Quantum Entanglement and Measurement

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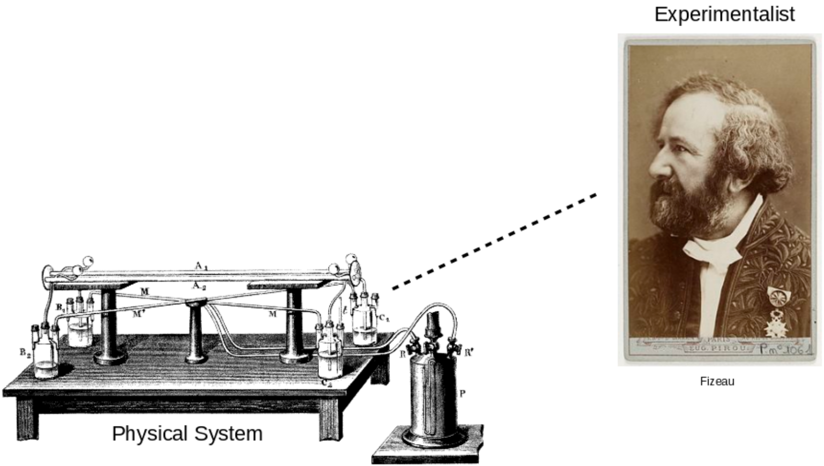
2nd Workhop on Quantum Information and Thermodynamics

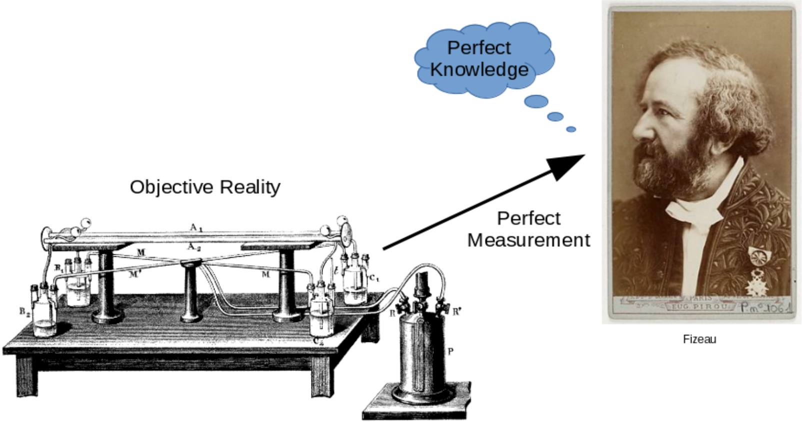
Korea Institute for Advanced Study
Phoenix Island, Jeju
November 2016

Outline

Classical Physics



Perfect Measurement in Classical Physics



Classical objective reality

Objective reality: Objects are in a definite state.

- The bit c *is* in the state $c = 0$ or $c = 1$.
- The particle *is* is in the state $(q(t), p(t))$ at time t

Ideal measurement:
Read off all information without disturbing the experiment.

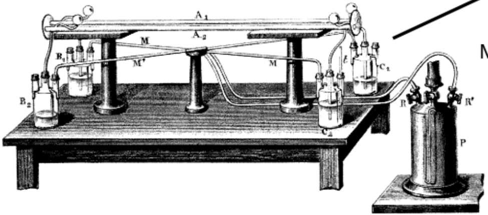
Deterministic dynamics:
Principle of least action, leading to equations of motion

$$\dot{p}(t) = -\partial_q H\,, \qquad \dot{q}(t) = \partial_p H$$

Realistic Measurement in Classical Physics

Partial Knowledge

Objective Reality



Realistic Classical Measurement

Fizeau

Needs classical probability theory

Subjective lack of knowledge: Classical probability theory

Incomplete knowledge caused by imperfect measurement:

- The bit c *is* in the state $c = 0$ or $c = 1$, but *we* don't know exactly in which one.
- The particle *is* is in the state $(q(t), p(t))$ at time t, but *we* don't know exactly where.

Classical probability theory:

- Think of an ensemble of identical systems
- Imagine a repeated uncorrelated measurement

Relative frequencies $\Rightarrow$ $\lim_{N\to\infty}$ $\Rightarrow$ **Probabilities**

Subjective lack of knowledge: Classical probability theory

Concept of probabilities

- The bit c *is* in the state $c = 0$ or $c = 1$, with probability

$$p(0), p(1) \text{ normalized by } p(0) + p(1) = 1$$

- The particle *is* in the state $(q(t), p(t))$ at time t with probability

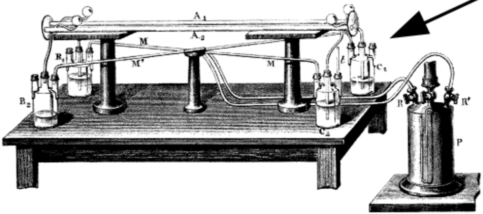
$$\rho(p, q, t) \text{ with } \int \mathrm{d}^3 q \int \mathrm{d}^3 p \, \rho(q, p, t) = 1$$

$\Rightarrow$ **Deterministic equations of motion for the probabilities**

Quantum Physics

What shall I ask?

What should I answer ?



No objective reality any more.

Quantum states

Classical mechanics:

- Of all possible configurations Nature selects the one with the minimal action $\delta S = 0$.

Quantum physics:

- All configurations "exist in parallel".
- Each configuration is weighted by a complex number $e^{\frac{i}{\hbar}S}$
- The probability of a measurement result is proportional to the squared absolute value of the sum of all weights of those configutations which are compatible with the result.

Quantum states

The catalog of all phases at a given time with respect to a measurement basis (e.g. position x) is summarized in a wave function $\psi(x)$, or equally, as a vector $|\psi\rangle$ in Hilbert space.

- Normalization $\int |\psi(x)|^2 \mathrm{d}x = |\psi\rangle\langle\psi| = 1$
- Deterministic evolution equation $i\hbar\partial_t|\psi\rangle = \mathbf{H}|\psi\rangle$

Two aspects:

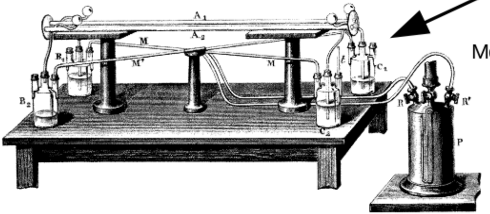
Probabilities represent a **subjective** uncertainty about facts.

Quantum amplitudes represent an **objective** uncertainty: Reality is only produced in the interaction / measurement.

Perfect Measurement in Quantum Physics

Perfect Knowledge
of |ψ>

Quantum State



Many ideal
Quantum
Measurements

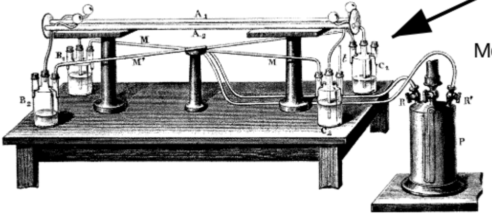
Fizeau

Needs textbook
Quantum mechanics

Realisitc Measurement in Quantum Physics

Partial knowledge:
Density Matrix

Quantum State



Realistic
Quantum
Measurements

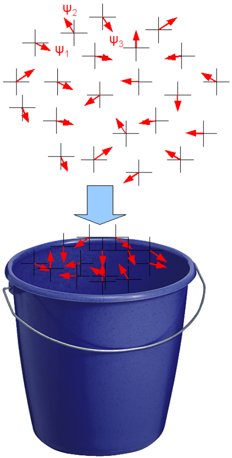
Fizeau

Needs Quantum
Information Theory

Quantum Ensembles

Probabilistic ensemble of quantum states:

$|\psi\rangle$ occurs with probability $p(\psi)$.



$$
\begin{aligned}
\langle M\rangle &= \int_\psi p(\psi)\langle\psi|M|\psi\rangle \\
&= \int_\psi p(\psi)\mathrm{Tr}\big[M|\psi\rangle\langle\psi|\big] \\
&= \mathrm{Tr}\Big[M\underbrace{\int_\psi p(\psi)|\psi\rangle\langle\psi|}_{=:\rho}\Big] = \mathrm{Tr}[M\rho]
\end{aligned}
$$

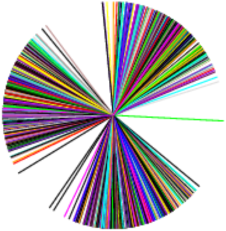
All information is encoded in the **density matrix** ρ.

Quantum Statistical Ensembles

Different ensembles may correspond to the same density matrix.
Example: $\rho = \int \mathrm{d}\alpha\, p(\alpha) |\psi_\alpha\rangle\langle\psi_\alpha|\,, \quad |\psi_\alpha\rangle = \cos\alpha |\uparrow\rangle + \sin\alpha |\downarrow\rangle$

p(α) = 1/2/π = const

p(α) ~ sin⁴(3α/2)



p(α) ~ δ(α)+δ(α−π/2)

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ in all cases.}$$

Quantum Statistical Ensembles

The density matrix ρ characterizes **equivalence classes** of statistical ensembles of quantum states that cannot be distinguished by measurements.

$$\rho = \rho^\dagger \,, \qquad \mathrm{Tr}\rho = 1$$

Spectral decomposition:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \rho_i$$

$\Uparrow$ The eigenvalues p_i are probabilities.

Time evolution

Schrödinger equation: $\partial_t|\psi\rangle = \frac{1}{i\hbar}\mathbf{H}|\psi\rangle$

$$\partial_t\rho \;=\; \sum_i p_i \partial_t |\psi\rangle\langle\psi| \;=\; \sum_i p_i \Big(\frac{1}{i\hbar}\mathbf{H}|\psi\rangle\langle\psi| - \frac{1}{i\hbar}|\psi\rangle\langle\psi|\mathbf{H}\Big).$$

$$i\hbar\partial_t\rho = [\mathbf{H},\rho]$$

Solution: Unitary evolution

$$\rho(t) \;=\; \underbrace{e^{-\frac{i}{\hbar}\mathbf{H}t}}_{\mathbf{U}(t)}\,\rho(0)\,\underbrace{e^{+\frac{i}{\hbar}\mathbf{H}t}}_{\mathbf{U}^{-1}(t)}$$

Density matrix formalism

vector formalism:	*operator formalism:*
$\vert\psi\rangle$	$\rho := \vert\psi\rangle\langle\psi\vert$ (pure)
	$\rho := \sum_i p_i \vert\psi_i\rangle\langle\psi_i\vert$ (mixed)
$\langle\psi\vert\psi\rangle = 1$	$\mathrm{Tr}[\rho] = 1$
$i\partial_t\vert\psi\rangle = H\vert\psi\rangle$	$i\partial_t\rho = [H,\rho]$
$\langle M\rangle_\psi = \langle\psi\vert M\vert\psi\rangle$	$\langle M\rangle_\rho = \mathrm{Tr}[M\rho]$
coherent superposition $\vert\psi\rangle = \alpha\vert\psi_1\rangle + \beta\vert\psi_2\rangle$ $\vert\alpha\vert^2 + \vert\beta\vert^2 = 1$	convex probabilistic mixture $\psi = p_1\rho_1 + p_2\rho_2$ $p_1 + p_2 = 1$

The smallest quantum system: The qubit

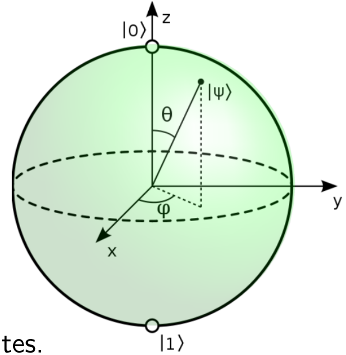
Classical bit: Configurations 0,1

Quantum bit: Amplitudes $\psi(0)$ and $\psi(1)$ with $|\psi(0)|^2 + |\psi(1)|^2 = 1$.

Bloch ball representation:
$\rho = \frac{1}{2}\left(\mathbf{1} + x\sigma^x + y\sigma^y + z\sigma^z\right)$

The vector (x, y, z) on the Bloch ball can be interpreted as expectation value of $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.

Points on the sphere represent pure states.



Positive operators

An operator ρ is called **positive** if one of the following equivalent statements holds:

- $\rho = \rho^\dagger$ and $\langle \rho\psi, \psi \rangle = \langle \psi|\rho|\psi \rangle \geq 0$
- $\rho = \rho^\dagger$ and all eigenvalues of ρ are non-negative.
- ρ can be written as $\rho = \mathbf{A}^\dagger \mathbf{A}$.

Density matrices are positive normalized operators.

Positive maps are functions mapping positive operators to other positive operators.

Von-Neumann entropy

Statistical mixture of quantum states:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i p_i \rho_i$$

The **von Neumann entropy** quantifies the uncertainty in the ensemble:

$$S(\rho) = -\sum_i p_i \ln p_i = -\mathrm{Tr}\Big[\rho \ln \rho\Big]$$

Pure states $\rho = |\psi\rangle\langle\psi|$ have entropy zero.
Mixtures have a positive entropy.

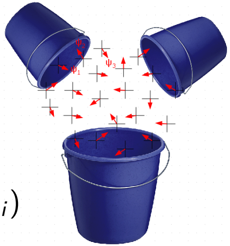
$$0 \leq S(\rho) \leq \ln d$$

Entropy "units"

- Physicists: $S(\rho) = -k_B \mathrm{Tr}\Big[\rho \ln \rho\Big]$ $\quad [S] = J/K$
- Mathematicians: $S(\rho) = -\mathrm{Tr}\Big[\rho \ln \rho\Big]$ $\quad [S] = 1$
- Information scientists: $S(\rho) = -\mathrm{Tr}\Big[\rho \log_2 \rho\Big]$ $\quad [S] = bit$

Properties of the von-Neumann entropy

$$S(\rho) = -\mathrm{Tr}\Big[\rho \ln \rho\Big]$$



- Concavity under probabilistic mixing:

$$S\Big(\sum_i p_i \rho_i\Big) \;\geq\; \sum_i p_i S(\rho_i)$$

- Invariance under unitary transformations

$$S(\rho) \;=\; S(\mathbf{U}\rho\,\mathbf{U}^\dagger)$$

⇒ Unitary transformations are information-preserving.

⇒ Schrödinger time evolution does not change entropy.

Thermostatics: Isolated quantum systems: Microcanonical ensemble

Entropy should be maximal.

Classical physics: $S = \ln |\Omega|$

Quantum physics: $S(\rho) = \ln \dim\mathcal{H}$

$$\Rightarrow \quad \rho = \frac{\mathbf{1}}{\dim\mathcal{H}}$$

(identity matrix normalized to trace 1)

Example: Microcanonical qubit: $\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

Systems in a heat bath: Canonical ensemble

If energy is exchanged with a heat bath, the entropy is maximised under the constraint that the energy average $\overline{E} = \langle \mathbf{H} \rangle$ is constant.

$$\delta S[\rho] = \delta \mathrm{Tr}[\rho \ln \rho] = 0\,, \qquad \delta \mathrm{Tr}[\rho] = 0\,, \qquad \delta \mathrm{Tr}[\mathbf{H}\rho] = 0$$

Introduce Lagrange multiplyer α, β:

$$\Rightarrow \delta\Big(\mathrm{Tr}[\rho \ln \rho] + \alpha \mathrm{Tr}[\rho] + \beta \mathrm{Tr}[\mathbf{H}\rho]\Big) = \mathrm{Tr}\Big[\delta\rho(\ln \rho + \alpha + \beta \mathbf{H})\Big] = 0\,.$$

$$\text{Solution:} \quad \rho \;=\; e^{-\alpha-\beta\mathbf{H}} \;=\; \frac{1}{Z} e^{-\mathbf{H}/k_B T}\,; \qquad Z = \mathrm{Tr}\big[e^{-\beta\mathbf{H}}\big]$$

$$\overline{E} \;=\; \langle \mathbf{H} \rangle \;=\; \mathrm{Tr}[\rho\mathbf{H}] \;=\; \frac{\mathrm{Tr}[\mathbf{H}e^{-\beta\mathbf{H}}]}{\mathrm{Tr}[e^{-\beta\mathbf{H}}]} \;=\; -\partial_\beta \ln Z$$

Outline

Von-Neumann measurement postulate

A Measurement is represented by a Hermitean operator

$$\mathbf{M} = \mathbf{M}^\dagger = \sum_m m|m\rangle\langle m|$$

If a system in a *pure* state $|\psi\rangle$ is measured by $\mathbf{M}$ it is *instantaneously projected* onto one of its eigenstates $|m\rangle$ with probability $|\langle\psi|m\rangle|^2$.

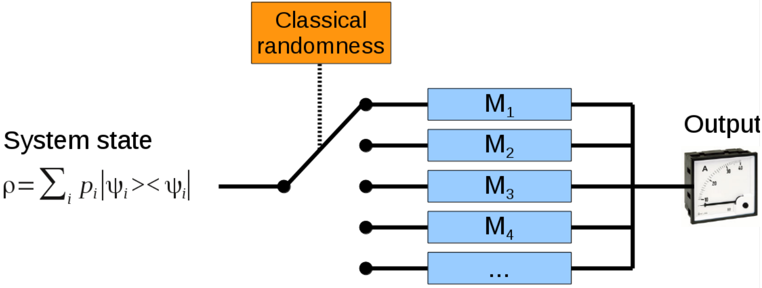
$$\rho \;\to\; \rho' = \sum_m |m\rangle\langle m|\,\rho\,|m\rangle\langle m|$$

A measurement usually generates classical randomness (=entropy).

Realistic measurements

A realistic measurement has a finite precision.

Simple model for a non-perfect measurement:



Realistic measurements

Usual projective measurement (von Neumann):

$$\mathbf{M} = \sum_m \lambda_m |m\rangle\langle m| \qquad \Rightarrow \qquad \rho \to \rho' = \sum_m |m\rangle\langle m|\,\rho\,|m\rangle\langle m|$$

Generalized measurement (take $\mathbf{M}_j$ with probability q_j):

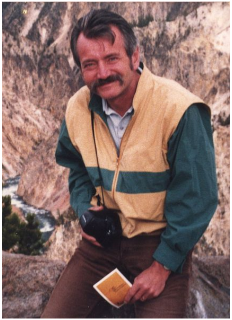
$$\rho \to \rho' = \sum_j q_j \sum_{m_j} |m_j\rangle\langle m_j|\,\rho\,|m_j\rangle\langle m_j|$$

- cannot be written as a projective measurement
- is a legal quantum operation (ρ' is again a density matrix)

Kraus Theorem

University of Würzburg

- Nobel prize for X-rays
- Nobel prize Quantum Hall effect
- Karl Kraus: Important theorem in QI



Kraus Theorem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces.

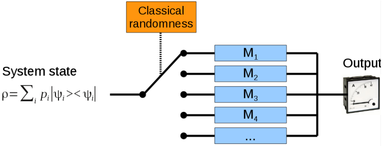
Let $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a quantum operation
(meaning that Φ applied to a density matrix is again a density matrix).

Then Φ can be written in the form

$$\boxed{\rho' = \Phi(\rho) = \sum_k \mathbf{B}_k \rho \mathbf{B}_k^\dagger}$$

with $\sum_k \mathbf{B}_k \mathbf{B}_k^\dagger = \mathbf{1}$.
The operators $\{\mathbf{B}_k\}$ are called **Kraus Operators**.
They are unique up to unitary transformations.

POVMs



$$\rho \to \rho' = \sum_j q_j \sum_{m_j} |m_j\rangle\langle m_j|\,\rho\,|m_j\rangle\langle m_j|$$

Applying the Kraus theorem this can be written as

$$\rho' = \Phi(\rho) = \sum_k \mathbf{B}_k \rho \mathbf{B}_k^\dagger$$

A measurement described by a set of operators $\{\mathbf{B}_k\}$ with $\sum_k \mathbf{B}_k {\mathbf{B}_k}^\dagger = \mathbf{1}$ is called a ***positive operator-valued measurement*** (POVM).

von Neumann measurement as a special case

$$\rho \;\rightarrow\; \rho' = \sum_m |m\rangle\langle m|\,\rho\,|m\rangle\langle m|$$

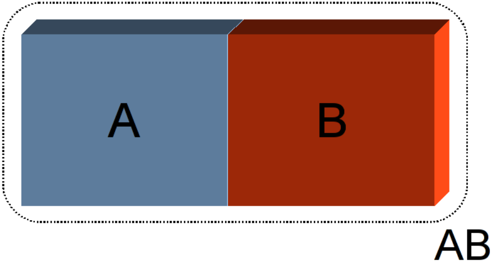
$$\rho' = \Phi(\rho) = \sum_m \mathsf{B}_m \rho \mathsf{B}_m{}^\dagger$$

$\Rightarrow \quad \mathsf{B}_m = |m\rangle\langle m|$ are projection operators.

Von-Neumann measurements are *projective* measurements.

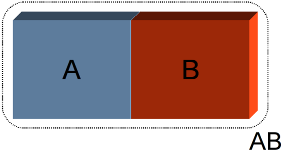
Outline

Bipartite systems



Hilbert space $\mathcal{H}_{\mathbf{AB}} = \mathcal{H}_{\mathbf{A}} \otimes \mathcal{H}_{\mathbf{B}}$

Bipartite systems



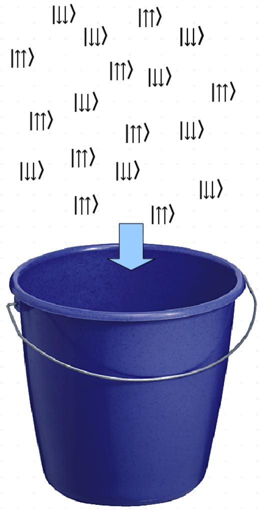
There are two different types of correlations between the two subsystems:

- **Quantum correlations** (entanglement)
- **Classical correlations** (probabilistic)

Classical correlations

Classical correlations are simply due to the composition of the ensemble:

$$\rho \;=\; \tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow|$$



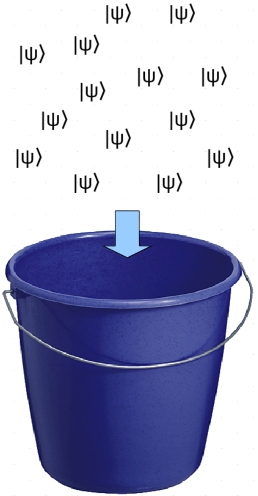
Quantum correlations (entanglement)

Quantum correlations are due to a superposition of amplitudes

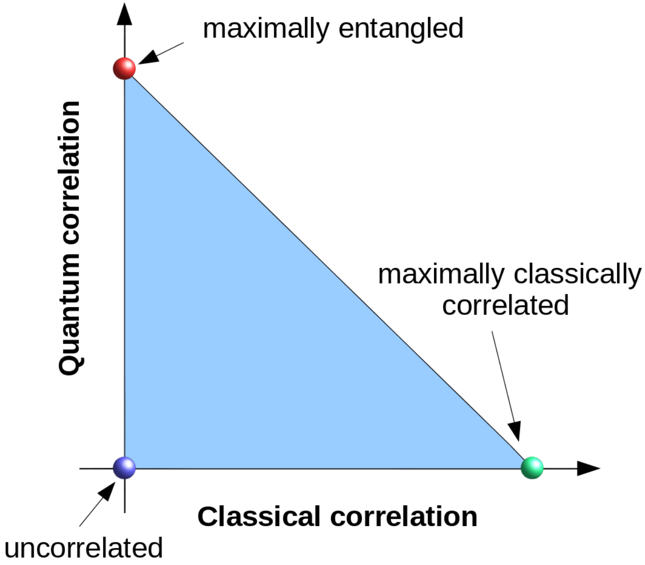
$$|\psi\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle$$

$$\rho \;=\; |\psi\rangle\langle\psi|$$

Quantum correlations can be present in pure quantum states.



Cartoon of correlation landscape



States of maximal classical and quantum corrleation

Two qubit system:

Maximal classical correlations:

$$\rho_{class} = \frac{1}{2}\Big(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\Big) = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Maximal quantum correlations:

$$\rho_{quant} = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & \mathbf{1}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \mathbf{1} & 0 & 0 & 1\end{pmatrix}$$

How to distinguish classical and quantum correlations

Two qubit system:

Measurement questions: $\uparrow\uparrow$ or $\uparrow\downarrow$ or $\downarrow\uparrow$ or $\downarrow\downarrow$?

Measurement operator: $\sigma^z \otimes \sigma^z$

Measurement projectors: $P_{\uparrow\uparrow} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow|$, $P_{\uparrow\downarrow} = |\uparrow\downarrow\rangle\langle\uparrow\downarrow|$,
$P_{\downarrow\uparrow} = |\downarrow\uparrow\rangle\langle\downarrow\uparrow|$, $P_{\downarrow\downarrow} = |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

$$
\begin{aligned}
\mathrm{Tr}[\rho_{class} P_{\uparrow\uparrow}] &= \mathrm{Tr}[\rho_{quant} P_{\uparrow\uparrow}] = 1/2 \\
\mathrm{Tr}[\rho_{class} P_{\uparrow\downarrow}] &= \mathrm{Tr}[\rho_{quant} P_{\uparrow\downarrow}] = 0 \\
\mathrm{Tr}[\rho_{class} P_{\downarrow\uparrow}] &= \mathrm{Tr}[\rho_{quant} P_{\downarrow\uparrow}] = 0 \\
\mathrm{Tr}[\rho_{class} P_{\downarrow\downarrow}] &= \mathrm{Tr}[\rho_{quant} P_{\downarrow\downarrow}] = 1/2
\end{aligned}
$$

In both cases we find only $\uparrow\uparrow$ and $\downarrow\downarrow$, each with probability 1/2.

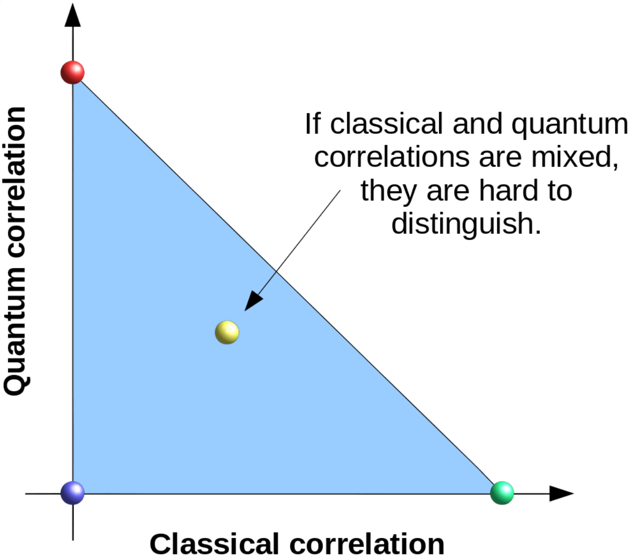
How to distinguish classical and quantum correlations

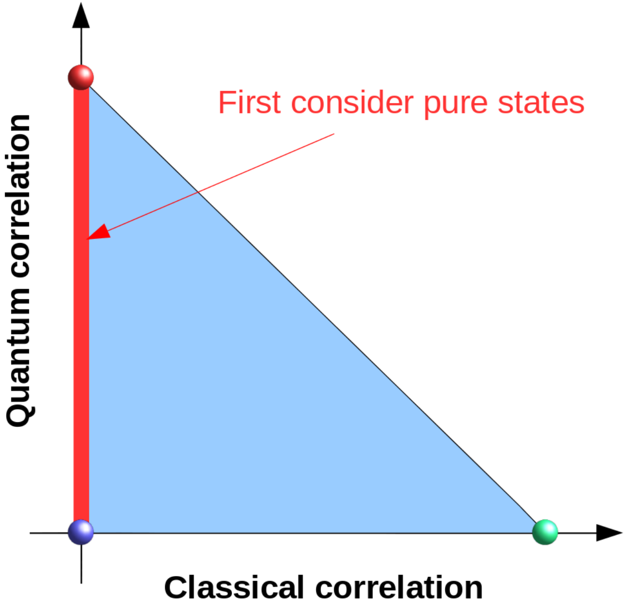
In order to detect quantum correlations, we have to measure several observables (different analyzer orientations):

$$\langle \sigma^z \otimes \sigma^z \rangle_{class} = \mathrm{Tr}[\rho_{class} \sigma^z \otimes \sigma^z] = 1$$
$$\langle \sigma^z \otimes \sigma^z \rangle_{quant} = \mathrm{Tr}[\rho_{quant} \sigma^z \otimes \sigma^z] = 1$$

$$\langle \sigma^x \otimes \sigma^x \rangle_{class} = \mathrm{Tr}[\rho_{class} \sigma^x \otimes \sigma^x] = \mathbf{0}$$
$$\langle \sigma^x \otimes \sigma^x \rangle_{quant} = \mathrm{Tr}[\rho_{quant} \sigma^x \otimes \sigma^x] = 1$$

To see the difference between classical and quantum correlations, one has to use several kinds of measurements (rotate the analyzer).


Quantum correlation
If classical and quantum correlations are mixed, they are hard to distinguish.
Classical correlation


First consider pure states
Quantum correlation
Classical correlation

Pure states

- Pure states $\rho = |\psi\rangle\langle\psi|$ are always classically uncorrelated (no classical statistics involved).

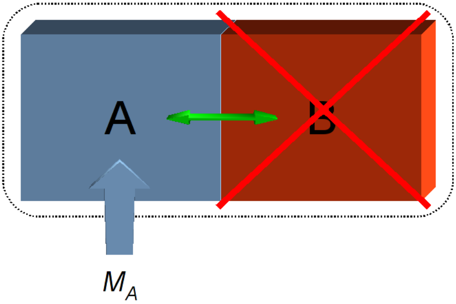
- A pure state is said to be **separable** if it factorizes:

$$|\psi\rangle = |\psi_{\mathbf{A}}\rangle \otimes |\psi_{\mathbf{B}}\rangle$$

$$\rho \;=\; |\psi_{\mathbf{A}}\rangle\langle\psi_{\mathbf{A}}| \;\otimes\; |\psi_{\mathbf{B}}\rangle\langle\psi_{\mathbf{B}}|$$

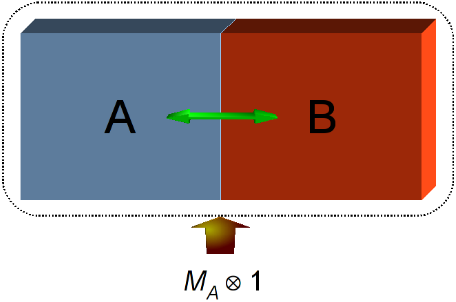
- A **pure** state is said to be **entangled** if it is not separable.

Measurement on subsystems



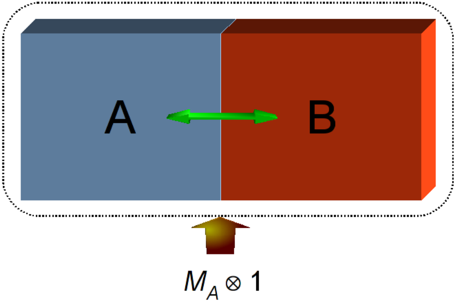
Measuring subsystem A with $M_{\mathbf{A}} = M_{\mathbf{A}}^{\dagger}$ ...

Measurement on subsystems



Measuring subsystem A with $M_{\mathbf{A}} = M_{\mathbf{A}}^{\dagger}$
is the same as measuring the combined system with $M = M_{\mathbf{A}} \otimes \mathbf{1}$.

Partial trace

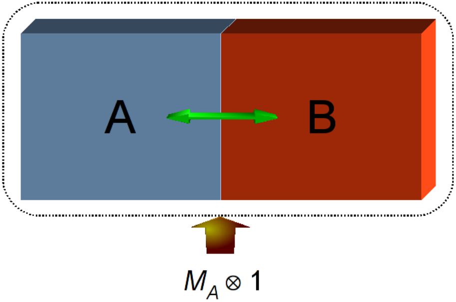


$$\langle M \rangle = \mathrm{Tr}\Big[\rho M\Big] = \mathrm{Tr}\Big[\rho(M_{\mathbf{A}} \otimes \mathbf{1})\Big] = \mathrm{Tr}\Big[\underbrace{\mathrm{Tr}_{\mathbf{B}}[\rho]}_{\Uparrow}\, M_{\mathbf{A}}\Big]$$

partial trace over B

Reduced density matrices: $\rho_{\mathbf{A}} = \mathrm{Tr}_{\mathbf{B}}[\rho]\,, \quad \rho_{\mathbf{B}} = \mathrm{Tr}_{\mathbf{A}}[\rho]$

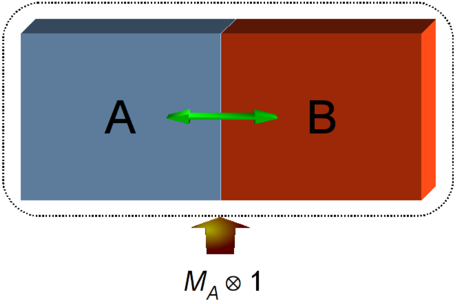
Partial trace



Pure state entanglement, seen from the perspective of a subsystem, looks like classical randomness.

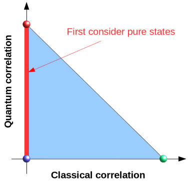
$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \rho_{\mathbf{A}} = \mathrm{Tr}_{\mathbf{B}}[\rho] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Partial trace



In quantum physics, subsystems may have a higher entropy than the composite system.

Measuring the entanglement of pure quantum states



$$S_{\mathbf{A}} = S(\rho_{\mathbf{A}}) = -\mathrm{Tr}[\rho_{\mathbf{A}} \ln \rho_{\mathbf{A}}]$$

$$S_{\mathbf{B}} = S(\rho_{\mathbf{B}}) = -\mathrm{Tr}[\rho_{\mathbf{B}} \ln \rho_{\mathbf{B}}]$$

$$S_{\mathbf{A}} = S_{\mathbf{B}}$$

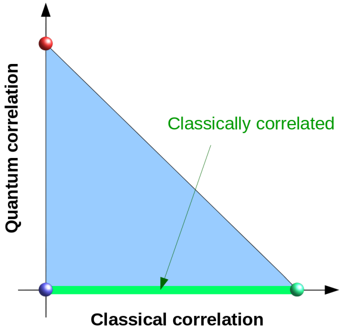
Undisputed unique entanglement measure for *pure* states:

$$E = S_{\mathbf{A}} = S_{\mathbf{B}}$$

For a maximally entangled state:

$$\rho_{quant} \;=\; \frac{1}{2}\big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\big)\big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\big) : \qquad E = \ln 2 \;[1 \text{ bit}]$$

Measuring classically correlated states



Use **mutual information**: $I_{A:B} = S_{\mathbf{A}} + S_{\mathbf{B}} - S_{\mathbf{AB}}$

For a maximally classically correlated state:

$$\rho_{quant} \;=\; \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) : \qquad I_{A:B} = \ln 2 \; [1 \text{ bit}]$$

The Problem

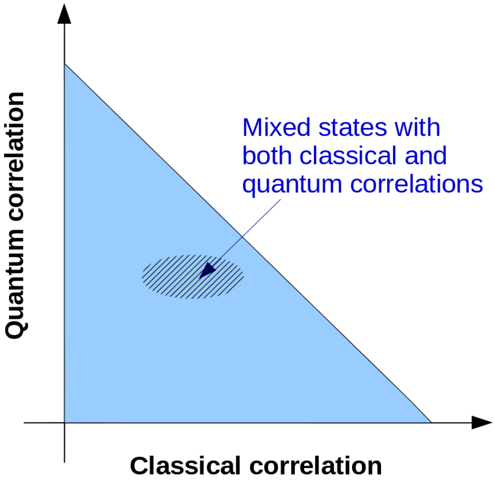
State	$I_{A:B}$	E
no correlations (pure & separable)	0	0
maximal classical correlations	1	1
maximal quantum correlations	**2**	1

"supercorrelated"

Problem: These measures cannot be combined to distinguish quantum and classical correlations for general mixed states. $\Rightarrow$

Different *entanglement measures* needed.

The Problem


Quantum correlation
Mixed states with both classical and quantum correlations
Classical correlation

Separable quantum states

- **Factorizing density matrix:**
  Neither classical nor quantum correlations:

$$\rho = \rho_{\mathbf{A}} \otimes \rho_{\mathbf{B}}$$

- Mixtures are said to be separable if they can be written as convex combinations of factorizing density matrices:

$$\rho = \sum_i p_i \, \rho_{\mathbf{A}}^{(i)} \otimes \rho_{\mathbf{B}}^{(i)}$$

Schmidt decomposition

Every pure state $|\psi\rangle$ of a bipartite system can be decomposed as

$$|\psi\rangle = \sum_{n=1}^{r} \alpha_n |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

with $r \leq \min(d_{\mathbf{A}}, d_{\mathbf{B}})$ and Schmidt numbers $\alpha_n \geq 0$ obeying

$$\sum_n \alpha_n^2 = 1$$

Schmidt decomposition

$$|\psi\rangle = \sum_{n=1}^{r} \alpha_n |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

For the density matrix of a pure state:

$$\Rightarrow \rho = |\psi\rangle\langle\psi| = \sum_{n,m=1}^{r} \alpha_n \alpha_m |n\rangle\langle m|_{\mathbf{A}} \otimes |n\rangle\langle m|_{\mathbf{B}}$$

For reduced density matrices:

$$\rho_{\mathbf{A}} = \mathrm{Tr}_{\mathbf{B}}[\rho] = \sum_{n=1}^{r} \alpha_n^2 |n\rangle\langle n|_{\mathbf{A}}\,, \qquad \rho_{\mathbf{B}} = \mathrm{Tr}_{\mathbf{A}}[\rho] = \sum_{n=1}^{r} \alpha_n^2 |n\rangle\langle n|_{\mathbf{B}}$$

The α_n^2 are just the probabilities in the reduced state.

Proof: Singular value decomposition

Purification

- Take an arbitrary mixed state $\rho_{\mathbf{A}}$ on the Hilbert space $\mathcal{H}_{\mathbf{A}}$:

$$\rho_{\mathbf{A}} = \sum_n p_n |n\rangle\langle n|_{\mathbf{A}}$$

- Extend $\mathcal{H}_{\mathbf{A}}$ by an auxiliary Hilbert space $\mathcal{H}_{\mathbf{B}}$ of the same dimension.
- Define an orthonormal basis $|n\rangle_{\mathbf{B}}$ in $\mathcal{H}_{\mathbf{B}}$.
- Define the **pure** state

$$|\psi\rangle = \sum_n \sqrt{p_n} |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

$$\Rightarrow \quad |\psi\rangle\langle\psi| = \sum_{n,m} \sqrt{p_n p_m}\, |n\rangle\langle m|_{\mathbf{A}} \otimes |n\rangle\langle m|_{\mathbf{B}}$$

Purification

$$\Rightarrow \quad |\psi\rangle\langle\psi| = \sum_{n,m} \sqrt{p_n p_m}\, |n\rangle\langle m|_{\mathbf{A}} \otimes |n\rangle\langle m|_{\mathbf{B}}$$

- Take the partial trace over the auxiary space $\mathcal{H}_{\mathbf{B}}$

$$\mathrm{Tr}_{\mathbf{B}}\left[|\psi\rangle\langle\psi|\right] \;=\; \sum_k \sum_{n,m} \sqrt{p_n p_m}\, |n\rangle\langle m|_{\mathbf{A}} \; \underbrace{\langle k||n\rangle\langle m||k\rangle_{\mathbf{B}}}_{=\delta_{kn}\delta_{kn}}$$

$$= \sum_k p_k |k\rangle\langle k|_{\mathbf{A}} \;=\; \rho_{\mathbf{A}}$$

The reduced density matrix is just the original mixed state.

In a suitably extended Hilbert space a mixed state can be represented as a pure state.

Outline

Recall definition of entanglement

- A **pure** state $|\psi\rangle$ is said to be separable if it factorizes:

$$|\psi\rangle = |\psi\rangle_{\mathbf{A}} \otimes |\psi\rangle_{\mathbf{B}}$$

$$\Rightarrow \quad \rho = |\psi\rangle\langle\psi|_{\mathbf{A}} \otimes |\psi\rangle\langle\psi|_{\mathbf{B}}$$

- A **mixed** state ρ is said to be separable if it can be expressed as a probabilistic combination of pure separable states:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \,, \quad |\psi_i\rangle \text{ separable.}$$

- One can show that a state is separable if and only if it can be written in the form

$$\rho \ = \ \sum_i p_i \ \rho_{\mathbf{A}}^{(i)} \otimes \rho_{\mathbf{B}}^{(i)} \,, \qquad 0 \le p_i \le 1, \quad \sum_i p_i = 1$$

- entangled $\equiv$ non-separable

Entanglement criteria vs. entanglement measures

Entanglement criteria are simple checks which provide a sufficient condition for the **existence** of entanglement.

- PPT criterion
- CCNR criterion
- ...

Entanglement measures are quantitative measures which tell us **how much entanglement** is there.

- Entanglement distance measures
- Entanglement of formation
- Quantum discord
- ...

PPT criterion

Definition of the partial transpose $T_{\mathbf{A}}, T_{\mathbf{B}}$:

For a factorizing operator $\mathbf{C} = \mathbf{C_A} \otimes \mathbf{C_B}$ the partial transpose is defined as the transposition of one of the tensor slots:

$$\mathbf{C}^{T_\mathbf{A}} := \mathbf{C}_\mathbf{A}^T \otimes \mathbf{C_B}\,, \qquad \mathbf{C}^{T_\mathbf{B}} := \mathbf{C_A} \otimes \mathbf{C}_\mathbf{B}^T\,.$$

A non-factorizing operator can be written as a linear combination of factorizing ones. So the partial transpose is also well-defined on general operators.

$$T_\mathbf{A} \circ T_\mathbf{B} = T_\mathbf{B} \circ T_\mathbf{A} = T\,, \qquad T \circ T_\mathbf{A} = T_\mathbf{A} \circ T = T_\mathbf{B}\,.$$

PPT criterion

Observation:

Transposition is a positive operation:
If ρ is a density matrix, then ρ^T is also a valid density matrix.

Peres-Horodecki-Criterion (positive partial transpose, PPT):

If ρ is separable, then ρ^{T_A} and ρ^{T_B} are positive operators, that is, they are both physically valid density matrices.

Or the other way round:

If ρ^{T_A} or ρ^{T_B} are **not** valid density matrices, then we know that the subsystems A and B are entangled.

PPT criterion

Example:

Maximally entangled state (Bell state):

$$\rho \;=\; \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}$$

$$\Rightarrow \qquad \rho^{T_\mathbf{A}} = \rho^{T_\mathbf{B}} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & \mathbf{1} & 0\\ 0 & \mathbf{1} & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Eigenvalues of ρ : $\{0, 0, 0, 1\}$

Eigenvalues of $\rho^{T_\mathbf{A}} = \rho^{T_\mathbf{B}}$: $\{0, -\mathbf{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}\}$

Interpretation of the PPT criterion

- Classical mechanics is invariant under time reversal

$$\big(q(t), p(t)\big) \quad \rightarrow \quad \big(q(-t), -p(-t)\big)$$

- Schrödinger unitary evolution is also invariant under time reversal

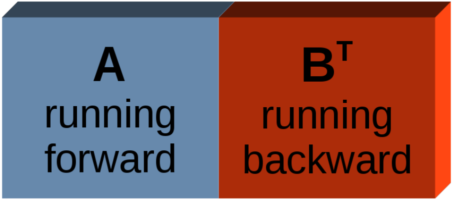
$$\psi(t), \mathbf{H} \quad \rightarrow \quad \psi(-t)^*, \mathbf{H}^*$$

which is the same as taking

$$\rho(t) \quad \rightarrow \quad \rho^*(-t) = \rho^T(-t)$$

Transposition $\sim$ Time reversal

Interpretation of the PPT criterion



PPT: If this is not a physically valid scenario,
then there must be entanglement between the two parts.

Side remark: Completely positive maps

- Completely positive maps $\Phi : \rho \to \Phi(\rho)$ are **physically realizable** positive maps.

- Not all positive maps are physically realizable.

  Example: Transposition $\rho \to \rho^T$ is positive but not physically realizable because it **could be entangled with another unknown external object**.

- Definition: Φ is called completely positive on $\mathcal{H}$ if $\Phi \otimes \mathbf{1}$ is positive on $\mathcal{H} \otimes \mathcal{H}_{aux}$ for every external Hilbert space $\mathcal{H}_{aux}$.

CCNR criterion

Operator realignment:
Let $|i\rangle_X$ und $|i\rangle_Y$ be a basis of the bipartite Hilbert space $\mathcal{H}_{\mathbf{AB}} = H_{\mathbf{A}} \otimes H_{\mathbf{B}}$ and let $\mathbf{C}$ be an operator with the matrix representation

$$\mathbf{C} = \sum_{ijkl} C_{ij,kl}\,|ij\rangle\langle kl|.$$

Define the realigned Matrix $\mathbf{C}^R$ by

$$\mathbf{C}^R = \sum_{ijkl} C_{ij,kl}|ik\rangle\langle jl| = \sum_{ijkl} C_{ik,jl}|ij\rangle\langle kl|$$

$$\boxed{C^R_{ij,kl} = C_{ik,jl}}$$

CCNR criterion

Operation	Components	Exchanged indices
Normal transpose T	$C^{T}_{ij,kl} = C_{kl,ij}$	$(12) \leftrightarrow (34)$
Partial transpose T_X	$C^{T_X}_{ij,kl} = C_{kj,il}$	$1 \leftrightarrow 3$
Partial transpose T_Y	$C^{T_X}_{ij,kl} = C_{il,kj}$	$2 \leftrightarrow 4$
Realignment R	$C^{R}_{ij,kl} = C_{ik,jl}$	$2 \leftrightarrow 3$

CCNR criterion

Operator Schmidt decomposition:

The vector Schmidt decomposition

$$|\psi\rangle = \sum_{n=1}^{r} \alpha_n |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

works also for operators

$$\mathbf{C} = \sum_n \alpha_n \mathbf{C}_n^X \otimes \mathbf{C}_n^Y,$$

where α_n are the singular values of $\mathbf{C}^R$
(the positive square root of the eigenvalues of ${\mathbf{C}^R}^T \mathbf{C}^R$)

Induced trace norm:

$$||\mathbf{C}||_s = \sum_n \alpha_n$$

CCNR criterion

Computable Cross Norm or Realignment Criterion (CCNR):

Consider a **separable pure** state:

$$\rho = |\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|_{\mathbf{A}} \otimes |\psi\rangle\langle\psi|_{\mathbf{B}}$$

$\Rightarrow$ Only a single Schmidt number $\alpha_1 = 1$ $\qquad \Rightarrow ||\rho||_s = 1$

Consider a **separable mixed** state. Then ρ is a probabilistic combination of pure separable states ρ_k:

$$||\rho||_s = ||\sum_k p_k \rho_k||_s \leq \sum_k p_k \underbrace{||\rho_k||}_{=1} = 1$$

meaning that $\sum_k \alpha_k \leq 1$. In opposite direction, we have CCNR:

$$\sum_{\alpha_k} > 1 \quad \Rightarrow \quad \text{non-separable} \quad \Leftrightarrow \quad \text{entangled}$$

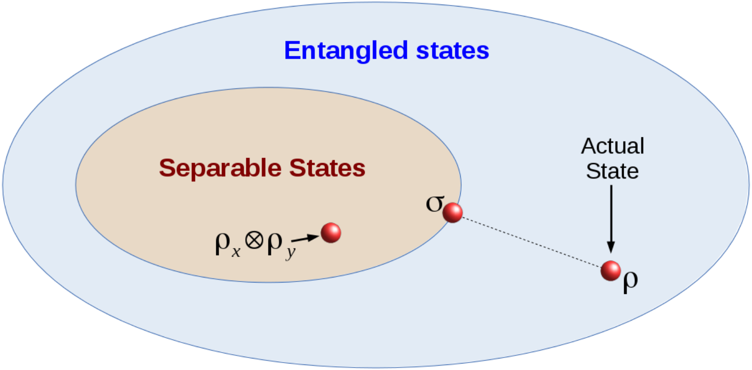
Outline

Entanglement measure $E(\rho)$ – List of desired properties

1. Separable state $\Leftrightarrow$ $E(\rho) = 0$.
2. EPR / Bell states $\Leftrightarrow$ $E(\rho)$ is maximal.
3. Pure states: $E(\rho) = S(\rho_{\mathbf{A}}) = S(\rho_{\mathbf{B}})$
4. $E(\rho)$ should be invariant under local unitary transformations.
5. $E(\rho)$ should not increase under LOCC operations.
6. Symmetry $A \leftrightarrow B$.
7. Convexity on probabilistic mixtures:

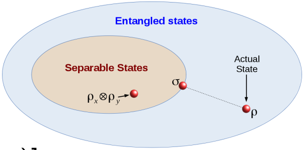
$$E\Big(\sum_k p_k \rho_k\Big) \;\leq\; \sum_k p_k E(\rho_k)$$

Entanglement measure based on distance



$$E_D(\rho) \;=\; \inf_{\sigma \text{ separabel}} D(\rho,\sigma)\,.$$

Entanglement measure based on distance



Example:

Relative entropy $D_R(\rho, \sigma) = \mathrm{Tr}[\rho(\ln \rho - \ln \sigma)]$
(Quantum-mechanical version of Kullback-Leibler divergence)

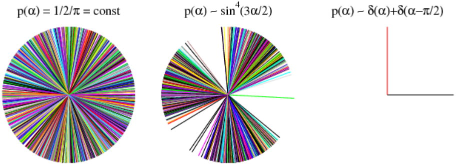
This allows us to define the

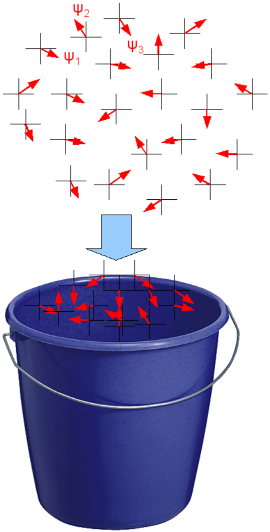
- Quantum mutual information: $S_{A:B} = D_R(\rho, \rho_A \otimes \rho_B)$
- Relative entanglement entropy: $E_R(\rho) = \inf_{\sigma \text{ separabel}} D_R(\rho, \sigma)$.

$$E_r(\rho) \leq S_{A:B}$$

Entanglement of formation

- A mixed state is represented by a collection of pure states.
- Each pure state has a well-defined entanglement.
- The representation is not unique.





Entanglement of formation

The entanglement of the representing pure states may be higher than the entanglement of the mixture.

Example:

$$|\pm\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\Big)$$

$$\Rightarrow \quad |\pm\rangle\langle\pm| \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho = \frac{1}{2}\Big(|+\rangle\langle+| + |-\rangle\langle-|\Big) \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(no correlation, no entanglement)

Entanglement of formation

Main idea:

Find the representation of the ensemble for which the averaged entanglement of the representing pure states is minimal:

$$E_f(\rho) = \inf\Big\{ \sum_i p_i \, E(|\psi_i\rangle\langle\psi_i|) \,\Big|\, \rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \Big\}$$

$$= \inf\Big\{ \sum_i p_i \, S_{\rho_{i,\mathbf{A}}} \,\Big|\, \rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \Big\}\,.$$

...very hard to compute!

Entanglement of formation

Exact formula for E_f for a 2-qubit system:

$$E_F(\rho) \;=\; S\left[\frac{1+\sqrt{1-C^2(\rho)}}{2}\right]$$

where

$$S[x] \;=\; -x\log_2 x-(1-x)\log_2(1-x)$$

$$C(\rho)=\max(0,\lambda_1-\lambda_2-\lambda_3-\lambda_4)$$

Here λ_i are the decreasingly sorted square roots of the eigenvalues of the following 4×4 matrix:

$$\Lambda=\rho(\sigma^y\otimes\sigma^y)\rho^*(\sigma^y\otimes\sigma^y)$$

Wooters et al.

Main message of this last part

Separable state:

$$\rho = \sum_i p_i \, \rho_{\mathbf{A}}^{(i)} \otimes \rho_{\mathbf{B}}^{(i)}$$

A bipartite system in a mixed state is *defined* to be entangled if the state is non-separable.

entangled $\Leftrightarrow$ non-separable

But, as we will see:

QUANTUM CORRELATIONS CAN BE PRESENT IN SEPARABLE STATES.

'NON-ENTANGLED' DOES NOT AUTOMATICALLY MEAN 'CLASSICAL'.

Separability vs. quantum correlation

Consider the following two-qubit state

$$\rho = \frac{1}{4}\Big(|+\rangle\langle+| \otimes |0\rangle\langle 0| \;+\; |-\rangle\langle-| \otimes |1\rangle\langle 1| \\ + |0\rangle\langle 0| \otimes |-\rangle\langle-| \;+\; |1\rangle\langle 1| \otimes |+\rangle\langle+|\Big)$$

where $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$ are four non-orthogonal states of each qubit.

Even though ρ is **separable** (i.e. non-entangled),
we will see that the **quantum correlation is non-zero**.

Dakić et al., PRL 105, 190502 (2010)

Definition of the quantum discord

Consider two subsystems A and B.

Classical information theory:
Let us define

- $I(A:B) = H(A) + H(B) - H(A,B)$
- $J(A:B) = H(B) - H(B|A)$

with the Shannon entropy $H(X) = -\sum_i p_X^{(i)} \log p_X^{(i)}$.

Thanks to Bayes rule these expressions are identical, i.e. we have two equivalent descriptions of the mutual information.

Definition of the quantum discord

Quantum information theory:
Analogously to the classical case we have defined the quantum mutual information as

$$\mathcal{I}_{A:B} = S(\rho_{\mathbf{A}}) + S(\rho_{\mathbf{B}}) - S(\rho)$$

with the von Neumann entropy $S(X) = -\mathrm{Tr}[X \log X]$.

The quantum mutual information $\mathcal{I}$ is a measure for the **total correlation** between A and B.

Definition of the quantum discord

How does J transform under a measurement?

We use a complete set of orthogonal projectors $\{\Pi_j\}$ to take measurements on subsystem A. After applying Π_j to A the state of subsystem B will be

$$\rho_{B|j} = \frac{1}{p_j}\mathrm{Tr}_{\mathbf{A}}[(\Pi_j \otimes \mathbf{1})\rho],$$

$$\text{where } p_j = \mathrm{Tr}[(\Pi_j \otimes \mathbf{1})\rho].$$

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This allows us to write

$$\mathcal{J}_{\{\Pi_j\}}(A:B) = S(\rho_{\mathbf{B}}) - \sum_j p_j S(\rho_{B|j}).$$

Note: $\mathcal{J}_{\{\Pi_j\}}$ depends on the chosen set of projectors $\{\Pi_j\}$.

Definition of the quantum discord

How does J transform?
Our aim is to cover **all classical information** in $\mathcal{J}$, so we take the maximum over all complete sets of orthogonal projectors $\{\Pi_j\}$:

$$\mathcal{J}(A:B) = \max_{\{\Pi_j\}} \mathcal{J}_{\{\Pi_j\}}(A:B) = S(\rho_{\mathbf{B}}) - \min_{\{\Pi_j\}} \sum_j p_j S(\rho_{B|j})$$

Definition of the quantum discord

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Unlike the classical case $\mathcal{I}$ and $\mathcal{J}$ do not coincide!

Definition of the quantum discord

Total amount of correlation:

$$\mathcal{I}(A:B) = S(\rho_{\mathbf{A}}) + S(\rho_{\mathbf{B}}) - S(\rho)$$

Classical correlation:

$$\mathcal{J}(A:B) = S(\rho_{\mathbf{B}}) - \min_{\{\Pi_j\}} \sum_j p_j S(\rho_{B|j})$$

Quantum correlation:

$$\mathcal{D}(A:B) = \mathcal{I}(A:B) - \mathcal{J}(A:B)$$

$\mathcal{D}(A:B)$ is called the **quantum discord**.

Properties of the quantum discord

- The quantum discord is not symmetric: $\mathcal{D}(A:B) \neq \mathcal{D}(B:A)$.
- The quantum discord is always non-negative: $\mathcal{D}(A:B) \geq 0$.
- The upper bound of the quantum discord is the von Neumann entropy of the measured subsystem: $\mathcal{D}(A:B) \leq S(A)$.
- The following equivalence holds:

$$\mathcal{D}(A,B) = 0 \quad \Leftrightarrow \quad \exists \text{ complete set of orthogonal}$$

$$\text{projectors } \{\Pi_k = |\Psi_k\rangle\langle\Psi_k|\} : \sum_k (\Pi_k \otimes \mathbf{1})\rho(\Pi_k \otimes \mathbf{1}) = \rho$$

Separability vs. quantum correlation

Let's get back to the introducing example:

$$\rho = \frac{1}{4}(|+\rangle\langle+| \otimes |0\rangle\langle 0| \; + \; |-\rangle\langle-| \otimes |1\rangle\langle 1| \\ + |0\rangle\langle 0| \otimes |-\rangle\langle-| \; + \; |1\rangle\langle 1| \otimes |+\rangle\langle+|).$$

The state ρ is separable (i.e. non-entangled), but computing the quantum discord gives $\mathcal{D}(A:B) = \frac{3}{4}\log\frac{4}{3} = 0.311 > 0$. Hence:

QUANTUM CORRELATIONS CAN BE PRESENT IN SEPARABLE STATES.

'NON-ENTANGLED' DOES NOT AUTOMATICALLY MEAN 'CLASSICAL'.

Werner states

Consider the two-qubit system

$$\rho_z = \frac{1-z}{4}\mathbf{1} + z|\Psi\rangle\langle\Psi|,$$

$$\text{where } 0 \leq z \leq 1 \text{ and } |\Psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

One can show:

- ρ_z is for $z \leq \frac{1}{3}$
- ρ_z is for $z > \frac{1}{3}$

$$\text{BUT } \forall\{\Pi_k\},\ z \in [0,1] : \sum_{k=1}^{2} (\Pi_k \otimes \mathbf{1})\rho_z(\Pi_k \otimes \mathbf{1}) \neq \rho_z$$

$$\Rightarrow \mathcal{D}(A:B) \neq 0$$

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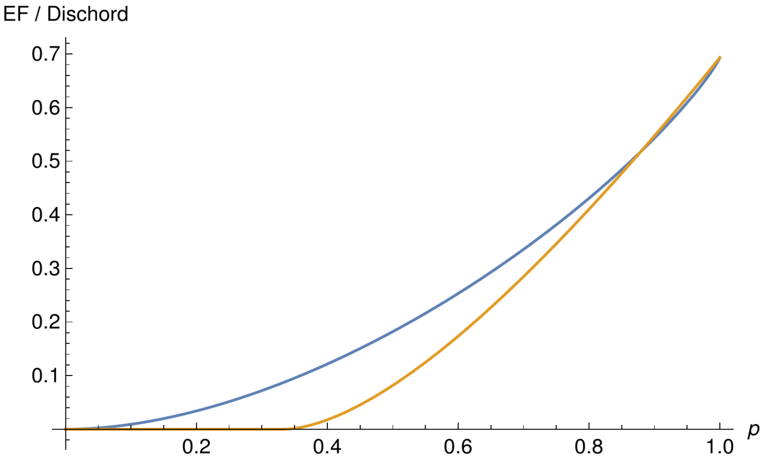
$$\Rightarrow \mathcal{D}(A:B) \neq 0$$

Entanglement of Formation vs. Quantum Discord in a Werner state



SUMMARY

- There are two types of correlations, namely classical and quantum-mechanical correlations.
- States are defined as entangled if they are not separable.
- There is no unique entanglement measure.
- The entanglement of formation is the standard choice, but hard to compute.
- Quantum correlations may even be present in non-entangled states.
- The quantum dicord is probably a better measure for quantum correlations.

Thank you !