = \frac{\theta_1(m \pm \epsilon_-)^2}{\theta_1(\epsilon_+ \pm \epsilon_-)^2}$$

expand in q,
read the coefficient at q

(3) Using the BPS coefficients from I_0, I_{10}, one can determine

$$\begin{aligned}\mathcal{I}_1' = \frac{1}{2^7 3^4}\frac{\theta_1(m \pm \epsilon_+)}{\theta_1(\epsilon_+ \pm \epsilon_-)\theta_1(-2\epsilon_+ \pm \alpha(a))}\big(&- 9E_4^2\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3 - 3E_4\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}\mathfrak{R}_2\mathfrak{R}_0^2 \\ &- 3E_4\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}\mathfrak{R}_2^3 - 12E_6\mathfrak{f}_{2,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{R}_0 - 8E_6\mathfrak{f}_{2,1}\mathfrak{g}_{0,1}\mathfrak{R}_2^3 - 4E_6\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^3 \\ &- \mathfrak{f}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_0^3 + 3\mathfrak{f}_{0,1}\mathfrak{g}_{0,1}\mathfrak{R}_2\mathfrak{R}_0^2 - 3E_4\mathfrak{f}_{0,1}\mathfrak{g}_{2,1}\mathfrak{R}_2^2\mathfrak{R}_0 - 9E_4\mathfrak{f}_{2,1}\mathfrak{g}_{0,1}\mathfrak{R}_2^2\mathfrak{R}_0 + \mathfrak{f}_{2,1}\mathfrak{g}_{0,1}\mathfrak{R}_0^3\big)\end{aligned}$$

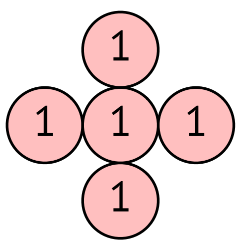
(4) This procedure can be repeated as much as wanted.

Results

- We worked out the 6d partition function for SU(3), SO(4), SO(6) cases.
- This procedure does *not* depend on 2d UV gauge theory descriptions of 6d strings, usually induced from the brane engineering of LSTs.

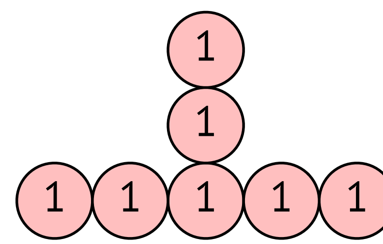
[Atiyah, Drinfeld, Hitchin, Manin] [Haghighat, Kozcaz, Lockhart, Vafa]

SO(8) LST



$$-\frac{\theta_1(m \pm \epsilon_+)\theta_1(m \pm \epsilon_-)^4}{\theta_1(\epsilon_+ \pm \epsilon_-)^5}$$

E6 LST



$$-\frac{\theta_1(m \pm \epsilon_+)\theta_1(m \pm \epsilon_-)^6}{\theta_1(\epsilon_+ \pm \epsilon_-)^7}$$

- We compare the elliptic genera of single SU(2), SU(3), SO(4), SO(6) instanton strings with those of 2d ADHM gauge theories.
  - SU(2) and SU(3): in agreement with U(1) ADHM gauge theory. [J.Kim, S.Kim, K.Lee]
  - SO(4) and SO(6): U(k) and Sp(k) ADHM gauge theories.
    Our results agree with U(1) ADHM gauge theory, not Sp(1) gauge theory.

Results

- The difference between Sp(1) and U(1) elliptic genera:
  extra BPS states for string configuration away from 6d spacetime.

$$\frac{1}{2^{11}3^6}\frac{\theta_1(m\pm\epsilon_-)\theta_1(-\epsilon_+\pm m)}{\theta_1(\epsilon_+\pm\epsilon_-)\theta_1(-2\epsilon_+\pm 2m)}\big(27E_4^3\mathfrak{f}_{2,1}^6-45E_4^2\mathfrak{f}_{2,1}^4\mathfrak{f}_{0,1}^2-24E_4E_6\mathfrak{f}_{2,1}^5\mathfrak{f}_{0,1}-15E_4\mathfrak{f}_{2,1}^2\mathfrak{f}_{0,1}^4-32E_6^2\mathfrak{f}_{2,1}^6-40E_6\mathfrak{f}_{2,1}^3\mathfrak{f}_{0,1}^3+\mathfrak{f}_{0,1}^6\big)$$

$SO(4)_R$ angular momentum tower

for an SO(6) instanton

It only carries a tower of full (non-fractional) circle momentum.

- Such comparison suggests that the Sp(k) ADHM elliptic genera are reliable in fractional momentum sectors.

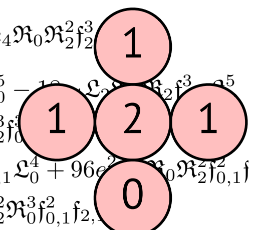
- Imposing the suggested pole structure, one can naturally excise them.
- However, it is *not* always the case, e.g., SO(8) case & heterotic LST.
  The bootstrap procedure *fails* with the suggested pole structure.

cf. [Y.Hwang, JK, S.Kim] for 5d instantons

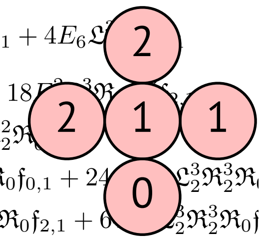
Results

- We continue to study the elliptic genera of *fractional* string chains in all SO(2n)-type (2,0) LSTs.

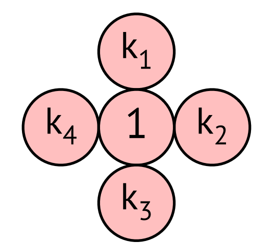
$$\mathcal{I}_{\{1,1,2,1,0\}} = \frac{1}{2^{17}3^9}\frac{\eta^{18}\,\theta_1(m\pm\epsilon_-)\theta_1(m\pm\epsilon_+)}{\theta_1(\epsilon_+\pm\epsilon_-)^4\theta_1(2\epsilon_+\pm 2\epsilon_-)}\big(\mathfrak{R}_2^3\mathfrak{f}_{0,1}^3\mathfrak{L}_0^6 + \mathfrak{R}_0^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^6 - 4e_6\mathfrak{R}_2^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^6 - 6e_4\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_2 \cdots$$
$$+ 3\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^6 - 3\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^6 - 18e_4^2\mathfrak{L}_2\mathfrak{R}_2^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^5 - 36e_6\mathfrak{L}_2\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{2,1}^3\mathfrak{L}_0^5 \cdots$$
$$- 6\mathfrak{L}_2\mathfrak{R}_0^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^5 + 12e_6\mathfrak{L}_2\mathfrak{R}_2^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^5 + 18e_4\mathfrak{L}_2\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^5 - 15e_4\mathfrak{L}_2^2\mathfrak{R}_2^3\mathfrak{f} \cdots$$
$$+ 9e_4\mathfrak{L}_2^2\mathfrak{R}_0^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^4 - 24e_4e_6\mathfrak{L}_2^2\mathfrak{R}_2^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^4 - 54e_4^2\mathfrak{L}_2^2\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{2,1}^3\mathfrak{L}_0^4 - 36e_6\mathfrak{L}_2^2\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{2,1}^3\mathfrak{L}_0^4 + 96e \cdots \mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^2\mathfrak{f} \cdots$$
$$+ 54e_4^2\mathfrak{L}_2^2\mathfrak{R}_2^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^4 + 108e_6\mathfrak{L}_2^2\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^4 + 9e_4\mathfrak{L}_2^2\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^4 + 6\mathfrak{L}_2^2\mathfrak{R}_0^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,} \cdots$$
$$- 12e_6\mathfrak{L}_2^2\mathfrak{R}_2^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^4 + 27e_4\mathfrak{L}_2^2\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^4 - 2\mathfrak{L}_2^3\mathfrak{R}_0^3\mathfrak{f}_{0,1}^3\mathfrak{L}_0^3 - 36e_6\mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{f}_{0,1}^3\mathfrak{L}_0^3 + 6e_4\mathfrak{L}_2^3\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^3\mathfrak{L}_0^3$$
$$+ 36e_6\mathfrak{L}_2^3\mathfrak{R}_0^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^3 + 54e_4^3\mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^3 - 64e_6^2\mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^3 - 24e_4e_6\mathfrak{L}_2^3\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{2,1}^3\mathfrak{L}_0^3 - 18e_4^2\mathfrak{L}_2^3\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{2,1}^3\mathfrak{L}_0^3$$
$$+ 72e_4e_6\mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^3 + 162e_4^2\mathfrak{L}_2^3\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^3 - 12e_6\mathfrak{L}_2^3\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^3 - 54e_4^2\mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^3$$
$$+ 12e_6\mathfrak{L}_2^3\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^3 - 54e_4\mathfrak{L}_2^3\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^3 - 27e_4^2\mathfrak{L}_2^4\mathfrak{R}_2^3\mathfrak{f}_{0,1}^3\mathfrak{L}_0^2 + 36e_6\mathfrak{L}_2^4\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^3\mathfrak{L}_0^2$$
$$+ 18e_4\mathfrak{L}_2^4\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}^3\mathfrak{L}_0^2 + 45e_4^2\mathfrak{L}_2^4\mathfrak{R}_0^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0^2 + 12e_6\mathfrak{L}_2^4\mathfrak{R}_0^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^2 - 162e_4^3\mathfrak{L}_2^4\mathfrak{R}_2^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^2$$
$$+ 192e_6^2\mathfrak{L}_2^4\mathfrak{R}_2^3\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^2 + 72e_4e_6\mathfrak{L}_2^4\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^2 - 81e_4^2\mathfrak{L}_2^4\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0^2 - 18e_4\mathfrak{L}_2^4\mathfrak{R}_0^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^2$$
$$- 72e_4e_6\mathfrak{L}_2^4\mathfrak{R}_2^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^2 - 27e_4^2\mathfrak{L}_2^4\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^2 - 108e_6\mathfrak{L}_2^4\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0^2 + 6e_4\mathfrak{L}_2^5\mathfrak{R}_0^3\mathfrak{f}_{0,1}^3\mathfrak{L}_0$$
$$+ 54e_4^2\mathfrak{L}_2^5\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^3\mathfrak{L}_0 + 36e_6\mathfrak{L}_2^5\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}^3\mathfrak{L}_0 + 24e_4e_6\mathfrak{L}_2^5\mathfrak{R}_0^3\mathfrak{f}_{2,1}^3\mathfrak{L}_0 - 72e_4e_6\mathfrak{L}_2^5\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2\mathfrak{L}_0$$
$$- 12e_6\mathfrak{L}_2^5\mathfrak{R}_0^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0 + 162e_4^3\mathfrak{L}_2^5\mathfrak{R}_2^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0 - 192e_6^2\mathfrak{L}_2^5\mathfrak{R}_2^3\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0 - 54e_4^2\mathfrak{L}_2^5\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\mathfrak{L}_0$$
$$+ 4e_6\mathfrak{L}_2^6\mathfrak{R}_0^3\mathfrak{f}_{0,1}^3 - 27e_4^3\mathfrak{L}_2^6\mathfrak{R}_2^3\mathfrak{f}_{0,1}^3 + 32e_6^2\mathfrak{L}_2^6\mathfrak{R}_2^3\mathfrak{f}_{0,1}^3 + 24e_4e_6\mathfrak{L}_2^6\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^3 + 18e_4^2\mathfrak{L}_2^6\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}^3$$
$$- 27e_4^3\mathfrak{L}_2^6\mathfrak{R}_0^3\mathfrak{f}_{2,1}^3 + 32e_6^2\mathfrak{L}_2^6\mathfrak{R}_0^3\mathfrak{f}_{2,1}^3 + 81e_4^3\mathfrak{L}_2^6\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2 - 96e_6^2\mathfrak{L}_2^6\mathfrak{R}_0^2\mathfrak{R}_2\mathfrak{f}_{0,1}\mathfrak{f}_{2,1}^2 - 81e_4^3\mathfrak{L}_2^6\mathfrak{R}_0\mathfrak{R}_2^2\mathfrak{f}_{0,1}^2\mathfrak{f}_{2,1}\big)$$

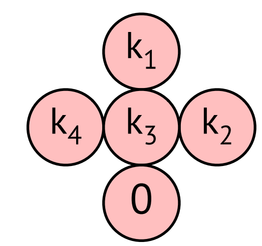


$$\mathcal{I}_{\{2,2,1,1,0\}} = \frac{1}{2^{26}3^{14}}\frac{\eta^{12}\,\theta_1(m\pm\epsilon_-)\theta_1(m\pm\epsilon_+)^3}{\theta_1(\epsilon_+\pm\epsilon_-)^4\theta_1(2\epsilon_+\pm 2\epsilon_-)^2}\big(3E_4\mathfrak{L}_2^3\mathfrak{R}_0^4\mathfrak{f}_{0,1} - 3\mathfrak{L}_0^2\mathfrak{L}_2\mathfrak{R}_0^4\mathfrak{f}_{0,1} + \mathfrak{L}_0^3\mathfrak{R}_0^4\mathfrak{f}_{2,1} + 4E_6\mathfrak{L} \cdots$$
$$+ 3E_4\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_0^4\mathfrak{f}_{2,1} + 2\mathfrak{L}_0^3\mathfrak{R}_2\mathfrak{R}_0^3\mathfrak{f}_{0,1} + 20E_6\mathfrak{L}_2^3\mathfrak{R}_2\mathfrak{R}_0^3\mathfrak{f}_{0,1} + 18E_4\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2\mathfrak{R}_0^3\mathfrak{f}_{0,1} + 18E \cdots$$
$$+ 12E_6\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2\mathfrak{R}_0^3\mathfrak{f}_{2,1} - 6E_4\mathfrak{L}_0^2\mathfrak{L}_2\mathfrak{R}_2\mathfrak{R}_0^3\mathfrak{f}_{2,1} + 36E_4^2\mathfrak{L}_2^3\mathfrak{R}_2^2\mathfrak{R}_0^2\mathfrak{f}_{0,1} + 36E_6\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2^2\mathfrak{R} \cdots$$
$$- 12E_4\mathfrak{L}_0^3\mathfrak{R}_2^2\mathfrak{R}_0^2\mathfrak{f}_{2,1} + 24E_4E_6\mathfrak{L}_2^3\mathfrak{R}_2^2\mathfrak{R}_0^2\mathfrak{f}_{2,1} - 36E_6\mathfrak{L}_0^2\mathfrak{L}_2\mathfrak{R}_2^2\mathfrak{R}_0^2\mathfrak{f}_{2,1} - 6E_4\mathfrak{L}_0^3\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{0,1} + 24 \cdots \mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{R} \cdots$$
$$+ 18E_4^2\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{0,1} - 12E_6\mathfrak{L}_0^2\mathfrak{L}_2\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{0,1} - 20E_6\mathfrak{L}_0^3\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{2,1} - 54E_4^3\mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{2,1} + 6 \cdots \mathfrak{L}_2^3\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f} \cdots$$
$$- 24E_4E_6\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{2,1} - 54E_4^2\mathfrak{L}_0^2\mathfrak{L}_2\mathfrak{R}_2^3\mathfrak{R}_0\mathfrak{f}_{2,1} - 4E_6\mathfrak{L}_0^3\mathfrak{R}_2^4\mathfrak{f}_{0,1} - 27E_4^3\mathfrak{L}_2^3\mathfrak{R}_2^4\mathfrak{f}_{0,1} + 32E_6^2\mathfrak{L}_2^3\mathfrak{R}_2^4\mathfrak{f}_{0,1} - 9 \cdots$$
$$- 9E_4^2\mathfrak{L}_0^3\mathfrak{R}_2^4\mathfrak{f}_{2,1} + 81E_4^3\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2^4\mathfrak{f}_{2,1} - 96E_6^2\mathfrak{L}_0\mathfrak{L}_2^2\mathfrak{R}_2^4\mathfrak{f}_{2,1} - 24E_4E_6\mathfrak{L}_0^2\mathfrak{L}_2\mathfrak{R}_2^4\mathfrak{f}_{2,1}\big)^2$$



and many others…





Conclusion

- We discussed the properties of 6d string elliptic genus:
  - A weak Jacobi form of weight 0 and index set by anomaly.
  - Casimir energy; Pole structure expected for 6d BPS spectrum; Numerator in the polynomial ring of Weyl invariant Jacobi forms.

- Using T-duality of LSTs, we studied the observables in LSTs.
  - 6d partition functions for (1,1) and (2,0) LSTs of SU(2), SU(3), SO(4), SO(6) types; Elliptic genera of *fractional* string chains in (2,0) LSTs of SO(8), E_6, E_7, E_8 types (cf. fractional strings in *heterotic* LSTs)
  - For full winding modes, the elliptic genus inevitably captures the extra BPS contributions carrying a tower of transverse SO(4) angular momentum. → How to properly obtain the 6d spectrum?

    We found it in (1,1) and (2,0) LSTs of SO(8) type & heterotic LSTs.