Autumn Symposium on String Theory
13.9.2017@KIAS

Hofstadter, Toda & Calabi-Yau

Yasuyuki Hatsuda
(Rikkyo University)
初田 泰之（立教大学）

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Hofstadter, Toda & Calabi-Yau

(2d electrons, integrable systems & geometry)

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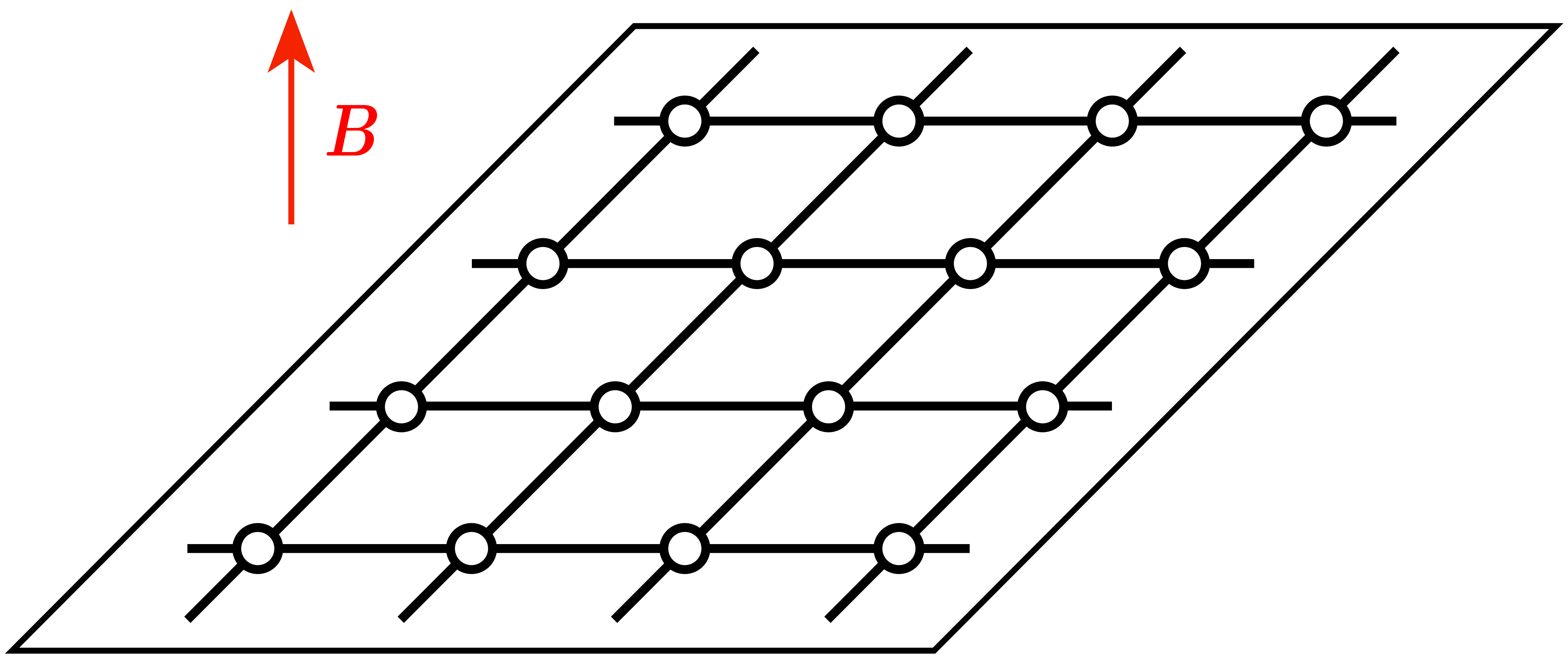
The aim of this talk is to introduce an attempt to relate string theory to relatively a realistic electron system

In 1976, Douglas Hofstadter considered an interesting 2d electron model in a magnetic field



He predicted a novel electron spectrum, which is now known as Hofstadter's butterfly

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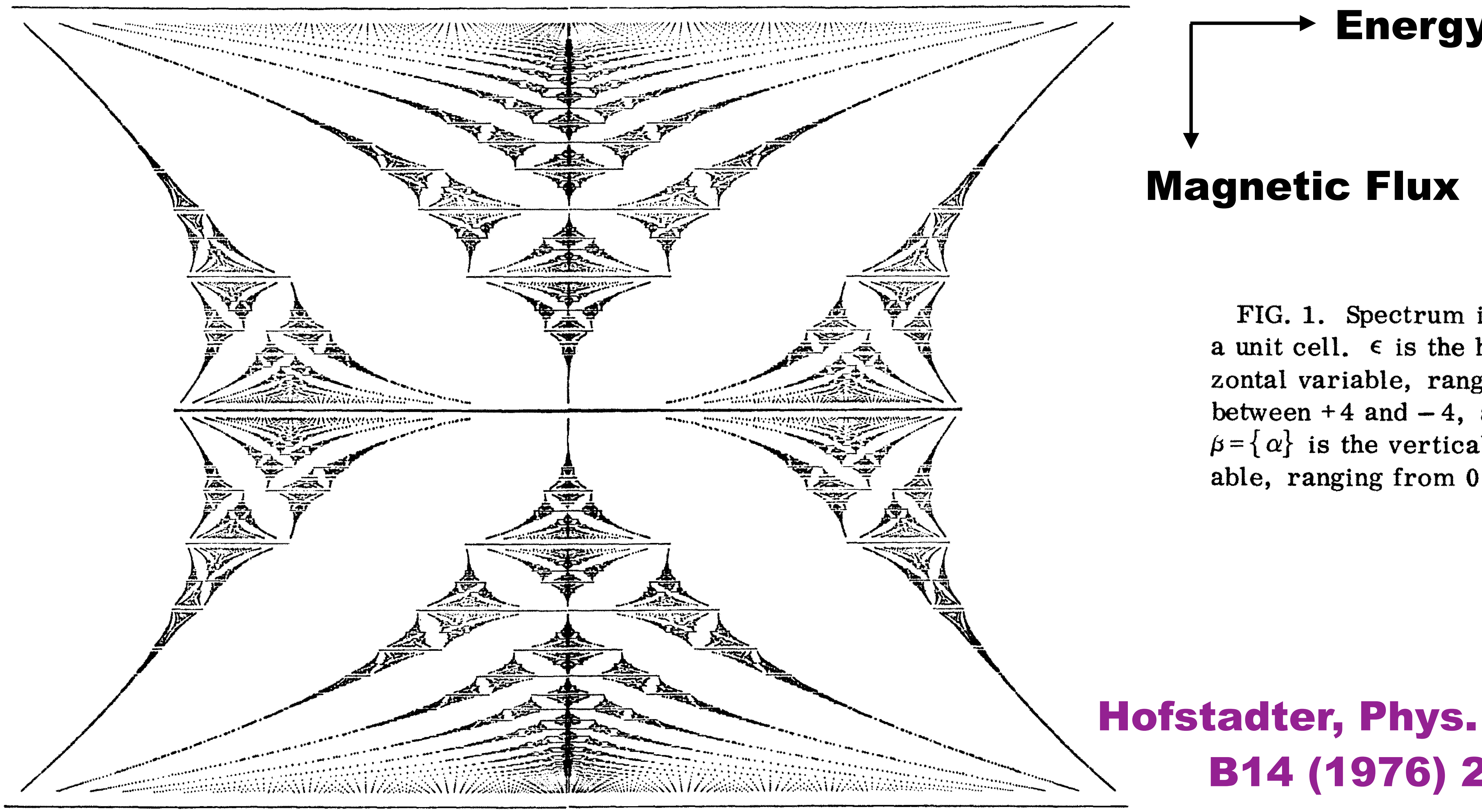

FIG. 1. Spectrum inside a unit cell. ϵ is the horizontal variable, ranging between $+4$ and -4, and $\beta = \{\alpha\}$ is the vertical variable, ranging from 0 to 1.

Hofstadter, Phys. Rev. B14 (1976) 2239.

40 years later, Katsura, Tachikawa and myself found that the completely same figure appears in the context of Calabi-Yau geometry

YH, Katsura & Tachikawa, arXiv:1606.01894
"Hofstadter's butterfly in quantum geometry"

I would like to explain its idea

Consequence

YH-Katsura-Tachikawa

New J.Phys. 18 (2016) 10, 103023

$$\rho(E;q) = \frac{1}{2\pi}\,\mathrm{Im}\left[\frac{\partial t(E;q)}{\partial E}\right]$$

Consequence

YH-Katsura-Tachikawa

New J.Phys. 18 (2016) 10, 103023

Density of states

$$\rho(E;q) = \frac{1}{2\pi}\,\mathrm{Im}\left[\frac{\partial t(E;q)}{\partial E}\right]$$

Magnetic effect

Consequence

YH-Katsura-Tachikawa
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Moduli of CYs

Density of states

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Moduli of CYs

Density of states

$$\rho(E;q) = \frac{1}{2\pi}\,\mathrm{Im}\left[\frac{\partial t(E;q)}{\partial E}\right]$$

Magnetic effect = “Quantum” parameter

Contents

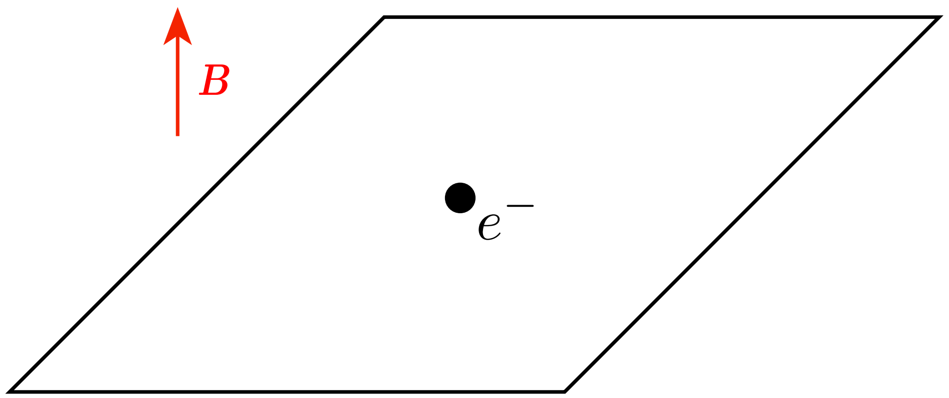
1. Hofstadter Model

2d Electron in Magnetic Field

$$H = \frac{1}{2m}\left(\boldsymbol{p} + e\boldsymbol{A}\right)^2 = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2)$$

Vector potential

$$\boldsymbol{\Pi} := \boldsymbol{p} + e\boldsymbol{A}$$



2d Electron in Magnetic Field

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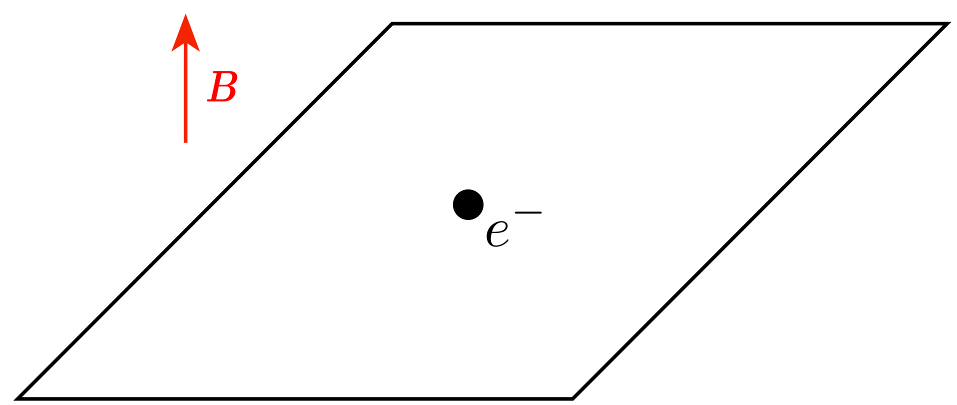


$$\boxed{[\Pi_x, \Pi_y] = \frac{\hbar e}{i}(\partial_x A_y - \partial_y A_x) = -i\hbar e B}$$

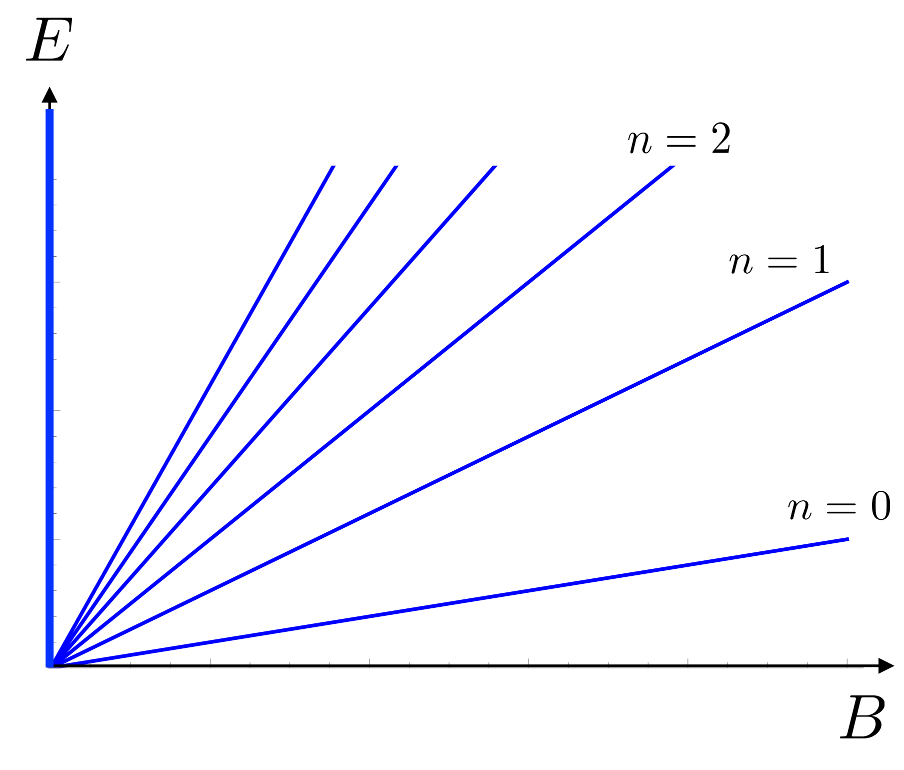
2d Electron in Magnetic Field

- **This Hamiltonian is the same as that for the harmonic oscillator**

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

$$\omega_c = \frac{eB}{m}$$

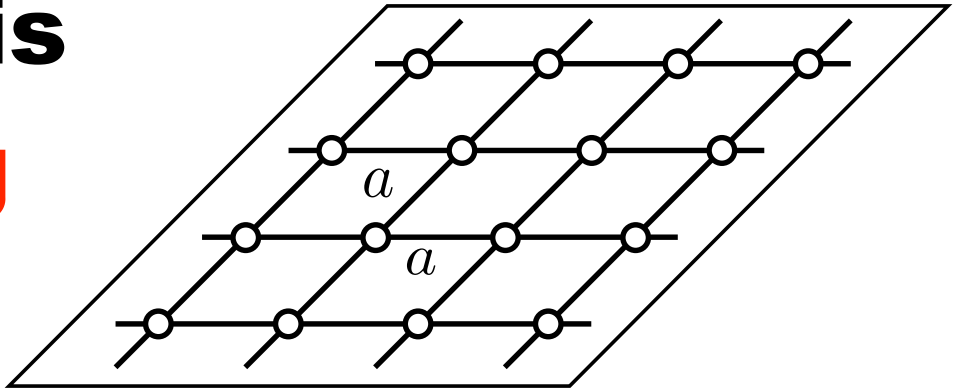
Landau level



- **The spectrum of the 2d electron is quantized by the magnetic effect**

An Electron on 2d Lattice

- In this case, the spectrum is obtained by the **tight-biding approximation**



$$E = 2\cos(k_x a) + 2\cos(k_y a)$$

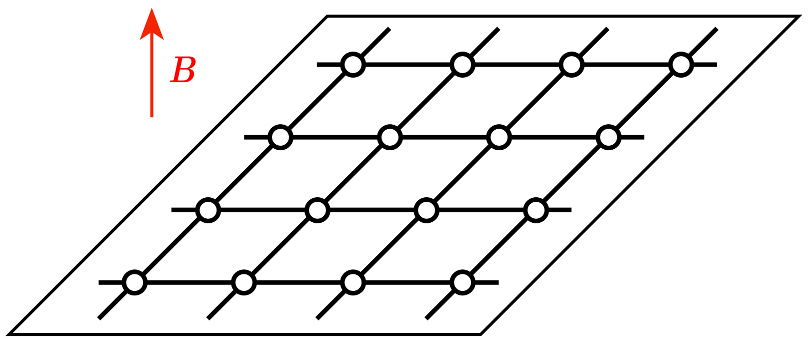
- The allowed range of the energy:

$$\boxed{-4 \leq E \leq 4}$$

Single energy band

The Hofstadter Model

- **An electron on a 2d lattice with a magnetic flux**



Free $E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) \xrightarrow{B} H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2)$

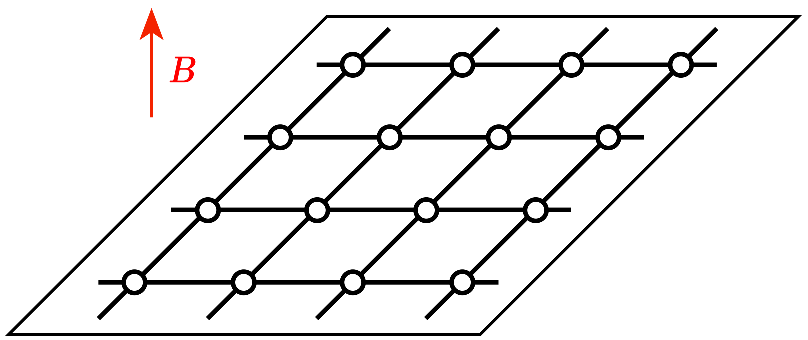
$$\boldsymbol{\Pi} := \boldsymbol{p} + e\boldsymbol{A}$$

Lattice

$$E = e^{ik_x a} + e^{-ik_x a} + e^{ik_y a} + e^{-ik_y a}$$

The Hofstadter Model

- **An electron on a 2d lattice with a magnetic flux**



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Lattice

$$E = e^{ik_x a} + e^{-ik_x a} + e^{ik_y a} + e^{-ik_y a}$$

$$\boxed{H = e^{\frac{ia}{\hbar}\Pi_x} + e^{-\frac{ia}{\hbar}\Pi_x} + e^{\frac{ia}{\hbar}\Pi_y} + e^{-\frac{ia}{\hbar}\Pi_y}}$$

The Hofstadter Model

- **By fixing the Landau gauge $A = (0, Bx, 0)$, the eigenvalue problem finally leads to Harper's equation**

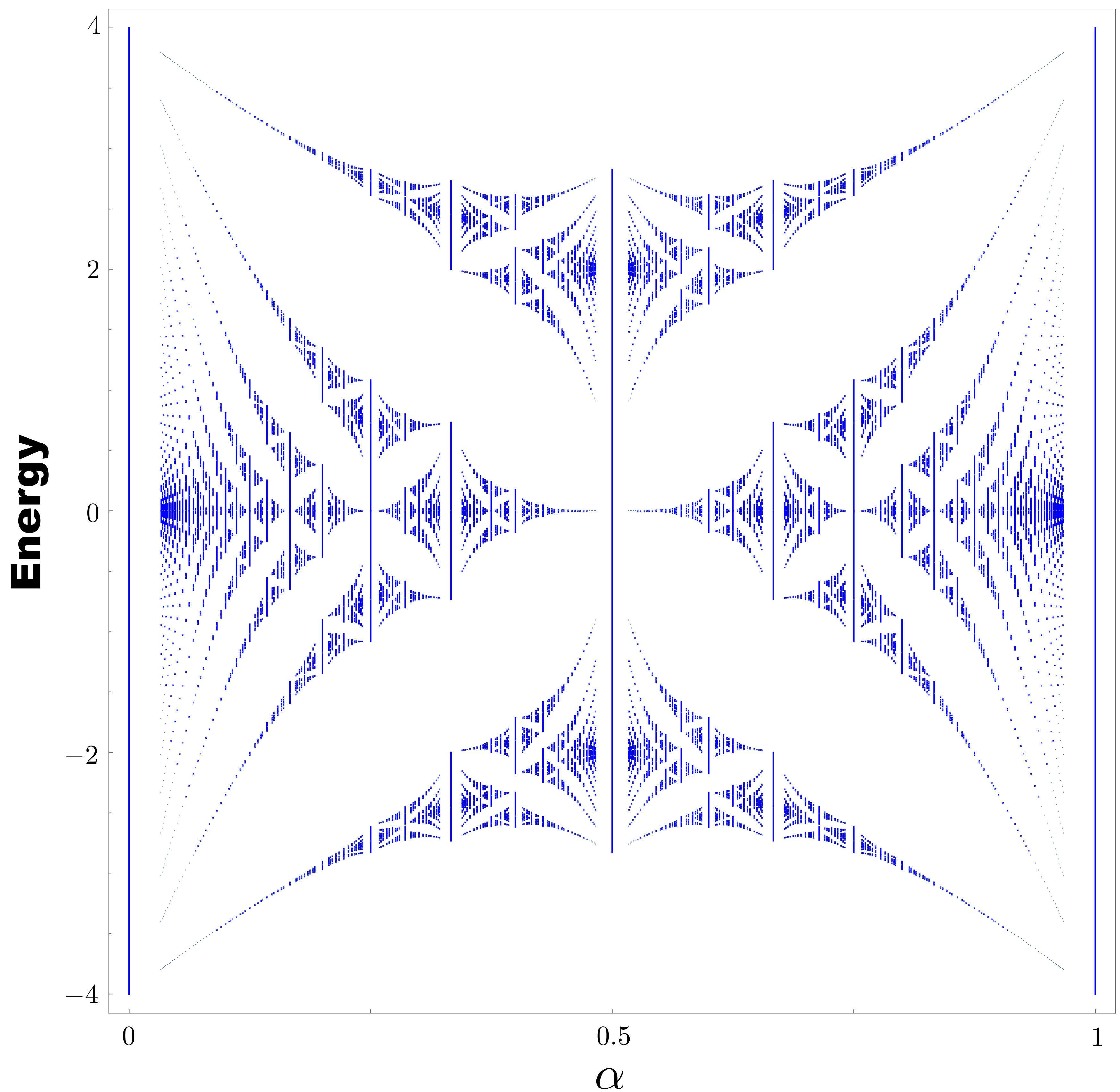
Hofstadter '76

$$\boxed{\psi(n+1) + \psi(n-1) + 2\cos(2\pi n\alpha + \nu)\psi(n) = E\psi(n)}$$

$$\alpha = \frac{a^2 eB}{2\pi\hbar}$$

- **If $\alpha = P/Q$ (rational), the spectrum of this equation gives Q energy bands**

Hofstadter's Butterfly



2. Toda Lattice

The Toda Lattice

- **The Toda lattice is the well-known quantum integrable system** **Toda '67**

$$H_1 = \sum_{n=1}^{N} \left(\frac{p_n^2}{2} + e^{x_n - x_{n+1}} \right), \quad x_{N+1} = x_1$$

- **There are N mutually commuting operators**

$$[H_n, H_m] = 0$$

- **The eigenvalue problem**

$$H_k \Psi(x_1, \ldots, x_N) = E_k \Psi(x_1, \ldots, x_N)$$

Generalized Toda Lattice

- **There is a one-parameter deformation of the Toda lattice** Ruijsenaars '90

$$H_1 = \sum_{n=1}^{N} \left(1 + q^{-1/2} R^2 e^{x_n - x_{n+1}}\right) e^{R p_n}$$

$$q = e^{iR\hbar}$$

Generalized Toda Lattice

- **There is a one-parameter deformation of the Toda lattice** **Ruijsenaars '90**

$$H_1 = \sum_{n=1}^{N} \left(1 + q^{-1/2} R^2 e^{x_n - x_{n+1}}\right) e^{R p_n}$$

$$q = e^{iR\hbar}$$

$$\Big\downarrow \; R \to 0$$

$$H_1 = N + R \sum_{n=1}^{N} p_n$$

$$+ R^2 \sum_{n=1}^{N} \left(\frac{p_n^2}{2} + e^{x_n - x_{n+1}}\right) + \mathcal{O}(R^3)$$

N=2 gToda

- **Let us consider the case of N=2**

$$H = e^{Rp_1} + e^{Rp_2} + R^2(e^{x_1 - x_2 + Rp_1} + e^{x_2 - x_1 + Rp_2})$$

$p_1 + p_2 = 0$ **(Center of mass frame)**

$$p := Rp_1, \quad x := x_1 - x_2 + Rp_1$$

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$p_1 + p_2 = 0$ **(Center of mass frame)**

$$p := Rp_1, \quad x := x_1 - x_2 + Rp_1$$

$$\boxed{H = e^p + e^{-p} + R^2(e^x + e^{-x})}$$

$$[x, p] = iR\hbar$$

N=2 gToda

- **The eigenvalue equation is thus a difference equation**

$$\boxed{\psi(x+iR\hbar)+\psi(x-iR\hbar)+2R^2\cosh x\,\psi(x)=E\psi(x)}$$

- **This equation is very similar to Harper's equation, but their spectra are quite different**

Harper $\rightarrow$ **Continuous (Bands)**

Toda $\rightarrow$ **Discrete**

3. Relation to Calabi-Yau

So far...

- **The Hofsdater model (Harper's equation)**

$$\boxed{\psi(n+1)+\psi(n-1)+2\cos(2\pi n\alpha+\nu)\psi(n)=E\psi(n)}$$

- **The generalized Toda lattice**

$$\boxed{\psi(x+i\hbar)+\psi(x-i\hbar)+2\cosh x\,\psi(x)=E\psi(x)}$$

- **The situation is similar to the difference between the Mathieu (cos) and the modified Mathieu (cosh) potentials**

To Calabi-Yau

$$E = e^{ik_x a} + e^{-ik_x a} + e^{ik_y a} + e^{-ik_y a}$$

$$E = e^{p} + e^{-p} + e^{x} + e^{-x}$$

$$\downarrow$$

$$X + X^{-1} + Y + Y^{-1} = E$$

This equation defines a genus one Riemann surface

To Calabi-Yau

- **The complex 3d space**

$$VW = X + X^{-1} + Y + Y^{-1} - E$$

describes a Calabi-Yau manifold

- **The Riemann surface has enough information to describe this CY manifold**
- **In this way, one can see a connection to the CY geometry**

Mirror Symmetry

- **The Calabi-Yau geometry has a remarkable hidden duality, called mirror symmetry**

Calabi-Yau A $\equiv$ **Calabi-Yau B**

Moduli t **Moduli** E

"Mirror map"

$$\mathrm{A} : \mathrm{Local}\ \mathbb{CP}^1 \times \mathbb{CP}^1$$

$$\mathrm{B} : VW = X + X^{-1} + Y + Y^{-1} - E$$

Spectral Solution 1

- **The spectral problem of the N=2 generalized Toda lattice is solved by the exact version of the quantization condition in terms of string theory**

Grassi, YH & Marino '14; Wang, Zhang & Huang '15

Conjecture

$$\frac{\partial}{\partial t} F_{\rm NS}^{\mathbb{P}^1\times\mathbb{P}^1}(t;\hbar) + \frac{\partial}{\partial \widetilde{t}} F_{\rm NS}^{\mathbb{P}^1\times\mathbb{P}^1}(\widetilde{t};\widetilde{\hbar}) = 2\pi\left(n+\frac{1}{2}\right)$$

$$t = t(E;q) \qquad q = e^{i\hbar}, \quad \widetilde{t} = \frac{2\pi t}{\hbar}, \quad \widetilde{\hbar} = \frac{4\pi^2}{\hbar}$$

[Sciarappa's talk]

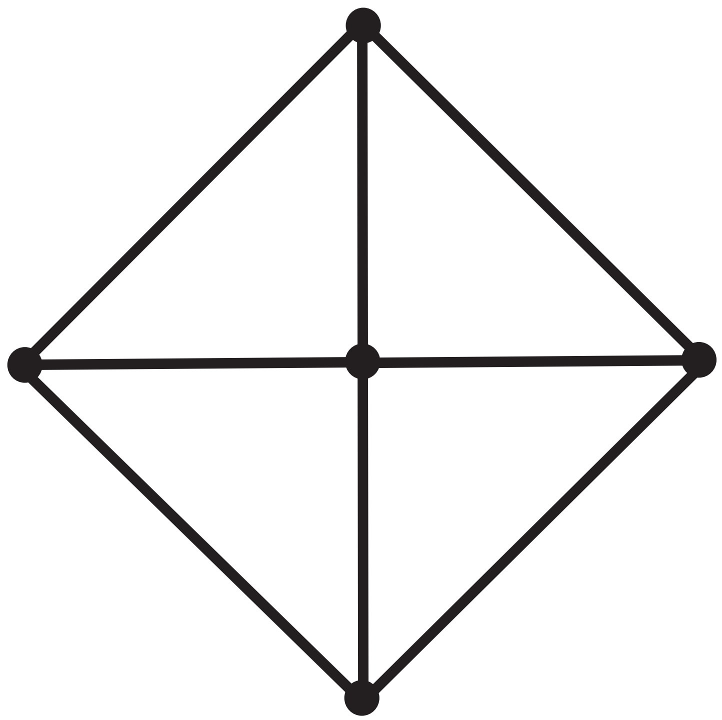
Spectral Solution 2

- **On the other hand, the spectrum of the Hofstadter problem is encoded in the quantum deformed mirror map**

YH, Katsura & Tachikawa '16

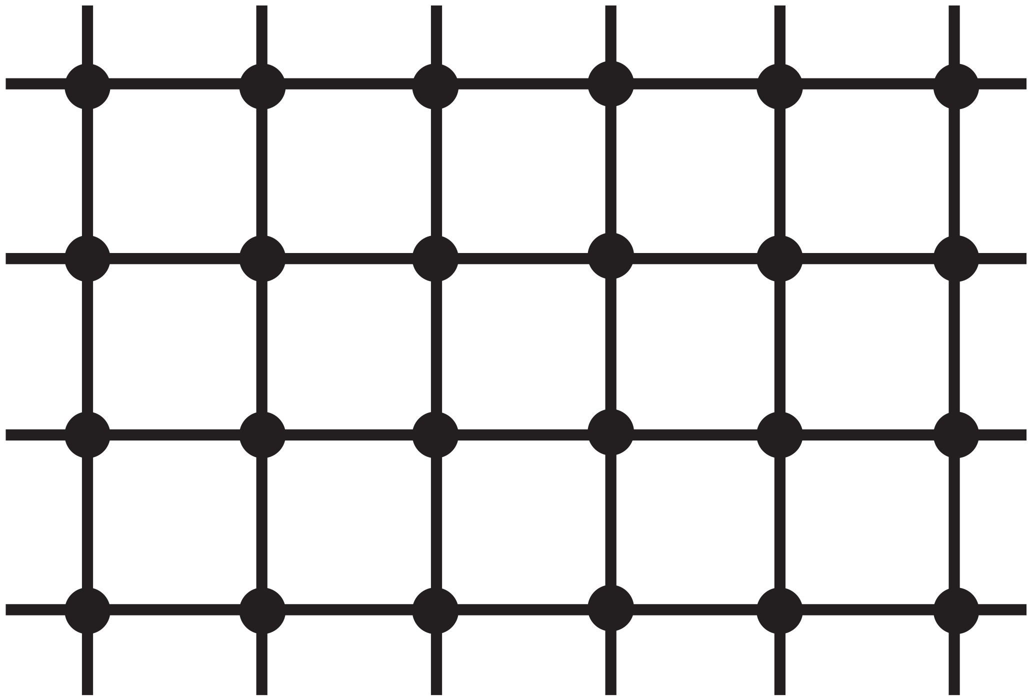
$$\boxed{\rho(E) = \frac{1}{2\pi} \mathrm{Im} \left[\frac{\partial t(E;q)}{\partial E} \right], \quad q = e^{2\pi i \alpha}}$$

Schematic Similarity

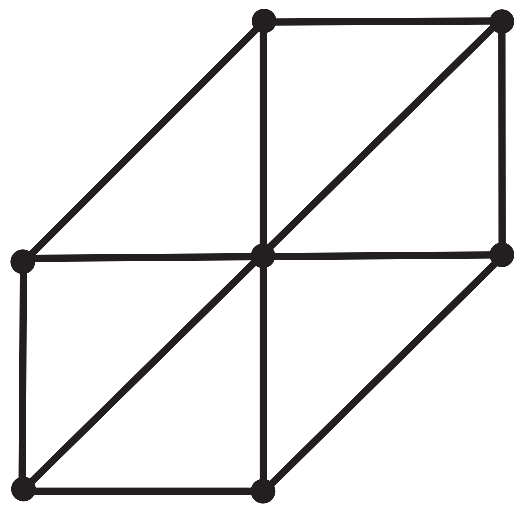


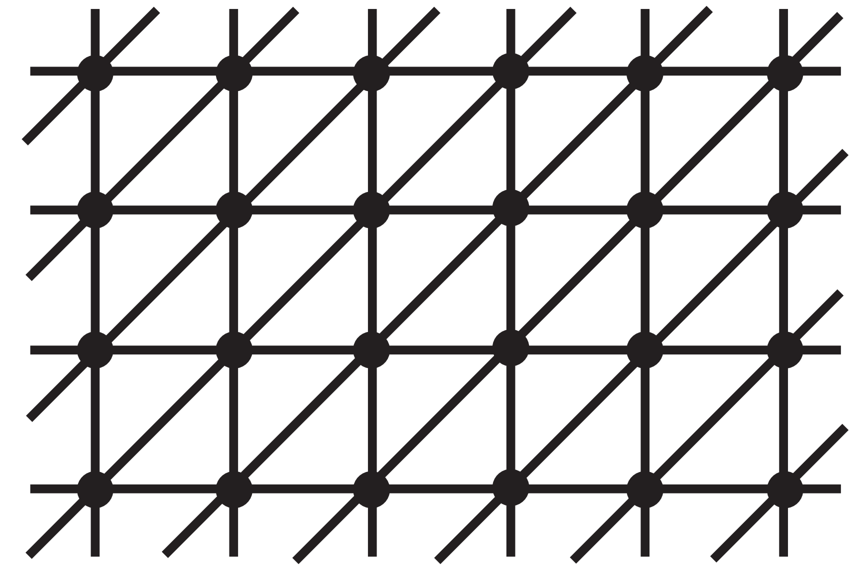
Local $CP^1 \times CP^1$

Toric diagram



Square lattice



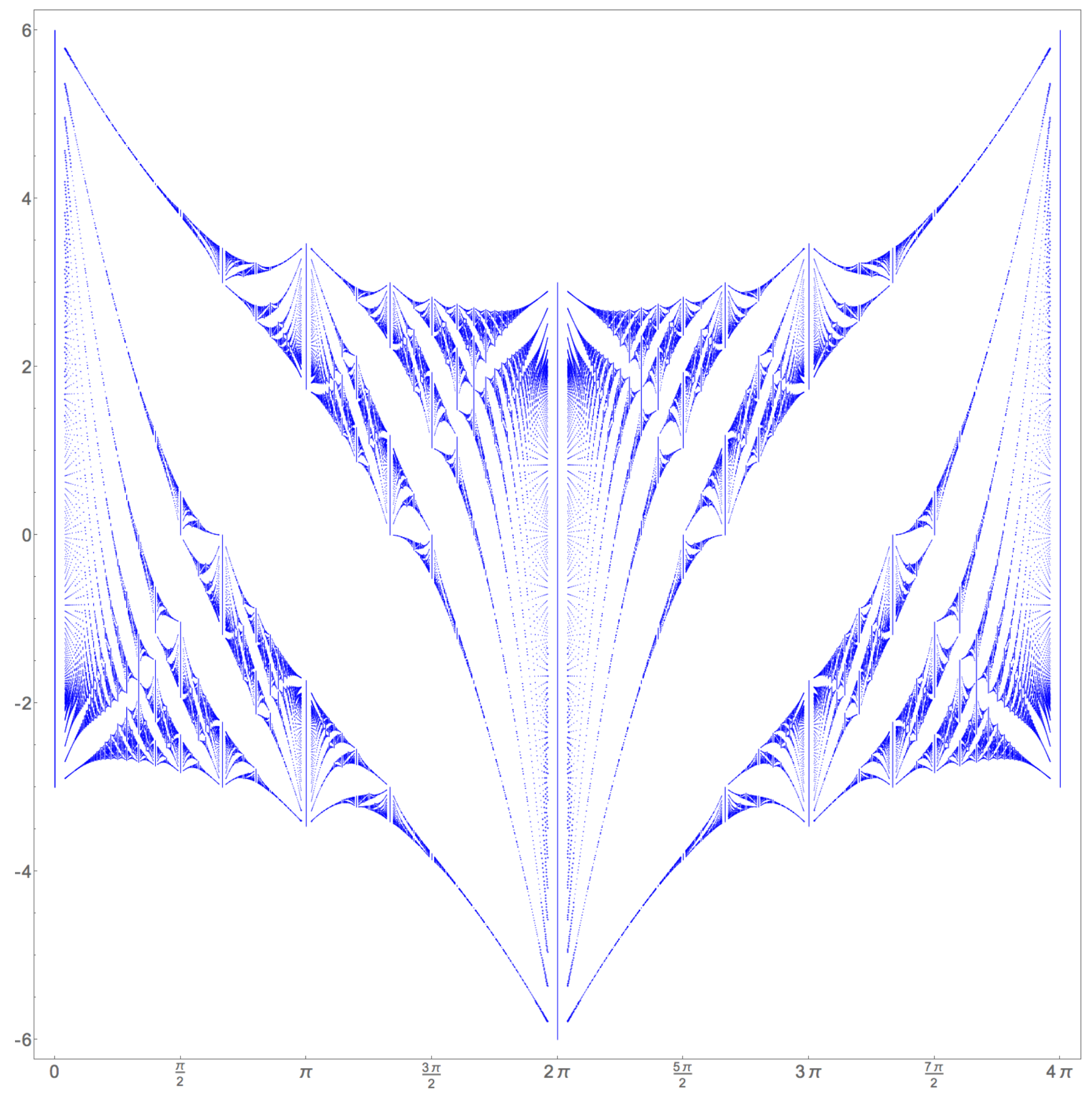


Local B_3

Triangular lattice

YH, Sugimoto & Xu '17

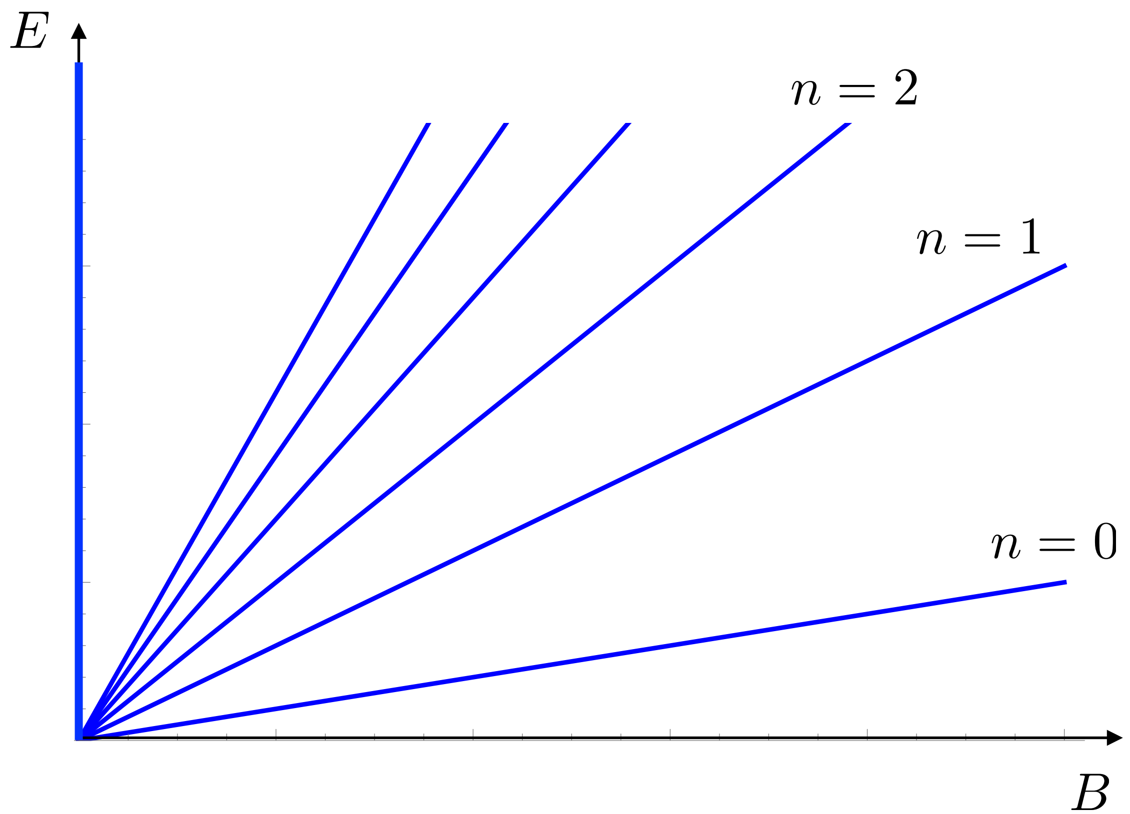
Claro & Wannier '79

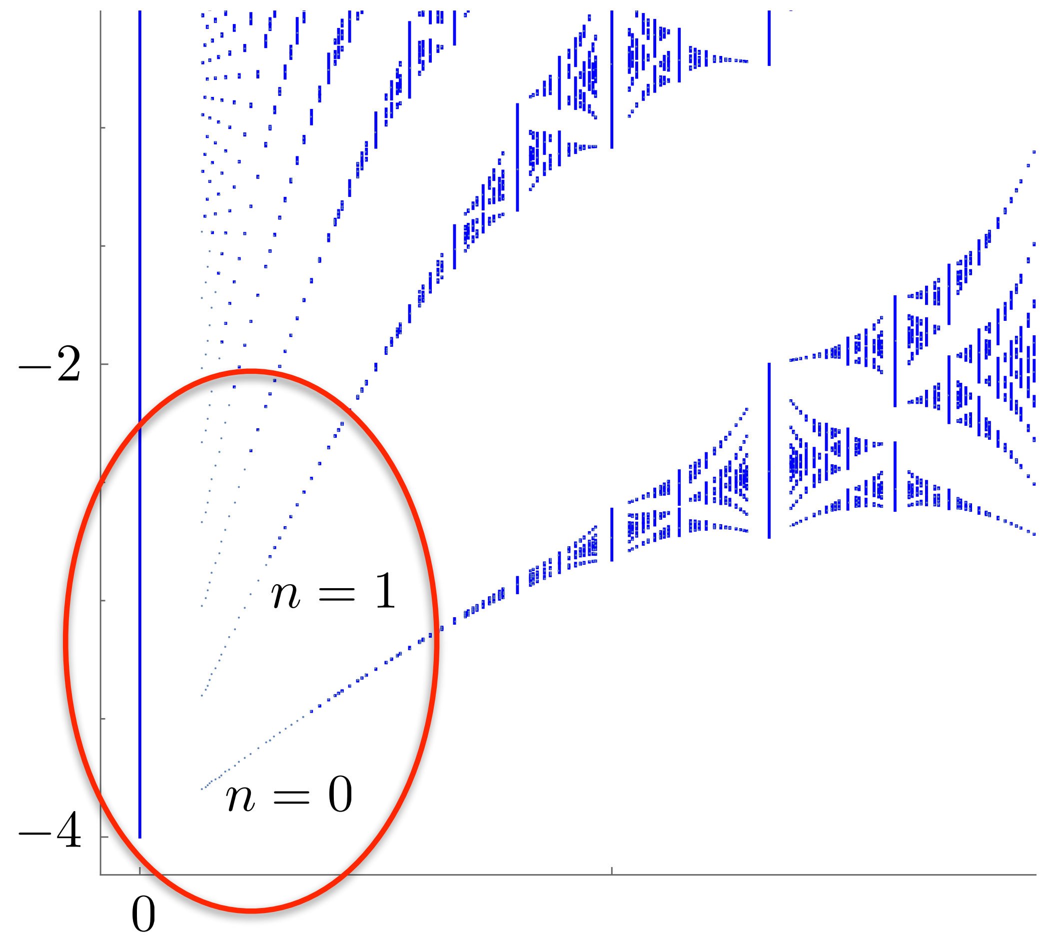


4. Weak Magnetic Expansion

@ Weak Coupling

- In the weak magnetic regime, the band width of the Hofstadter model is extremely narrow, and one can see Landau level splitting





@ Weak Coupling

- **This fact is easily seen by the Hamiltonian analysis**

$$H = e^{\frac{ia}{\hbar}\Pi_x} + e^{-\frac{ia}{\hbar}\Pi_x} + e^{\frac{ia}{\hbar}\Pi_y} + e^{-\frac{ia}{\hbar}\Pi_y}$$

$$= 4 - \frac{a^2}{\hbar^2}(\Pi_x^2 + \Pi_y^2)$$

$$+ \frac{a^4}{12\hbar^4}(\Pi_x^4 + \Pi_y^4) + \cdots$$

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$$E_n = 4 - (2n+1)\phi + \frac{2n^2+2n+1}{8}\phi^2 + \mathcal{O}(\phi^3)$$

$$(\phi = 2\pi\alpha)$$

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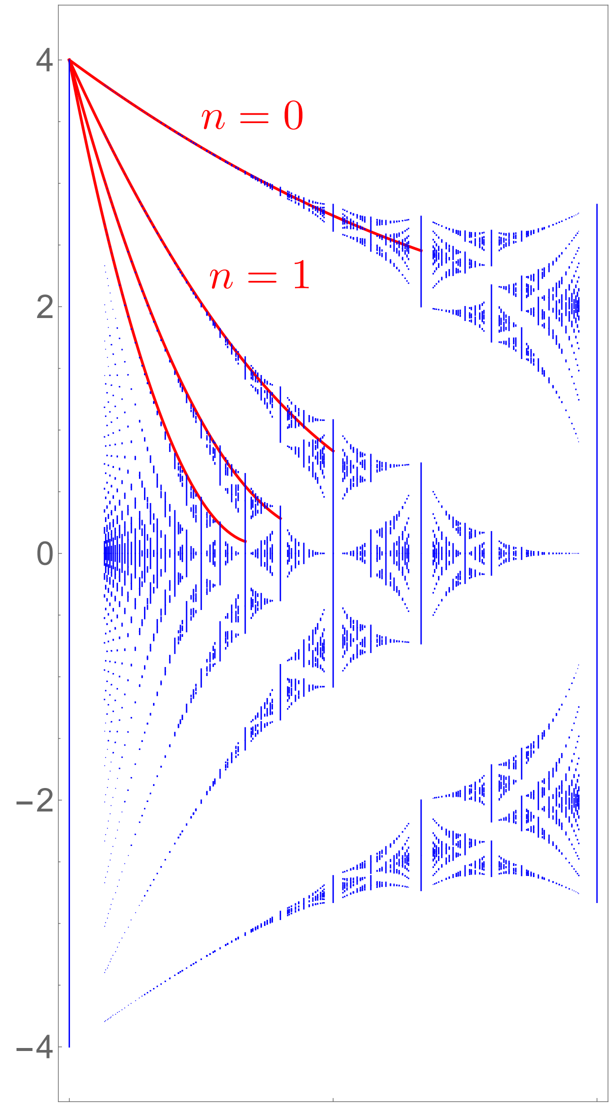

@ Weak Coupling

- **There is a systematic way to compute the weak magnetic expansion**

MATHEMATICA package: Sulejmanpasic & Ünsal '16

Extension: Gu & Sulejmanpasic '17

$$E_0 = 4 - \phi + \frac{\phi^2}{8} - \frac{\phi^3}{192} + \frac{\phi^4}{768} + \frac{67\phi^5}{245760} + \frac{653\phi^6}{5898240}$$

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$$+ \cdots + 6.79 \times 10^{39}\phi^{99} + 4.59 \times 10^{40}\phi^{100} + \mathcal{O}(\phi^{101})$$

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$$+ \cdots + 6.79 \times 10^{39}\phi^{99} + 4.59 \times 10^{40}\phi^{100} + \mathcal{O}(\phi^{101})$$

- **Obviously, this expansion looks divergent**
- **One needs a resummation method**

Borel Sum

- **The standard way to resum a factorially divergent series is the Borel sum**
- **Let us review it briefly**
- **Consider the following formal divergent series**

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \qquad f_n \sim n!$$

- **Borel transform**

$$\mathcal{B}f(\zeta) := \sum_{n=0}^{\infty} \frac{f_n}{n!} \zeta^n$$

Convergent series!

- **Borel sum**

$$\mathcal{S}f(z) = \frac{1}{z} \int_0^{\infty} d\zeta \, e^{-\zeta/z} \mathcal{B}f(\zeta)$$

- **The Borel sum gives a meaning to a formal divergent series**
- **One has to be careful about singularities of the Borel transform**

Borel(-Pade) Singularities

$$\mathcal{B}E_0(\zeta) = \sum_{n=0}^{100} \frac{E_0^{(n)}}{n!}\zeta^n$$

$$\approx \frac{P_{50}(\zeta)}{Q_{50}(\zeta)}$$

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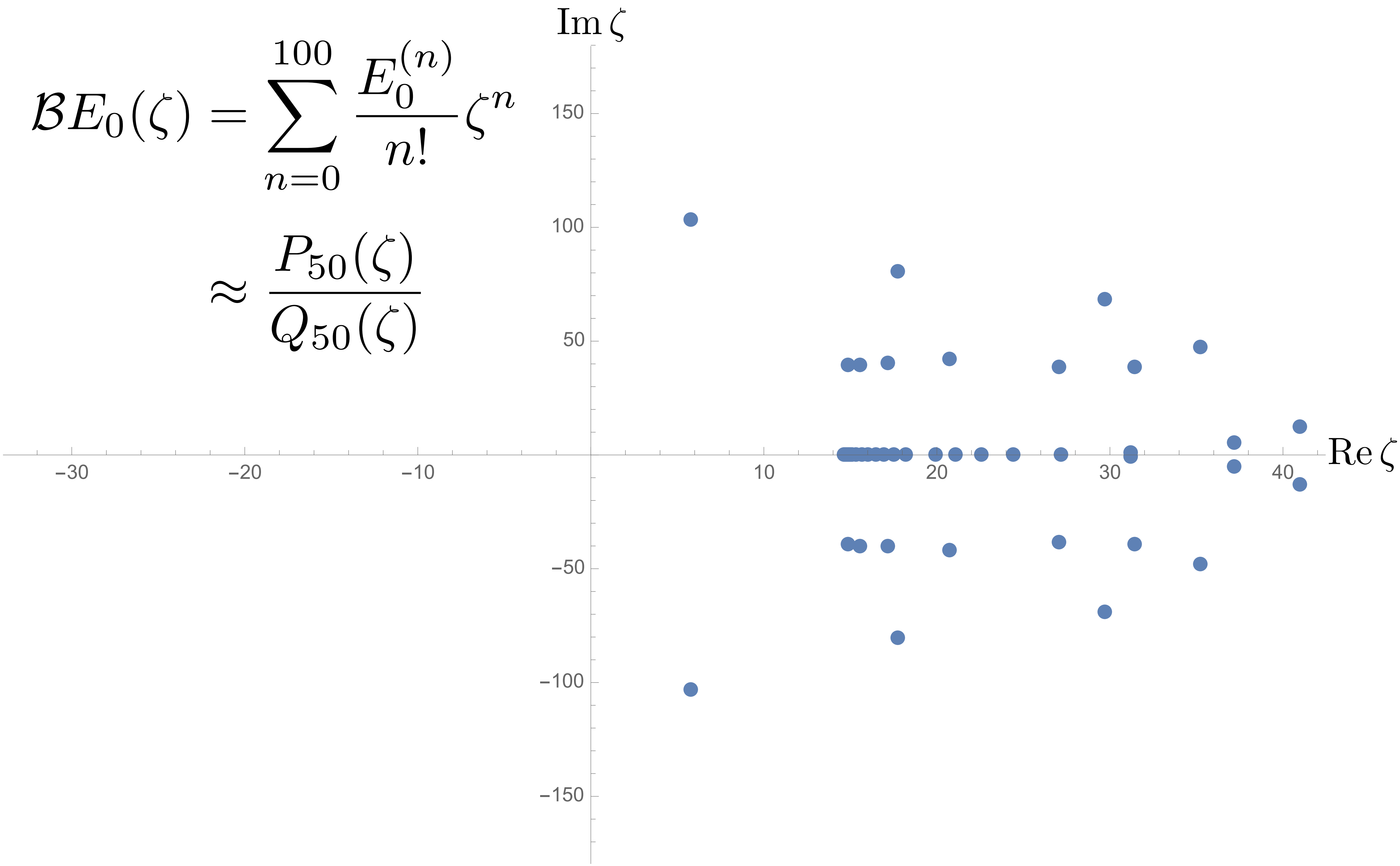

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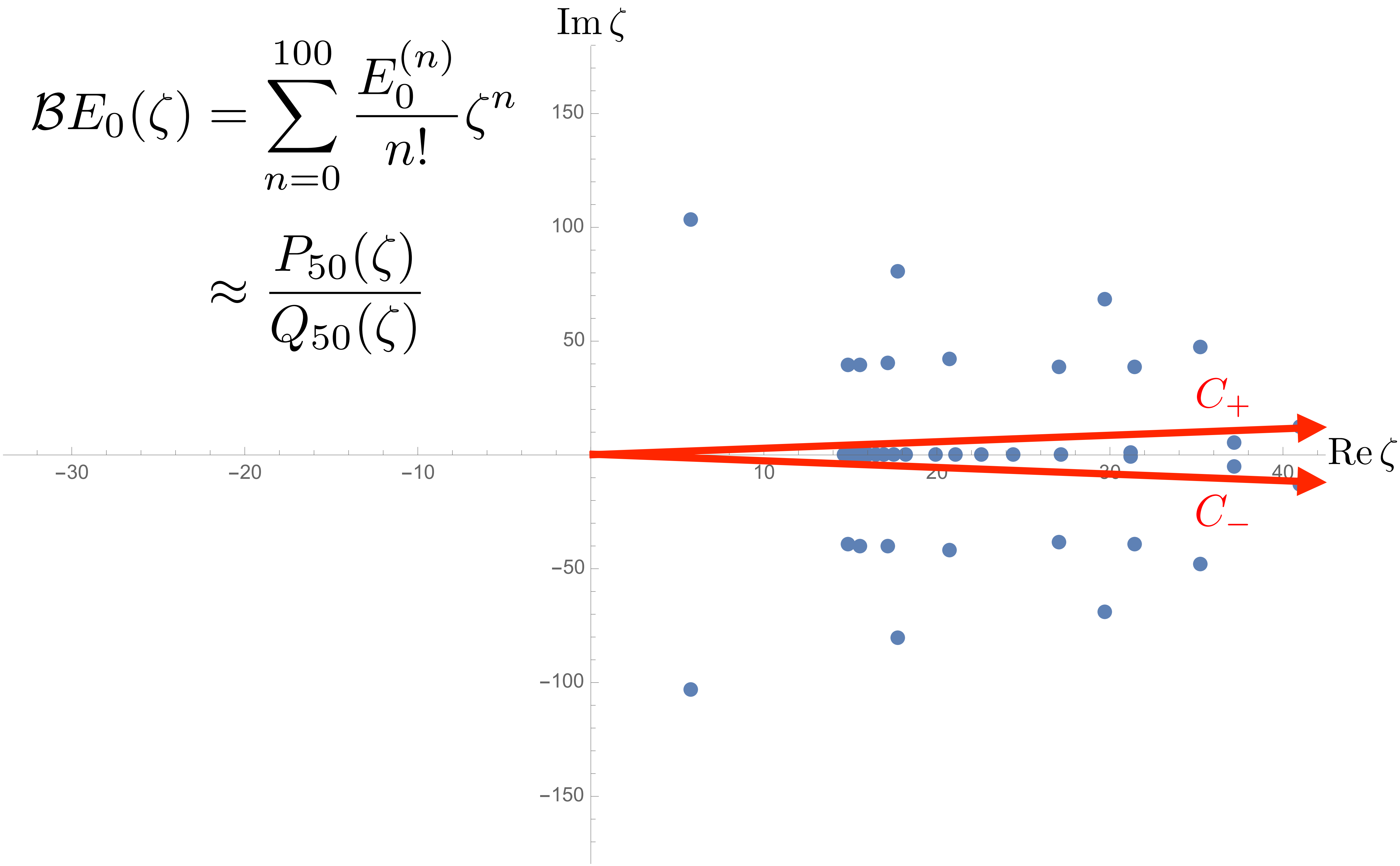
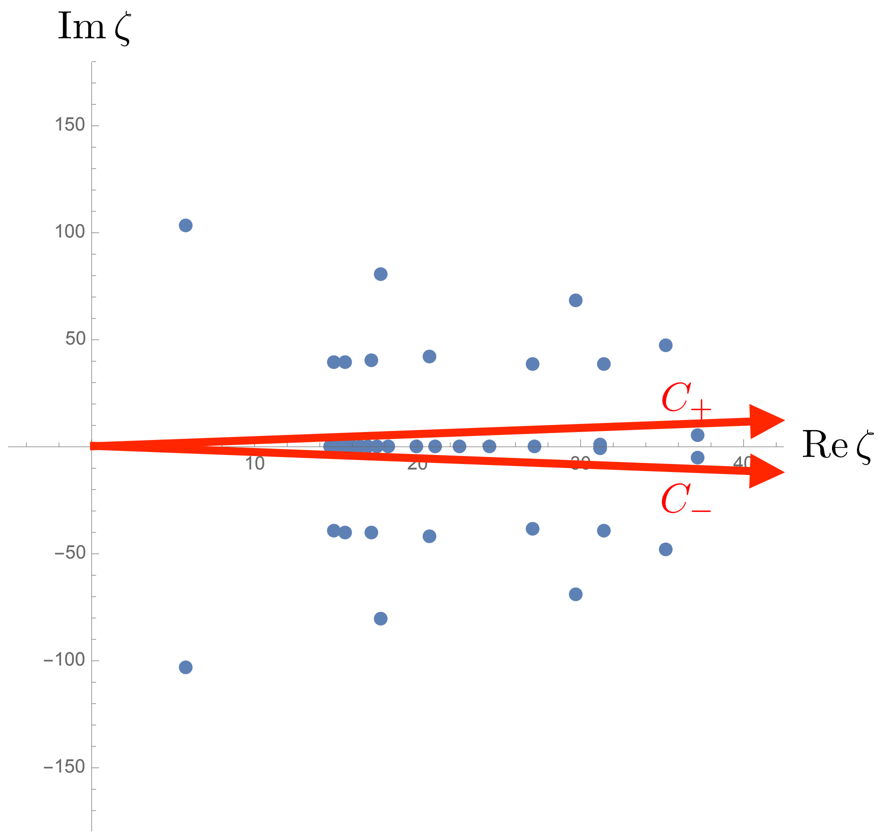


Borel(-Pade) Resum

$$\mathcal{S}_{\pm} E_0 := \frac{1}{\phi} \int_{C_{\pm}} d\zeta \, e^{-\zeta/\phi} \mathcal{B} E_0(\zeta)$$



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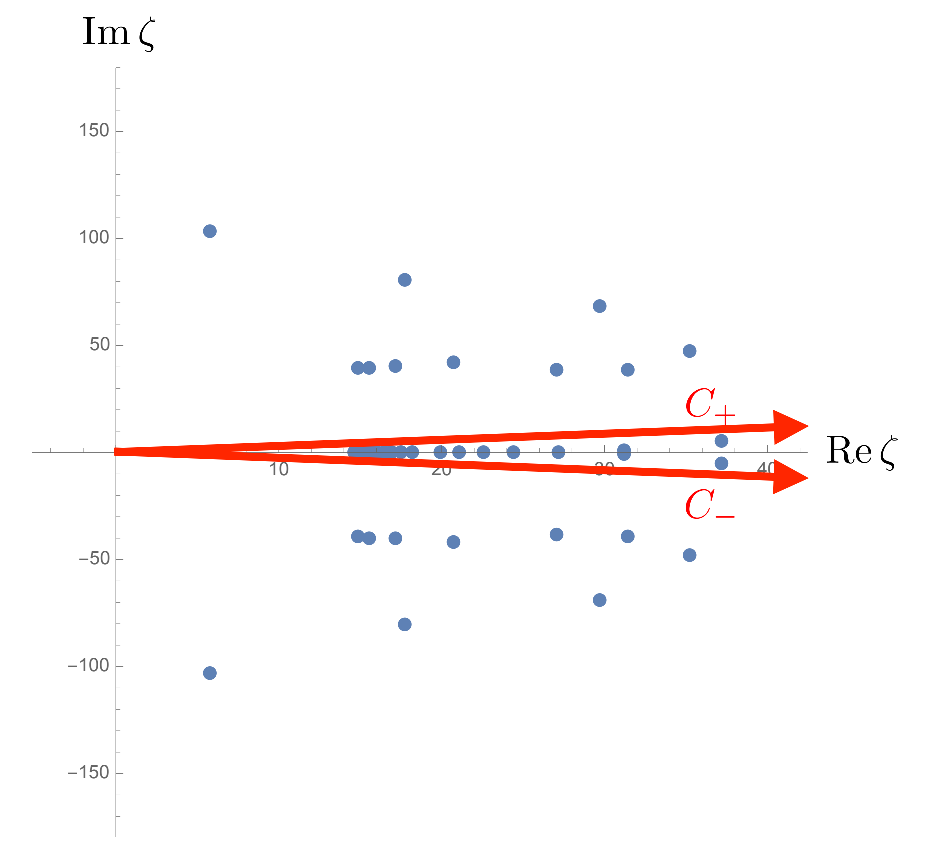

$$\phi = \frac{2\pi}{5} \quad \left(\alpha = \frac{1}{5}\right)$$

$$\mathcal{S}_{\pm} E_0 = 2.935649214 \pm 0.000378867 i$$

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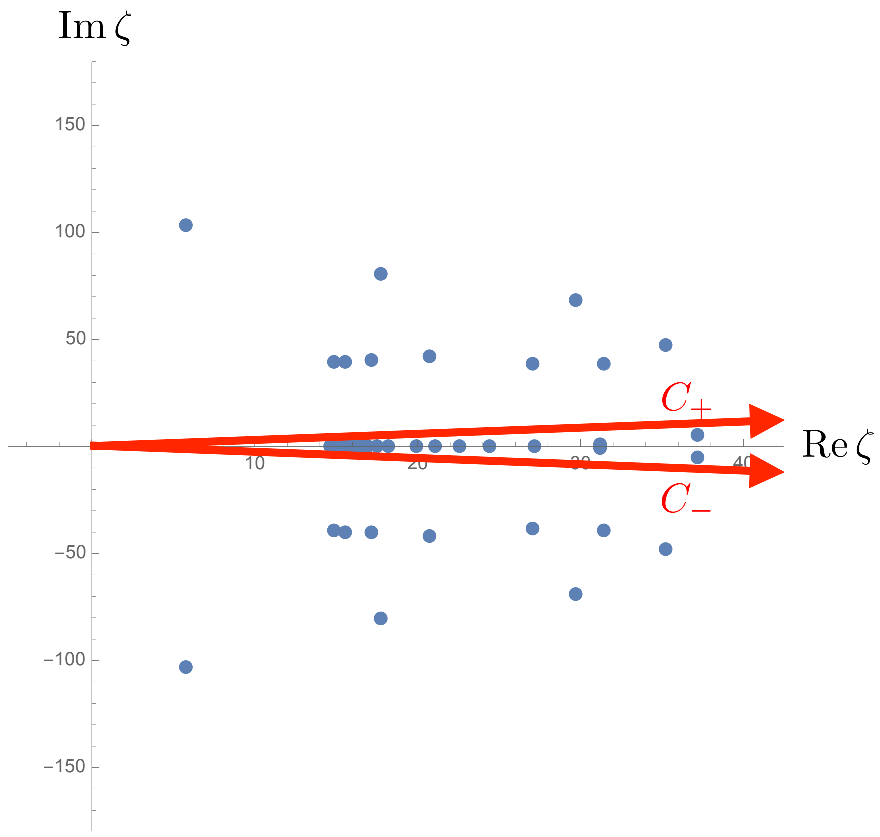


$$\phi = \frac{2\pi}{5} \quad \left(\alpha = \frac{1}{5}\right)$$

$$\mathcal{S}_{\pm} E_0 = 2.935649214 \pm 0.000378867 i$$

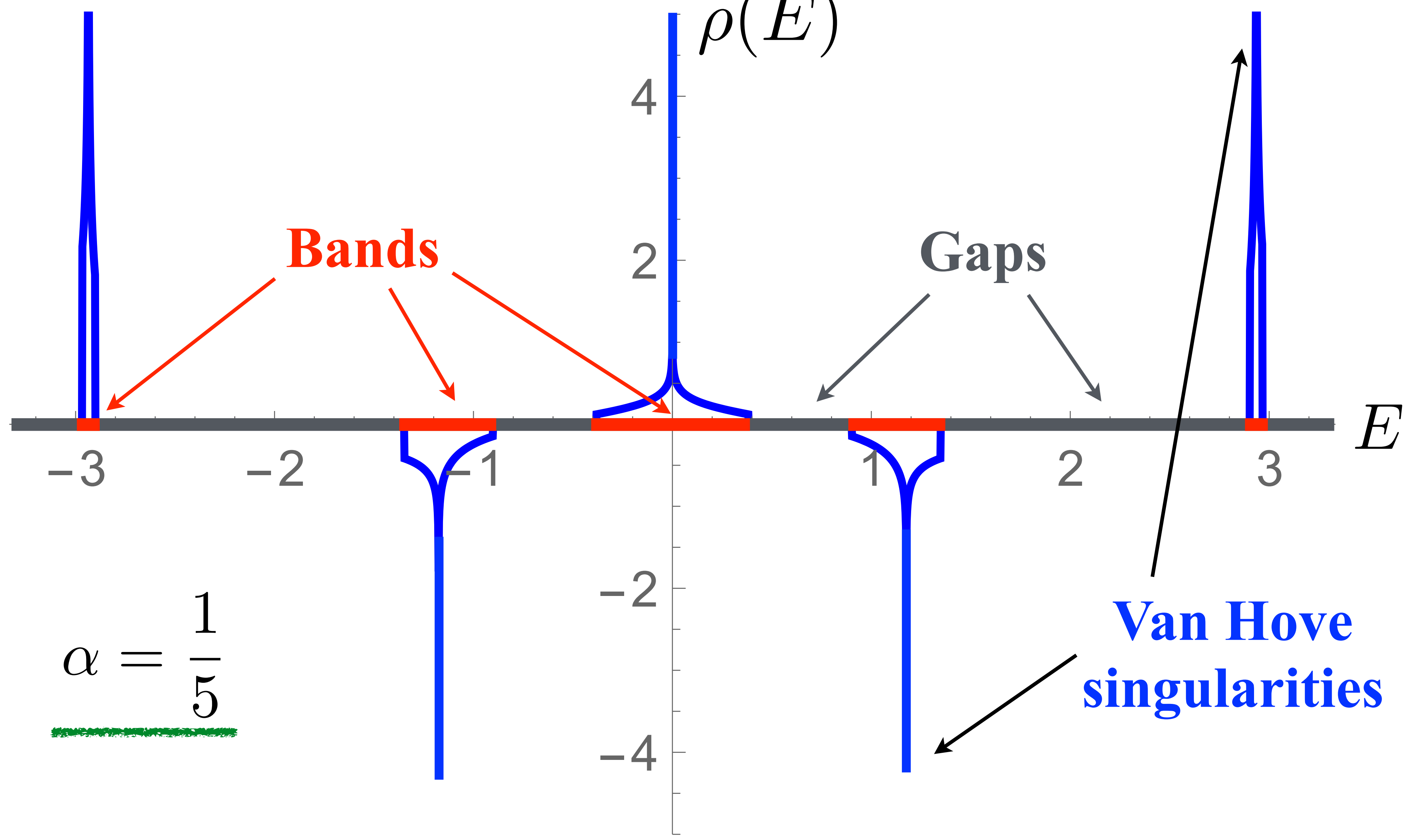
This ambiguity should be canceled by “nonperturbative” corrections

What does this value mean?

Observation

YH, in progress

- **The Borel resummed value is very close to the energy at the Van Hove singularity**



Observation

- **For $\alpha = 1/5$, the positions of the Van Hove singularities are analytically determined by**

$$E\left(E^4 - 10E^2 + \frac{35 - 5\sqrt{5}}{2}\right) = 0$$

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$$E_{\mathrm{VHS}} = \pm 2.935648819,\ \pm 1.175570505,\ 0$$

Observation

- **For $\alpha = 1/5$, the positions of the Van Hove singularities are analytically determined by**

$$E\left(E^4 - 10E^2 + \frac{35 - 5\sqrt{5}}{2}\right) = 0$$

$$E_{\mathrm{VHS}} = \pm\underline{2.935648}819,\ \pm1.175570505,\ 0$$

$$\mathcal{S}_{\pm}E_0 = \underline{2.935649}214 \pm 0.000378867i$$

Observation

- **The coincidence is probably not accidental**

$\alpha = 1/10$

$\alpha = 1/15$

Observation

- **The coincidence is probably not accidental**

$\alpha = 1/10$

$$E^{\max}_{\mathrm{VHS}} = \mathbf{3.4199769502011856}6$$

$$\mathcal{S}_{\pm}E_0 = \mathbf{3.4199769502011857}6 \pm 4.0 \times 10^{-9} i$$

$\alpha = 1/15$

Observation

- **The coincidence is probably not accidental**

$\alpha = 1/10$

$$E_{\rm VHS}^{\rm max} = \mathbf{3.4199769502011856}6$$

$$\mathcal{S}_{\pm} E_0 = \mathbf{3.4199769502011857}6 \pm 4.0 \times 10^{-9} i$$

$\alpha = 1/15$

$$E_{\rm VHS}^{\rm max} = \mathbf{3.602714983890327032980205}$$

$$\mathcal{S}_{\pm} E_0 = \mathbf{3.60271498389032703298020}7 \pm 3.6 \times 10^{-14} i$$

Summary

- **The Hofstadter model (2d electrons) and the generalized Toda lattice (integrable system) has a nontrivial relation to the Calabi-Yau geometry**
- **The weak magnetic expansion in the Hofstadter model is not Borel summable**
- **The resummation seems to reproduce the value at the Van Hove singularity**

Open Questions

- **The full weak magnetic expansion must be a transseries expansion**
- **P/NP (Dunne-Ünsal) relations?**
- **If α is irrational, the spectrum is much more involved**
- **What does the Borel resum for the irrational case mean?**

Thank you!