Note Title 2015-01-23

2d $\mathcal{N}=(2,2)$ SUSY gauged LSM

$\uparrow$

4d $\mathcal{N}=1$ by dim red.

$\nwarrow$ 4 supercharges, 1 R-symmetry

$$Q_\pm ,\ \bar{Q}_\pm = Q_\pm^\dagger ,\ H ,\ P ,\ M ,\ F_V ,\ F_A$$

$$\{ Q_\pm , \bar{Q}_\pm \} = H \pm P \ , \quad \text{other } \{ Q , \overset{(-)}{Q} \} = 0$$

$$[F_V , Q_\pm] = - Q_\pm \ , \quad [F_V , \bar{Q}_\pm] = Q_\pm$$

$$[F_A , Q_\pm] = \mp Q_\pm \ , \quad [F_A , \bar{Q}_\pm] = \pm \bar{Q}_\pm$$

$$Q_A = \bar{Q}_+ + Q_- \quad , \quad Q_B = \bar{Q}_+ + \bar{Q}_-$$

$$(Q, H, F) = (Q_A, H, F_A) \text{ or } (Q_B, H, F_V)$$

$$\begin{cases} Q^2 = 0 = Q^{+2} \\ \{Q, Q^+\} = 2H \\ [F, Q] = Q \end{cases}$$

$$\boxed{\begin{array}{c} A \leftrightarrow B \\ Q_- \leftrightarrow \bar{Q}_- \\ F_A \leftrightarrow F_V \end{array}}$$

mirror autom.

X : Kähler ω

(0,2)

$[\bar{Q}_+, \mathcal{O}] = 0$

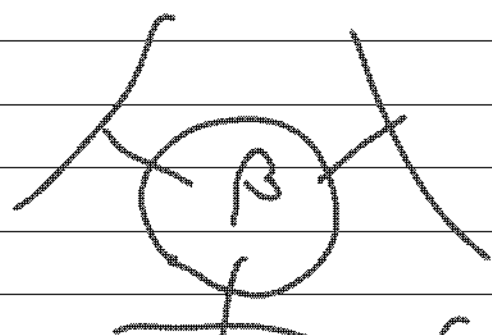
- Chiral operator $\mathcal{O}$: $\bigoplus_{p,q} H^{0,p}(X, \Lambda^q T_X)$

$$[\bar{Q}_\pm, \mathcal{O}] = 0$$

Q_B-cohomology ring of operators "chiral ring"

B-ring

- twisted chiral operator $\tilde{\mathcal{O}}$ $\bigoplus H^\bullet(X)$



$$[\bar{Q}_+, \tilde{\mathcal{O}}] = [Q_-, \tilde{\mathcal{O}}] = 0$$

$\sum_\beta n_\beta e^{-\int_\beta \omega}$

Q_A-coh. ring of oprs "twisted chiral ring"

A-ring

chiral $\quad \phi, \psi_\pm, F$

$\uparrow$ chiral

$$x^0 \leftarrow \underbrace{x^0 \overbrace{x^1 \; x^2}^{\text{reduce}} x^3}_{1+1 \text{ dim}} \rightarrow x^1$$

twisted chiral $\quad \tilde{\phi}, \tilde{\psi}_\pm, \tilde{F}$

$\uparrow$ twisted chiral

$$\boxed{\sigma = v_1 - i v_2}$$

Vector $\quad v_\mu, \sigma, \lambda_\pm, D \qquad (v_\mu = v_0, v_1)$

$$\hookrightarrow \sigma, \lambda_\pm, D - i v_{01} \leftarrow \text{twisted chiral}$$

$$v_{01} = \partial_0 v_1 - \partial_1 v_0 + i[v_0, v_1]$$

gauge group G $\leftarrow$ compact $\qquad \mathfrak{g} = \mathrm{Lie}(G)$

$\phi \in$ matter repr V $\leftarrow$ f.d. rep$/\mathbb{C}$ of G

Superpotential $W(\phi)$ $\leftarrow$ gauge inv.

Twisted superpotential $\widetilde{W}(\sigma)$

G-gauge field $\quad \mathfrak{g}_{\mathbb{C}} \quad \mathfrak{g}$

$(\sigma, v_\mu, \lambda_\pm, D)$

☆ contents : • $\phi, \psi_\pm, F$ in V + • vector of G

☆ $$L = \int d^4\theta \left(\bar{\Phi} e^V \Phi - \frac{1}{2e^2} \mathrm{Tr}(\Sigma^\dagger \Sigma) \right)$$

$$+ \mathrm{Re} \int d^2\theta \, W(\Phi) \Big|_{\bar{\theta}^\pm = 0} + \mathrm{Re} \int d^2\tilde{\theta} \, \widetilde{W}(\Sigma) \Big|_{\bar{\theta}^+ = \theta^- = 0}$$

<u>Assumption</u> $\widetilde{W}(\sigma) = -t(\sigma)$;

$t = \zeta - i\theta : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}$ G-inv linear form.

---

component $D_\mu = \partial_\mu + i v_\mu$

$$L = |D_\mu \phi|^2 + i\bar{\psi}\not{D}\psi + \bar{F}F$$
$$+ \frac{1}{2e^2}\mathrm{Tr}\, v_{01}^2 + \frac{1}{e^2}|D_\mu\sigma|^2 + \frac{i}{e^2}\mathrm{Tr}\,\bar{\lambda}\not{D}\lambda + \frac{1}{2e^2}\mathrm{Tr}\, D^2$$
$$- \frac{1}{2}\bar{\phi}\{\bar{\sigma},\sigma\}\phi - \frac{1}{4e^2}\mathrm{Tr}([\sigma,\bar{\sigma}]^2) + \bar{\phi}D\phi$$
$$- \bar{\psi}\sigma\psi + \bar{\phi}\lambda\psi + \bar{\psi}\lambda\phi + \mathrm{tr}\,\bar{\lambda}[\sigma,\lambda]/e^2$$
$$+ (F\partial W + W''\psi\psi + \text{c.c.})$$
$$- \zeta(D) + \theta(v_{01})$$

eliminate F, D

$$\overset{\downarrow}{=} \quad \text{kin} + \text{Yukawa} - U(\sigma,\phi) + \theta(v_{01})$$

$$U(\sigma,\phi) = \frac{1}{2}|\sigma\phi|^2 + \frac{1}{2}|\bar{\sigma}\phi|^2 + \frac{1}{4e^2}\mathrm{Tr}[\sigma,\bar{\sigma}]^2$$

$$+ \frac{e^2}{2}\left(\phi\bar{\phi} - \xi\right)^2 \leftarrow \text{D-term potential}$$

$$+ |dW(\phi)|^2 \leftarrow \text{F-term potential}$$

$\langle \phi\bar{\phi}, \underset{\in \mathfrak{g}}{X} \rangle$

$:= \bar{\phi} X \phi$

"moment map"

$: V \to \mathfrak{g}^*$

1st main Example :

$N, d \in \mathbb{N}$

$G = U(1)$

$V \ni p$ charge $-d$, $X_1, \cdots, X_N$ charge 1

$W = p f(x_1, \cdots, x_N)$ degree d polynomial

$\widetilde{W} = -t \cdot \sigma \qquad t = \zeta - i\theta \in \mathbb{C}/2\pi i \mathbb{Z}$

$$U = |-d p \sigma|^2 + \|\sigma X\|^2$$
$$+ \frac{e^2}{2}\left(-d|p|^2 + \|X\|^2 - \zeta\right)^2$$
$$+ |f(x)|^2 + \|p\, df(x)\|^2$$

$2d \qquad 2(d-1)$

$$\phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F$$

R-symmetry $\quad q \quad \phi, \psi_\pm, F$

$$\xrightarrow{V} e^{iq\alpha}\phi,\ e^{i(q-1)\alpha}\psi_\pm,\ e^{i(q-2)\alpha}F$$

$$\xrightarrow{A} e^{i\tilde{q}\beta}\phi,\ e^{i(\tilde{q}\mp 1)\beta}\psi_\pm,\ e^{i\tilde{q}}F$$

<u>$U(1)_V$</u> $\quad W = p f(x)$

$$\left\{\begin{array}{lcccc} & P, & x_1, & \cdots, & x_N \\ q: & 2 & 0 & \cdots & 0 \\ \tilde{q}: & 0 & 0 & \cdots & 0 \end{array}\right. \qquad \checkmark$$

<u>$U(1)_A$</u> $\quad \tilde{W} = -t\sigma \quad \tilde{q} \quad \tilde{\phi}, \tilde{\psi}_\pm, \tilde{F}$

$$\xrightarrow{A} e^{i\tilde{q}\beta}\tilde{\phi},\ e^{i(\tilde{q}-1)\beta}\tilde{\psi}_\pm,\ e^{i(\tilde{q}-2)\beta}\tilde{F}$$

$$\xrightarrow{V} e^{iq\alpha}\tilde{\phi},\ e^{i(q\mp 1)\alpha}\tilde{\psi}_\pm,\ e^{iq\alpha}\tilde{F}$$

$$\left\{\begin{array}{l} \sigma \\ \tilde{q}\ 2,\ q=0 \end{array}\right.$$

<u>anomaly</u>

$$\left.\begin{array}{l} -d + \underbrace{1+\cdots+1}_{N} = -d+N \\ \text{Only when } d=N:\ \exists\, U(1)A \end{array}\right\} U(1)_A \longrightarrow \mathbb{Z}_{2(d-N)}$$

$U = 0$?

$\underline{\zeta > 0}$ $\quad \|X\|^2 \neq 0$: $U(1) \xrightarrow{\text{is broken to}} \{1\}$; $\sigma = 0$

$\boxed{\text{Assume}} \to$ genericity of $f(x)$: $\quad f(x) = df(x) = 0 \Rightarrow X = 0$

e.g. $f(x) = x_1^d + \dots + x_N^d$

$\Longrightarrow \quad p = 0$

size of X_f

$\zeta \to$ metric

$\theta \to$ B field

remain: $\|X\|^2 = \zeta$, $\quad f(x) = 0$

without fixed pt.

$$\{\text{Vac config}\}/\cong \;=\; \left\{ x \in \mathbb{C}^N \,\middle|\, \begin{array}{l} \|x\|^2 = \zeta \\ f(x) = 0 \end{array} \right\} / U(1)$$

$$\cong \{ x \in \mathbb{C}^N \setminus 0 \mid f(x) = 0 \} / \mathbb{C}^\times$$

$$= \text{hypersurface } f = 0 \text{ of } \mathbb{CP}^{N-1} =: X_f$$

· Transverse to $\underline{\{vac\}}$ are all massive

$\longrightarrow$ At low energies, we expect to have

$t = \zeta - i\theta \rightsquigarrow [\omega - iB]$

non-linear σ-model

with target $= X_f \subset \mathbb{CP}^{N-1}$

<u>Remarks</u> : <u>space-time dim > 2</u> : $\{vac\} \approx$ moduli space of theories.

<u>dim = 2</u> : IR divergence : $\{vac\} =$ target space.

$\underline{\zeta < 0}$ $\quad p \neq 0 : U(1) \to \mathbb{Z}_d \; ; \; \sigma = 0$

genericity of $f \Longrightarrow X = 0$

$\mathbb{Z}_d$ trivially

$$\{\text{Vac}\} = \left\{ p \;\middle|\; |p|^2 = -\frac{\zeta}{d} \right\} / U(1)$$

$$= \text{one point} / \mathbb{Z}_d$$

Out of the transverse to $\{\text{vac}\}$ (for $d \geq 3$)

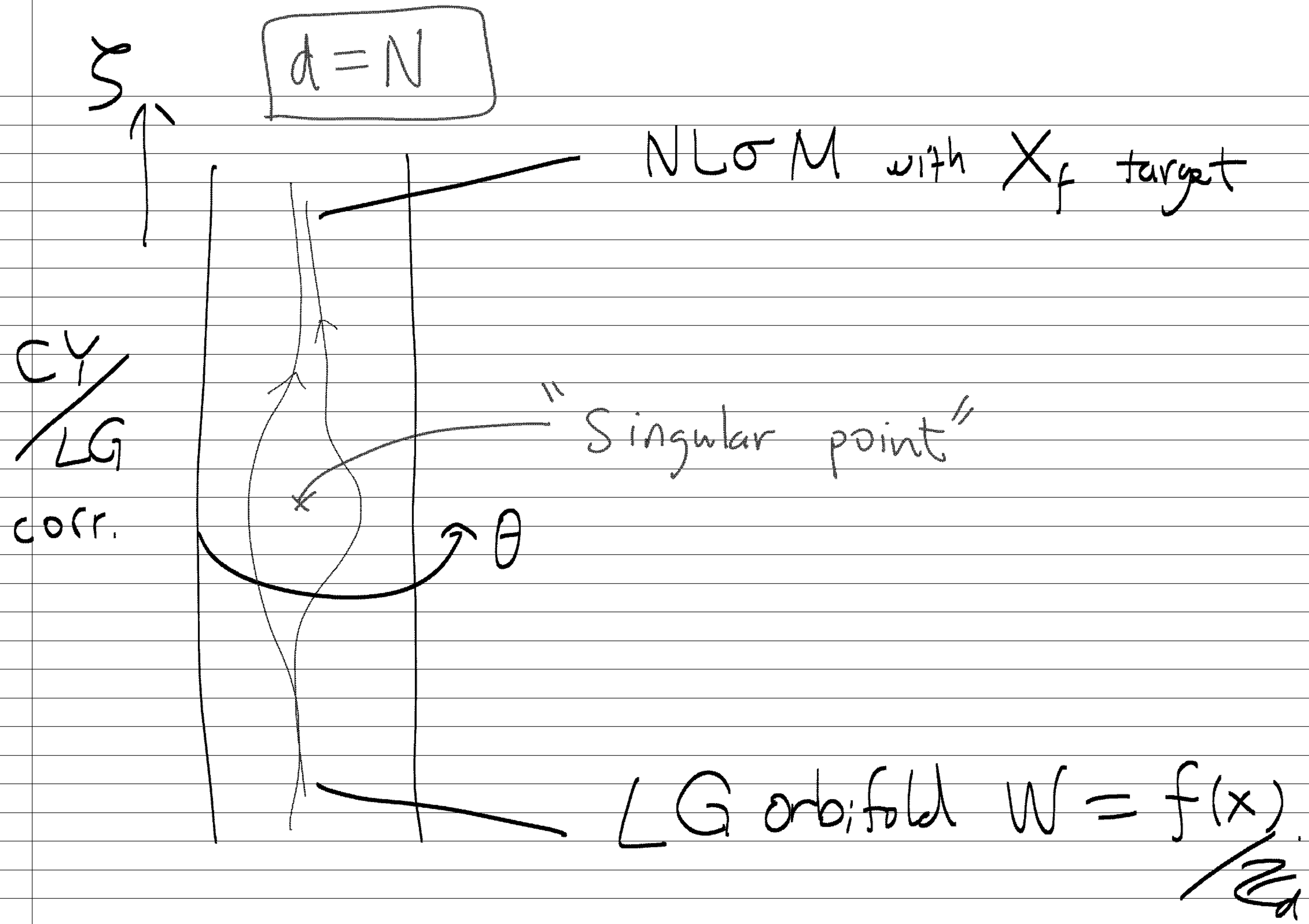
X's are massless.

At low energies, we expect to have

a theory of X's, $\mathbb{Z}_d$ gauge sym.

with $W = \langle p \rangle f(X_1, \cdots, X_N)$

$\longrightarrow$ "Landau-Ginzburg orbifold".



$d = N$
ζ
NLσM with X_f target
CY/LG corr.
"Singular point"
θ
LG orbifold $W = f(x)/\mathbb{Z}_d$

$d > N$

$\underbrace{\cdot \; \cdot \; \cdots \; \cdot}_{d-N}$ massive vacua

NLSM X_f

RG flow

LG orb $f/\mathbb{Z}_d$

$d < N$

X_f

$f/\mathbb{Z}_d$

$\underbrace{\cdot \; \cdot \; \cdots \; \cdot}_{N-d}$ massive vacua

D = spacetime dim

$$\int d^D x \, \partial_\mu \phi \, \partial^\mu \phi$$

$\frac{1}{x^2}$

$D > 2$: ∞ energy to change ϕ at ∞

$D \leq 2$:

$$\langle \phi(x) \phi(0) \rangle \sim \frac{1}{|x|^{D-2}} \quad D > 2 \quad : \phi \to \phi_{vac} \text{ at } |x| \to \infty \text{ is possible}$$

$$\log |x| \quad D = 2 \quad : \phi \to \phi_{vac} \quad |x| \to \infty \text{ is impossible}$$

$\underline{\zeta = 0}$

$\left.\begin{array}{l} \text{D-term \& F-term} \\ |x|^2 \propto |p|^2, \quad \check{p}\, df(x) = 0 \end{array}\right\}$ $x = p = 0$

$U(1) \to U(1)$ unbroken

"Singular"

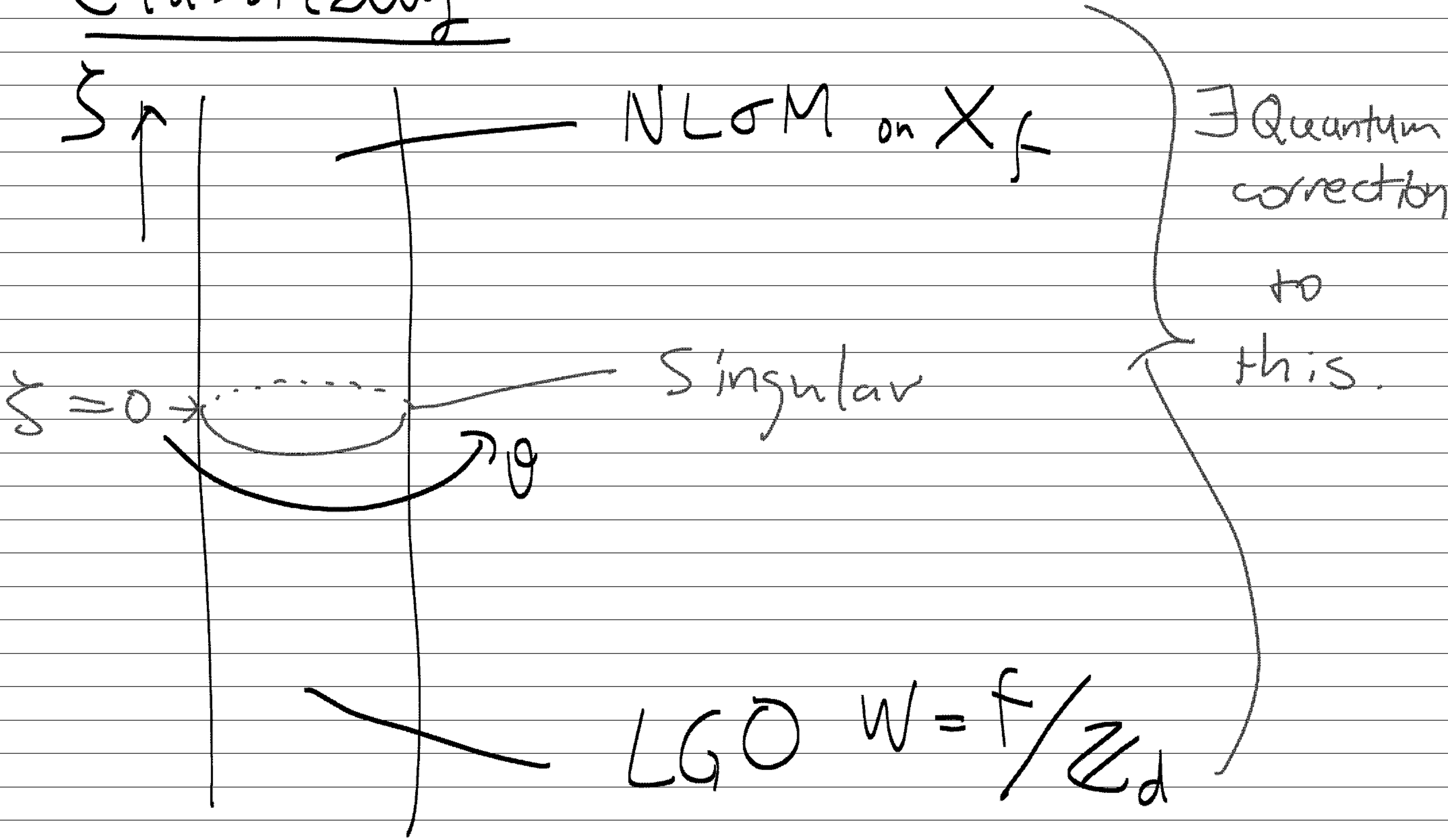
$\boxed{\sigma : \text{free}}$

Coulomb branch $\quad \sigma \in \mathbb{C}$ ⋯ non-compact

continuous spectrum

Recall $\left.\begin{array}{l} \zeta > 0 : X_f \text{ compact} \\ \zeta < 0 : U = \|df(x)\|^2 \end{array}\right\}$ discrete spectrum

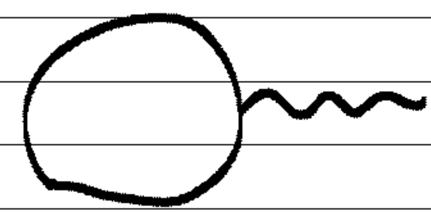
Classically

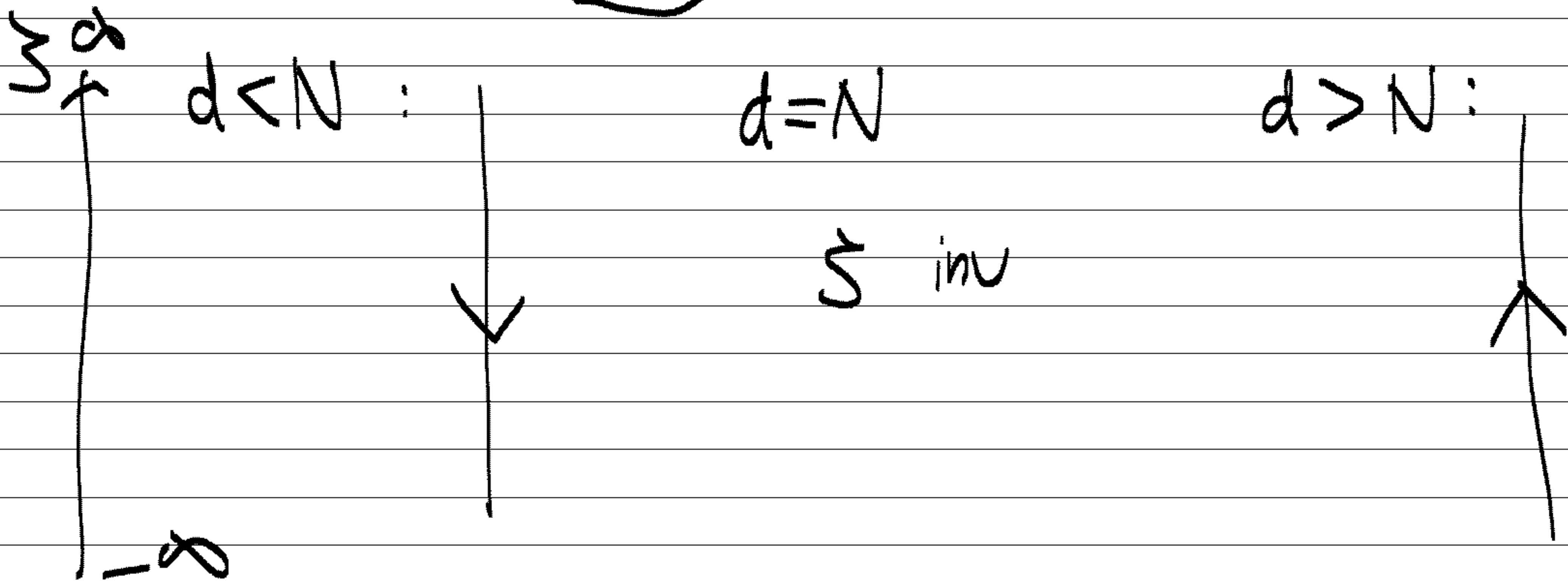


- Aixal anomaly $\propto N-d$

  $N \neq d$: θ is unphysical (absorbed by field redef.)

- Running of FI : $\mu \dfrac{d\xi}{d\mu} = N-d$





- Electrostatic energy :

$\theta =$ background electric field (Coleman)

$$\rightsquigarrow E = `\frac{e^2}{2}\theta^2' = \min_{n\in\mathbb{Z}} \frac{e^2}{2}(\theta+2\pi n)^2$$

Important to see if Coulomb branch is a true flat direction

$$U \underset{\substack{\uparrow \\ \text{large } |\sigma|}}{=} \frac{e^2}{2}\zeta^2 + \frac{e^2}{2}\theta^2$$

real flat direction $\exists$ when $\zeta = 0,\ \theta \equiv 0 \ (2\pi\mathbb{Z})$

More precisely we need to integrate over

X's & P.

$$t_{eff} = \zeta_{eff} - i\,\theta_{eff}$$

$|\sigma|$ large

$$= -\frac{\partial \widetilde{W}_{eff}}{\partial \sigma} = \underbrace{N}_{t+} \cdot \log\sigma + (-d)\log(-d\sigma)$$

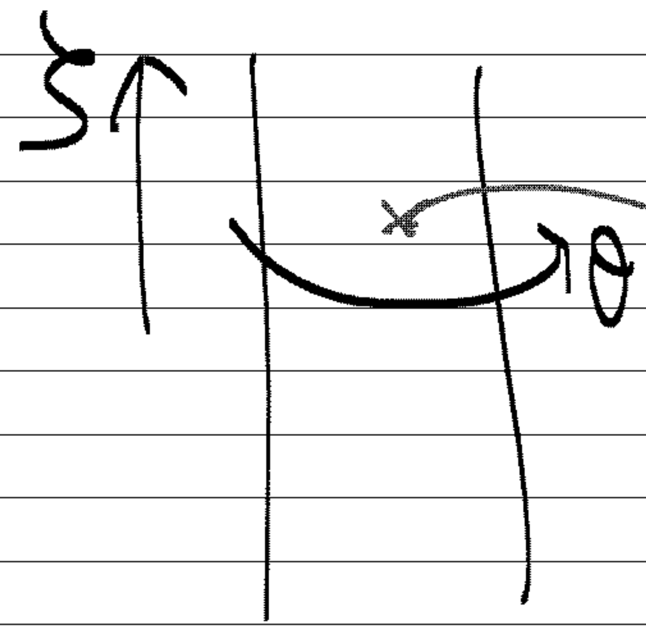
large σ is a true SUSY ground state when $t_{eff} \equiv 0 \pmod{2\pi i}$

$$e^{-t} = \frac{\sigma^N}{(-d\sigma)^d} = (-d)^{-d}\sigma^{N-d}$$

$d \neq N$: $|d-N|$ solutions to the vac eqn.

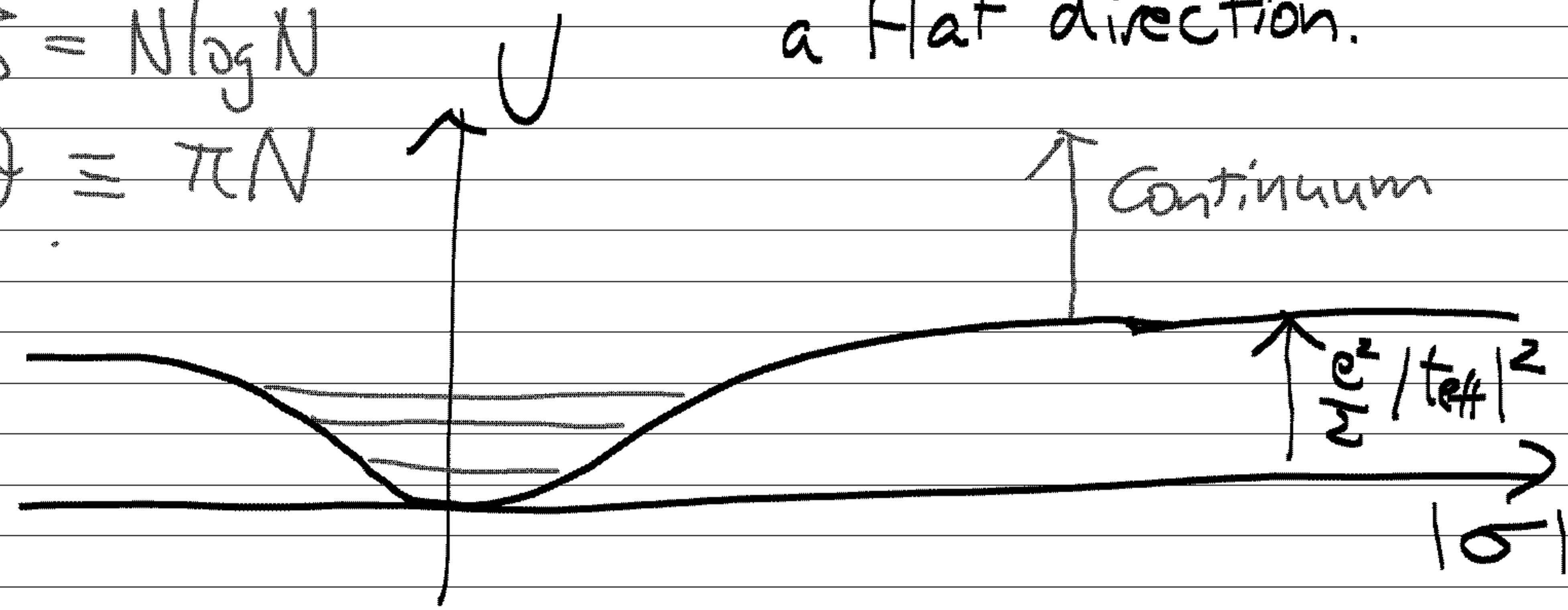
all massive.

$d = N$: $e^{-t} = (-N)^{-N}$ ← only at this point Coulomb branch is a flat direction.



$\xi = N \log N$

$\theta \equiv \pi N$



X_f a hypersurface of degree d in $\mathbb{CP}^{N-1}$ $\Big\}$ LGO $W = f/\mathbb{Z}_d$

$\hat{c} = N-2$ $\qquad\qquad$ $\hat{c} = N\left(1 - \frac{2}{d}\right)$

· $d = N$ $\quad$ It is Calabi-Yau.

· $d < N$ $\quad$ Fano … Ricci > 0

$\longrightarrow$ LGO $W = f/\mathbb{Z}_d$ (or $N-d$ massive vacua)

· $d > N$ $\quad$ general type ·· Ricci < 0 (or $d-N$ massive vacua)

$$\mu \frac{d g_{\mu\nu}}{d\mu} \underset{\text{1-loop}}{=} * R_{\mu\nu} \longleftarrow \text{LGO } W = f/\mathbb{Z}_d$$

2nd main example

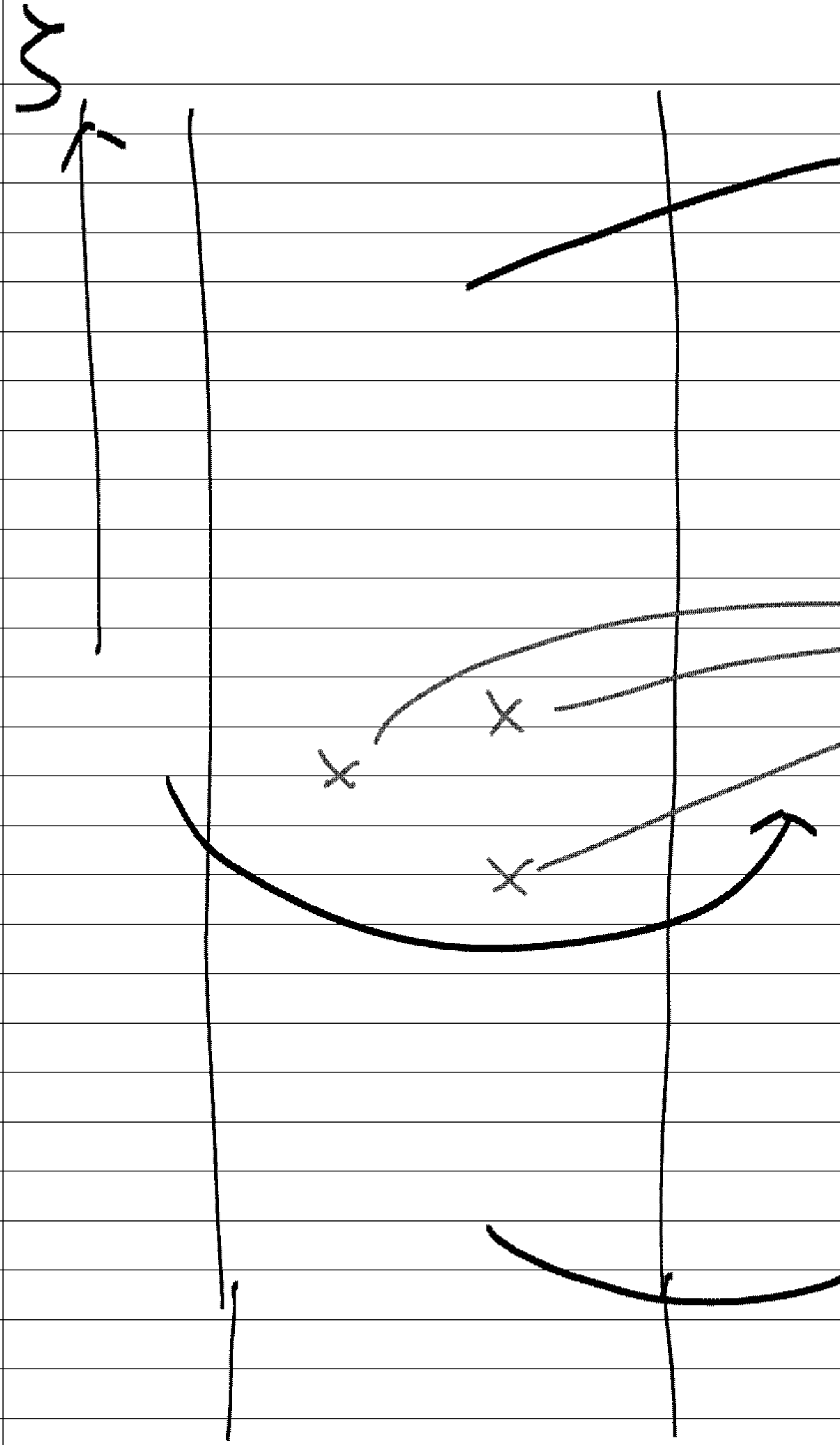
$G = U(2) \ni g$ $\qquad p^i \longrightarrow \det^{-1} g \, p^i$

$V \ni \underbrace{p^1, \ldots, p^7}_{\det^{-1}}, \underbrace{x_1, \ldots, x_7}_{\text{fundamental}}$ $\qquad x_i \longrightarrow g x_i$

$$W = \sum_{i,j,k} A^{ij}_k p^k [x_i x_j] \qquad [x_i x_j] = x_i^1 x_j^2 - x_i^2 x_j^1$$

$$\left(= \sum_{i,j} A^{ij}(p) [x_i x_j] = \sum_k p^k A_k(x) \right)$$

$$\widetilde{W} = -t \cdot \mathrm{tr}\,\sigma \qquad t = \zeta - i\theta \in \mathbb{C}/2\pi i \mathbb{Z}$$



NLσM with target

$U(2) \to \{1\}$: $X_A = (A_1(x) = \cdots = A_7(x) = 0)$

$\subset G(2,7)$

Singular points

$\exists$ true Coulomb branch

$U(2) \to SU(2) \rightsquigarrow$ NLσM with target

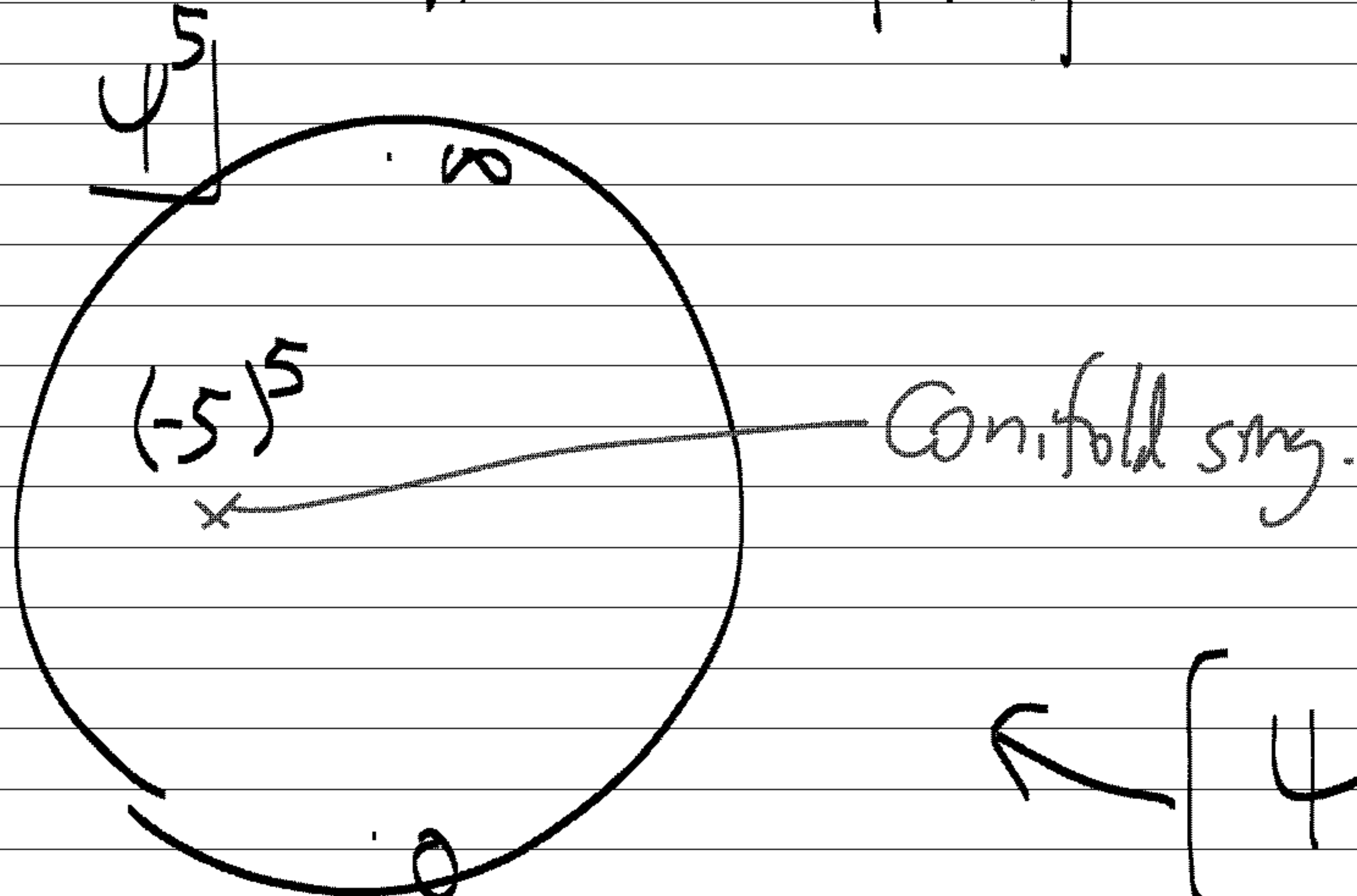
$Y_A = \{ p \mid \mathrm{rank}\, A(p) = 4 \}$

$\subset \mathbb{CP}^6$

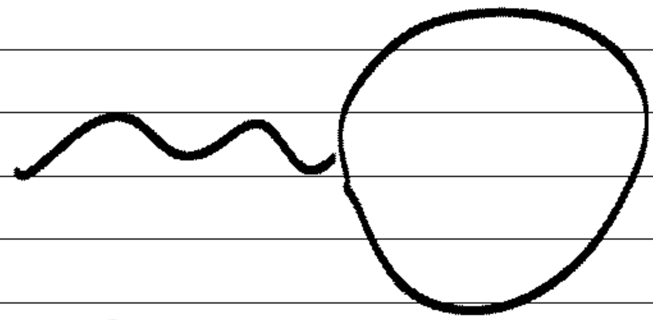
$d = N = 5$: X_f quintic CY 3-fold

<u>Candelus et al</u> (1991 ?)

mirror of X_f ···· $\dfrac{\{ z_1^5 + \cdots + z_5^5 + \psi z_1 \cdots z_5 = 0 \}}{\mathbb{Z}_5^3 \ : \ z_i \to \omega_i z_i \quad \omega_i^5 = 1, \ \omega_1 \cdots \omega_5 = 1}$



$\leftarrow \left[\psi = e^{t/5} \right]$



$$\frac{1}{4e^2}\mathrm{Tr}[\sigma,\bar{\sigma}]^2 \rightsquigarrow \sigma \in \mathfrak{t}_{\mathbb{C}}$$

$$\widetilde{W}_{eff} = -t(\sigma) + \sum_{\alpha>0} \pm \pi i \alpha(\sigma) - \sum_i Q_i(\sigma)\left(\log Q_i(\sigma) - 1\right)$$

$$V = \bigoplus_i V_{Q_i}$$

$G = U(1)$

$$V \ni \begin{array}{cccc} P & X_1 & \cdots & X_N \\ -d & 1 & \cdots & 1 \end{array}$$

$$\widetilde{W}_{eff} = -t\sigma - (-d\sigma)\left(\log(-d\sigma) - 1\right) - N \cdot \sigma\left(\log\sigma - 1\right)$$

$$t_{eff} = -\frac{\partial \widetilde{W}_{eff}}{\partial \sigma} = t - d\log(-d\sigma) + N\log\sigma$$

$V \ni p^1 \cdots p^7, \, x_1 \cdots x_7$

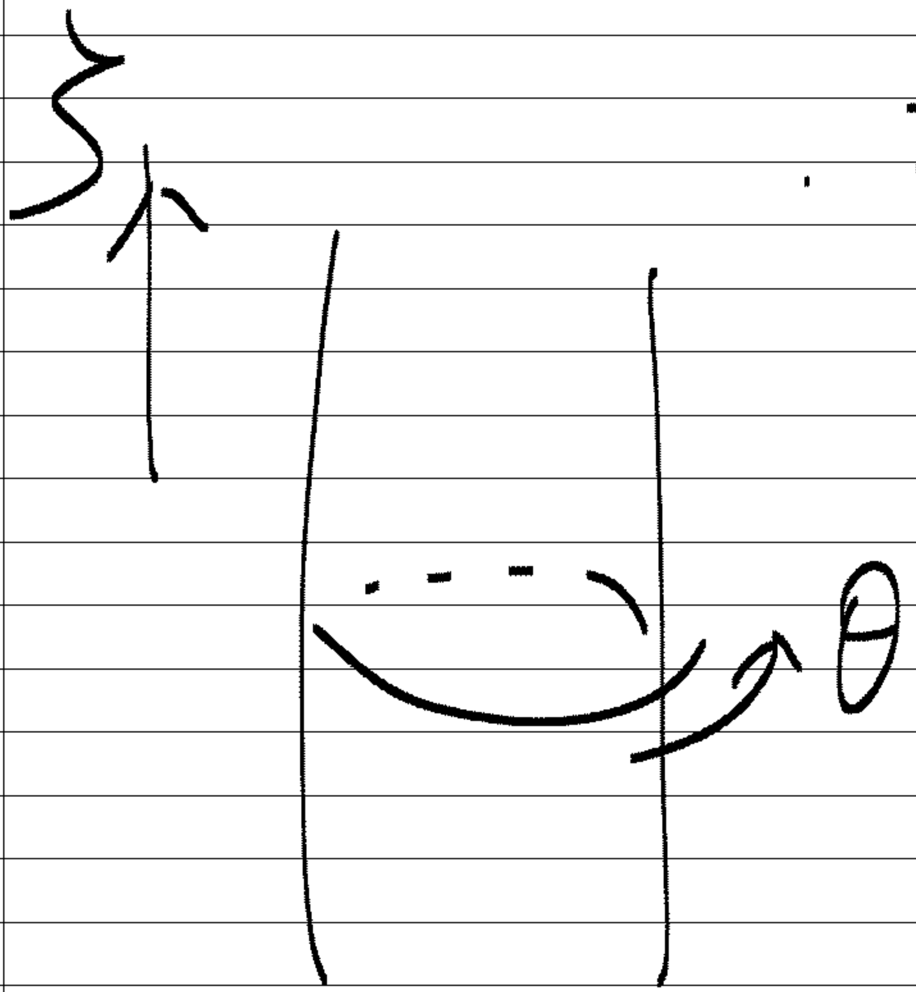
$G = U(2) \subset SL(V)$

$(\det^{-1} g)^7 (\det g)^7 = 1$

"Calabi–Yau cond."

$\rightsquigarrow$ $U(1)_A$ anomaly free (θ physical)

FI ζ does not run



$$U(\sigma, p, x) = \left\| - \mathrm{tr}(\sigma)\cdot p \right\|^2 + \frac{1}{2}\left\| \sigma X \right\|^2 + \frac{1}{2}\left\| \bar{\sigma} X \right\|^2$$

$$+ \frac{1}{4e^2} \mathrm{Tr}\,[\sigma, \bar{\sigma}]^2$$

2×7 :

$$X = \begin{pmatrix} x_1^1 & \cdots & x_7^1 \\ x_1^2 & & x_7^2 \end{pmatrix} \quad + \frac{e^2}{2} \mathrm{Tr}\left(XX^{\dagger} - \|p\|^2 \mathbb{1}_2 - \zeta \mathbb{1}_2 \right)^2$$

$$+ \underbrace{\sum_k \left| A_k(x) \right|^2}_{X^4} + \sum_{i,a} \underline{\left| A^{ij}(p)\, X_j^a \right|^2}$$

$$\underline{\zeta > 0} \qquad X \text{ rank } 2 : U(2) \to \{1\}$$

$$; \ \sigma = 0$$

$$A^{ij}_{k} \text{ generic} : \xRightarrow{X \text{ rk} 2} \ p = 0$$

$$\{\text{vac}\}/\sim \ = \left\{ X : 2\times 7 \ \middle| \ \begin{array}{c} XX^{\dagger} = \zeta 1_2 \\ A_1(X) = \cdots = A_7(X) = 0 \end{array} \right\} / U(2)$$

$$\cong \{ X : \text{rk} 2 \ 2\times 7 \mid A_1(X) = \cdots = A_7(X) = 0 \} / GL(2,\mathbb{C})$$

$$X = (X_1 \cdots X_7) = \begin{pmatrix} \vec{X}^1 \\ \vec{X}^2 \end{pmatrix} \sim \text{rk} 2 \text{ subsp} \subset \mathbb{C}^7$$

$$= (A_1(X) = \cdots = A_7(X) = 0) \subset G(2,7)$$

all transv to vac are massive.

$\therefore$ Non-linear σ-model

with target X_A

$:= (A_1 = \cdots = A_7 = 0) \subset G(2,7)$.

$$\begin{cases} \zeta \sim [\omega] \\ \theta \sim [B] \end{cases}$$

$\underline{\zeta < 0}$

$$\left.\begin{array}{l} p \neq 0 \\ X = 0 \end{array}\right\} U(2) \to SU(2)$$

$$\boxed{\mathrm{Tr}\,\sigma = 0}$$

$\rightsquigarrow$ We need to understand $SU(2)$ gauge theory to find the low energy behaviour.

(later)

$\underline{\zeta = 0}$ $\quad \underline{X = \rho = 0}$ allowed $\rightsquigarrow$ required.

$$U = \frac{1}{2e^2} \mathrm{Tr}[\sigma, \bar{\sigma}]^2$$

$$\Longrightarrow [\sigma, \bar{\sigma}] = 0 \quad \text{i.e. } \sigma \text{ diagonalizable}$$

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \; : U(2) \to U(1)^2$$

Coulomb branch

Quantum: int-out X & P

$$
\begin{aligned}
\widetilde{W}_{eff} = & -t(\sigma_1+\sigma_2) + \pi i(\sigma_1-\sigma_2) \\
& - 7 \cdot \sigma_1 (\log\sigma_1 - 1) \\
& - 7 \cdot \sigma_2 (\log\sigma_2 - 1) \\
& - 7(-\sigma_1-\sigma_2)(\log(-\sigma_1-\sigma_2) - 1)
\end{aligned}
$$

$$t^1_{eff} = -\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_1} = t - \pi i + 7\log\sigma_1 - 7\log(-\sigma_1-\sigma_2)$$

$$t^2_{eff} = -\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_2} = t + \pi i + 7\log\sigma_2 - 7\log(-\sigma_1-\sigma_2)$$

<u>Vac eq</u> these $\equiv 0 \mod 2\pi i\mathbb{Z}$

$$e^{-t} = -\frac{\sigma_1^7}{(-\sigma_1-\sigma_2)^7} = -\frac{\sigma_2^7}{(-\sigma_1-\sigma_2)^7}$$

$$\sigma_1^7 = \sigma_2^7 \quad \therefore \sigma_2/\sigma_1 = \omega \ ; \omega^7 = 1.$$

$$e^t = (1+\omega)^7$$

Weyl of $U(2)$: $\sigma_1 \leftrightarrow \sigma_2$ ie $\omega \leftrightarrow \omega^{-1}$

$$\omega = 1, \ \omega_0, \ \omega_0^2, \ \omega_0^3, \ \omega_0^4, \ \omega_0^5, \ \omega_0^6$$

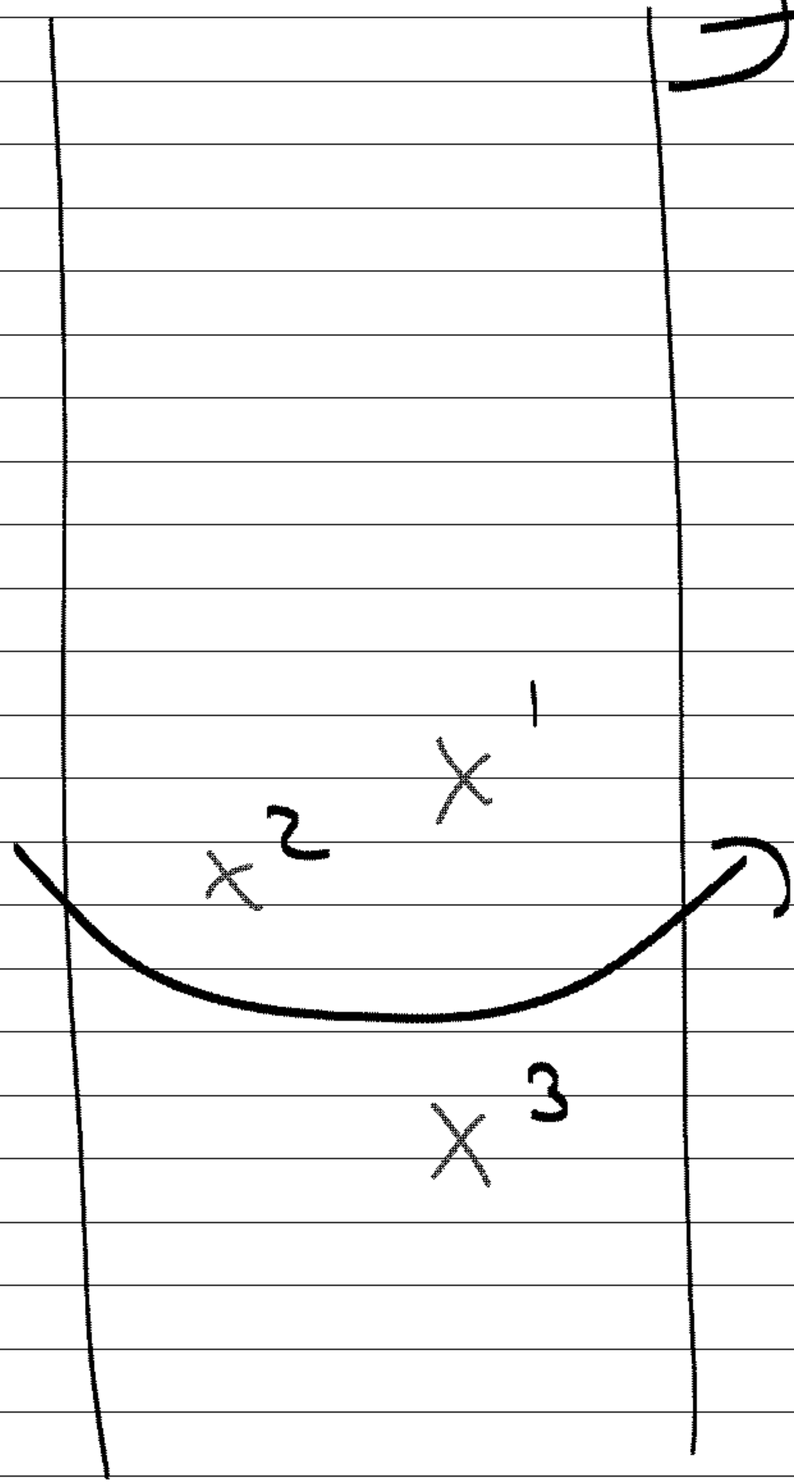
$\exists$ evidences that

SUSY ground state

is not supported near

$$\sigma_1 = \sigma_2 .$$

$\underline{w = 1 \text{ : excluded}}$



$\exists$ Three singular points:

$$w = (1+w_0)^7 \sim e^{4.1}$$

$$(1+w_0^2)^7 \sim e^{1.5}$$

$$(1+w_0^3)^7 \sim -e^{-5.6}$$

Back to $\zeta < 0$

Vac eq $\Rightarrow$ $P \neq 0$, $X = 0$: $U(2) \to SU(2)$

Theory $\sim$ $SU(2)$ with 7 X's mass $A(P)$

fibered over $\{P\} = \mathbb{C}P^6$

First solve $(SU(2), X)$ part

$A(p)$: 7×7 antisymmetric matrix

$\xrightarrow{\text{rk}}$ $6,\ 4,\ \cancel{2},\ \cancel{0}$

$\mathrm{Pf}\neq 0$
genericity of A^{ij}_k

$\mathbb{CP}^6$

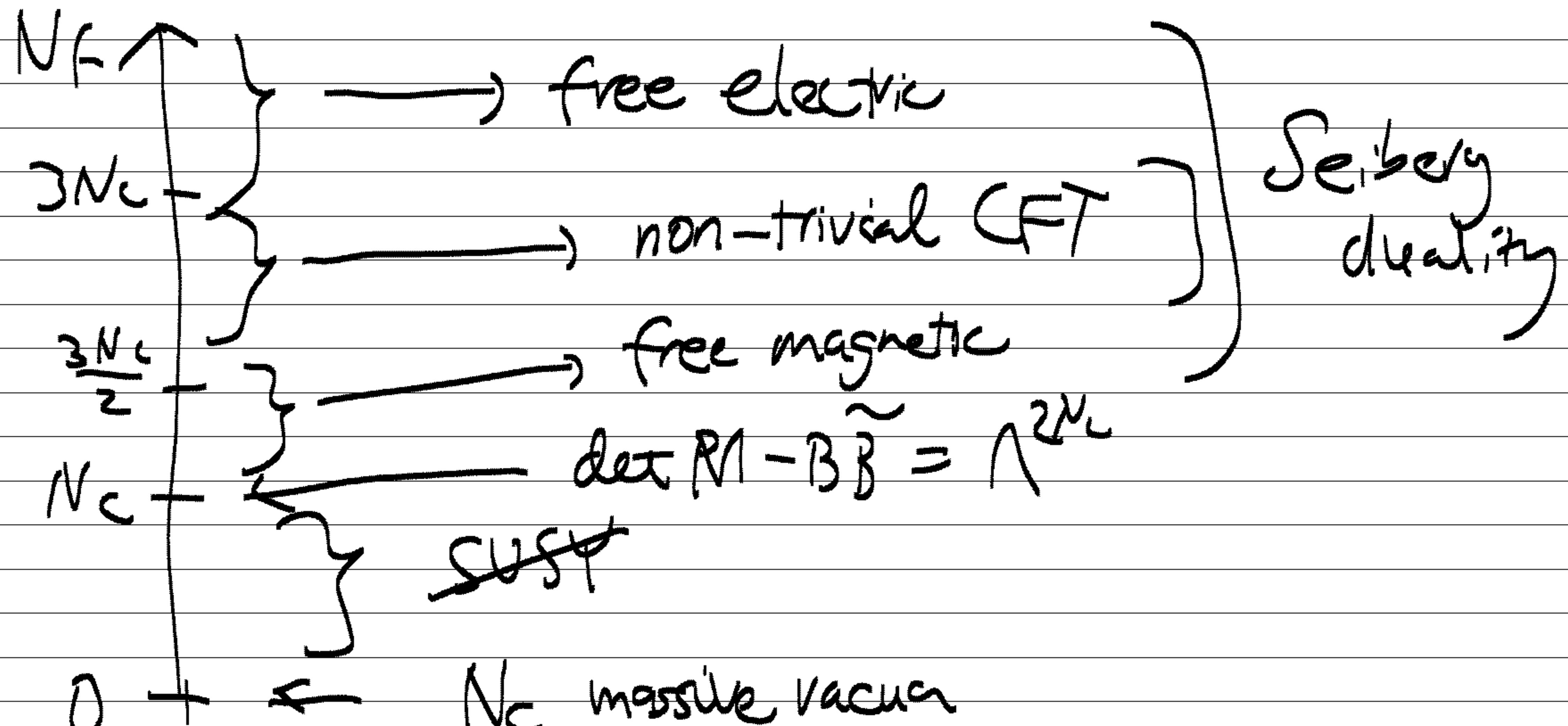
$\text{rk}\, A(p)=6$: $N_{\text{eff}}=1$

$\text{rk}\, A(p)=4$: $N_{\text{eff}}=3$

Y_A : Pfaffian CY^3

SU(2) with N fundamentals

<u>aside</u> 4d $\mathcal{N}=1$ $SU(N_c)$, N_f $Q, \tilde{Q}$



2d (2,2) SU(2), N funds

N even : $\exists$ Coulomb branch "Singular"

N odd : Coulomb branch is lifted "regular"

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}$$

$$\widetilde{W}_{eff} = \pi i (2\sigma_1) - N\sigma_1(\log\sigma_1 - 1) - N(-\sigma_1)(\log(-\sigma_1) - 1)$$

$$= \pi i (2-N)\sigma_1 \equiv N\pi i \sigma_1$$

N

$\vdots$

$N \geq 4$: non-trivial CFTs

$\exists$ duality $\begin{cases} SU(N-2) \\ USp(N-2-1) \leftarrow \underline{N \text{ odd}} \end{cases}$

3 $\leftarrow$ "Quantum deformed ~~moduli~~ target space"

2

1

0

$N = 0, 1, 2$: $\nexists$ normalizable SUSY ground state

$SU(2)$ $N=3$ fundamentals

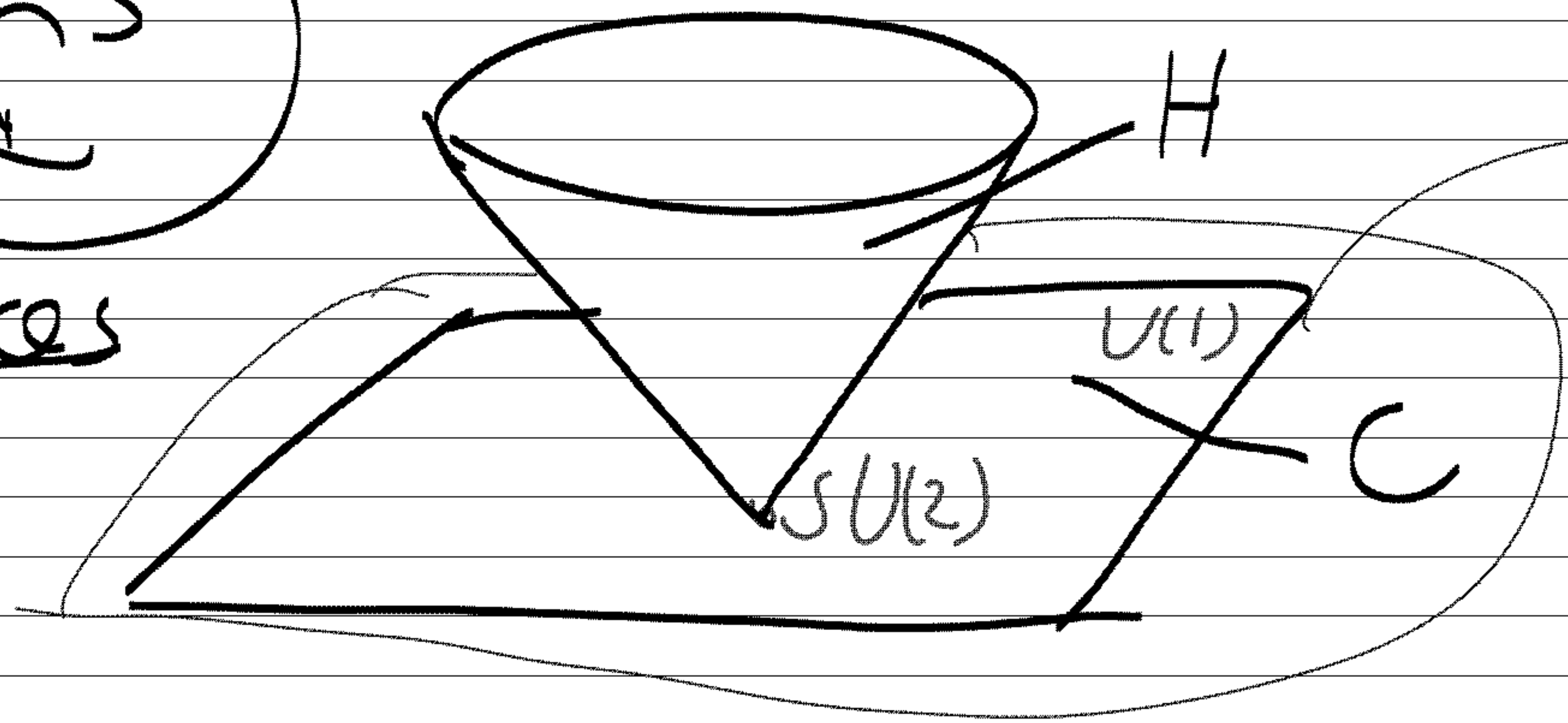
$\xrightarrow{RG}$ The free theory of baryons (meson)

$SU(2) = USp(2)$

$\epsilon_{kij}[X_i X_j] = B_k$ $k=1,2,3$

$\mathbb{C}^3$

evidences



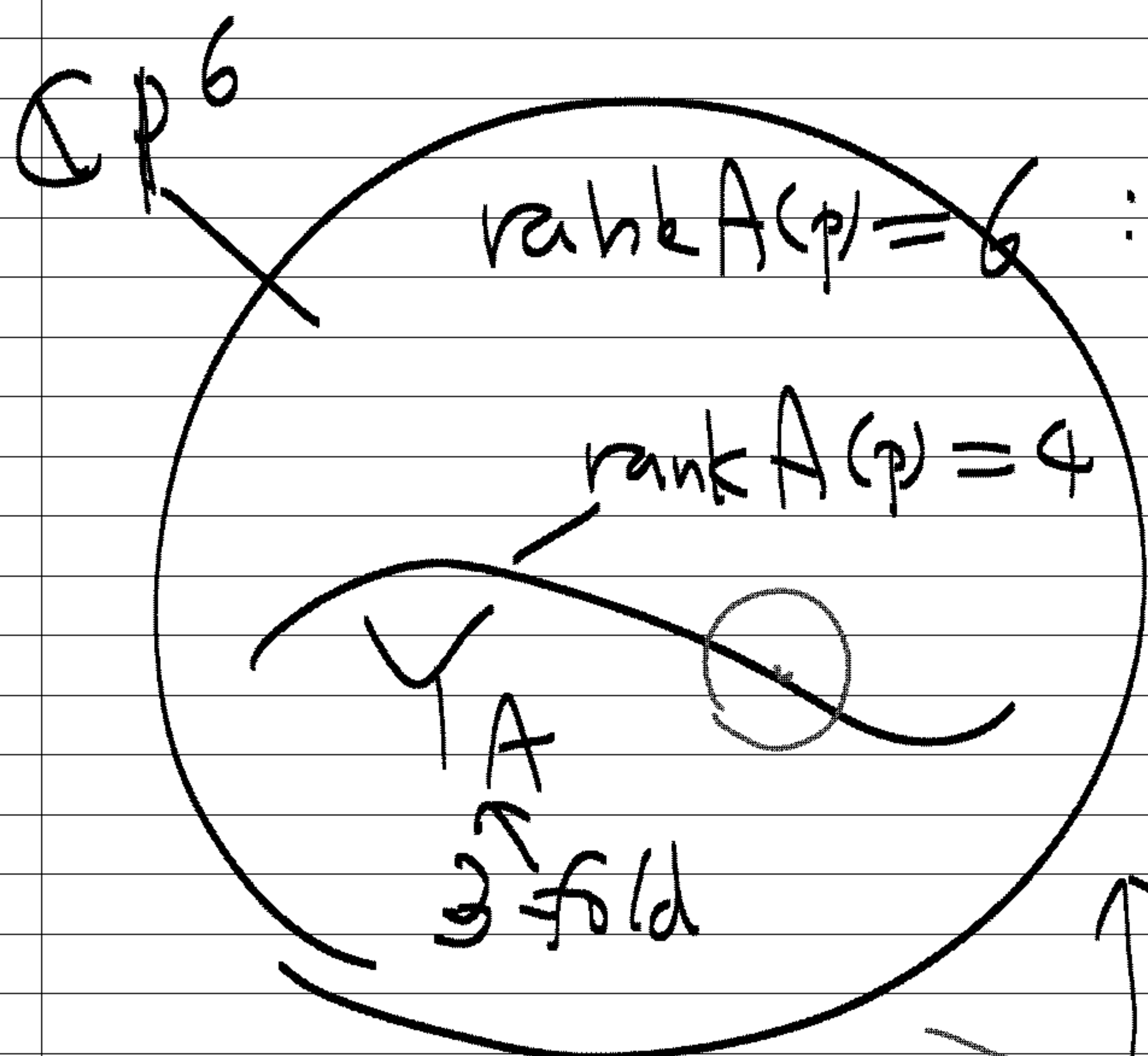
$$\text{holo/alg Functions (Higgs branch)}$$

$$\cong \mathbb{C}[X_1, X_2, X_3]^{SL(2,\mathbb{C})}$$

$$\cong \mathbb{C}\big[\underbrace{[X_1 X_2]}_{=B_3}, \underbrace{[X_2 X_3]}_{=B_1}, \underbrace{[X_3 X_1]}_{=B_2}\big]$$

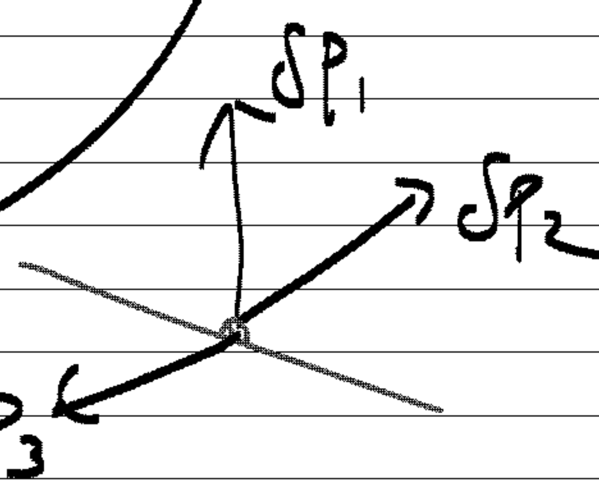
$$\leftarrow \mathbb{C}^3$$

Back to $\zeta < 0$



$\text{rank}\, A(p) = 6$: $N_{eff} = 1$: ~~SUSY~~

$\text{rank}\, A(p) = 4$: $N_{eff} = 3$ $\rightarrow$ free theory of B_1, B_2, B_3



$$A(p) = \begin{pmatrix} \begin{matrix} 0 & +\delta p_3 & - \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & 0 \end{matrix} & 0 \\ 0 & (*)^{4}_{4} \end{pmatrix}$$

$$
A(p) = \left(\begin{array}{ccc|c}
0 & \delta p_3 & -\delta p_2 & \\
-\delta p_3 & 0 & \delta p_1 & 0 \\
\delta p_2 & -\delta p_1 & 0 & \\
\hline
 & 0 & & *
\end{array}\right) \quad 4\times 4
$$

$$
W = A^{ij}(p)\,[X_i X_j] \supset \delta p_3\,[X_1 X_2] - \delta p_2\,[X_1 X_3] + \delta p_1\,[X_2 X_3]
$$

$$= \delta P_3 \cdot B_3 + \delta P_2 \cdot B_2 + \delta P_1 \cdot B_1$$

a function of $\delta P_{1,2,3}$, $B_{1,2,3}$

which has a single critical pt (i.e. $\delta P_{1,2,3} = B_{1,2,3} = 0$) whose

$\det W'' \neq 0$. i.e. a single massive vacuum

i.e. all transv. to Y_A are massive

Left with tangent modes only.

I.e. we have non-linear σ-model with target

$$Y_A = \left\{ p \in \mathbb{P}^5 \;\middle|\; \operatorname{rk} A(p) = 4 \right\}$$

$$SU(2) = USp(2)$$

$$\rightsquigarrow \quad \underset{\text{even}}{USp(k)} = \left\{ g : k \times k \;\middle|\; \begin{array}{l} \text{unitary} \\ g^T J_k g = J_k \end{array} \right\}$$

$$J_k = \left(\begin{array}{c|c} 0 & -\mathbb{1}_{k/2} \\ \hline \mathbb{1}_{k/2} & 0 \end{array} \right)$$

$$SO(k) = \left\{ g : k \times k \;\middle|\; \begin{array}{l} \text{real} \\ g^T \mathbb{1}_k g = \mathbb{1}_k \\ \det g = 1 \end{array} \right\}$$

$USp(k)$ with N fund's $X_1, \cdots, X_N$; $W=0$

- regular (~~Coulomb~~) $\Longleftrightarrow$ N odd
- $N \leq k$: ~~SUSY~~
- $N = k+1$ $\xrightarrow{RG}$ Free theory of $[X_i X_j]$ $\quad [X_i X_j] = X_i^T J_k X_j$
- $N \geq k+3$ (N : odd)

  $\cdots$ $\exists$ duality

$$USp(\overbrace{N-k-1}^{\tilde{k}})$$

N funds $\tilde{X}^1, \ldots, \tilde{X}^N$

4d: -4

3d: -2

$\frac{N(N-1)}{2}$ singlets $(a_{ij})_{i,j=1}^{N}$

antisym.

$$W = \sum_{i,j} a_{ij} [\tilde{X}^i \tilde{X}^j]$$

$$[X_i X_j] \longleftrightarrow a_{ij}$$

Evidences :

- $\underset{\sim}{\hat{C}}$

	$SU(N)$	$U(1)_B$	$U(1)_V$	$U(1)_A$
X	$\square$	1	0	0
$\tilde{X}$	$\bar{\square}$	-1	1	0
a	$\boxminus$	2	0	0

$0 \leftrightarrow 0$

$$\hat{C} = kN - \frac{k(k+1)}{2}$$

$$\parallel$$

$$\hat{C} = \frac{N(N-1)}{2} - \frac{\tilde{k}(\tilde{k}+1)}{2}$$

$$\tilde{k} = N - k - 1$$

- Flow by complex masses

$$W = m\,[X_{N-1} X_N] \longleftrightarrow W = \sum a_{ij} [\tilde{X}^i \tilde{X}^j] + m\, a_{N-1\,N}$$

$$\begin{array}{ccc} USp(k),\ N & \longleftrightarrow & USp(\tilde{k}),\ N,\ \frac{N(N-1)}{2} \\ \downarrow & & \downarrow \quad [\tilde{X}^{N-1}\tilde{X}^{N}] + m = 0 \\ USp(k),\ N-2 & \longleftrightarrow & USp(\tilde{k}-2),\ N-2,\ \frac{(N-2)(N-3)}{2} \end{array}$$

$$\tilde{X}^{N-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i\sqrt{m} \\ 0 \end{pmatrix}, \quad \tilde{X}^{N} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ i\sqrt{m} \end{pmatrix}$$

$$\tilde{X}^{i} = \begin{pmatrix} \tilde{X}^{i}_{1} \\ \vdots \\ \tilde{X}^{i}_{\tilde{k}-2} \\ \tilde{X}^{i}_{\tilde{k}-1} \\ \tilde{X}^{i}_{\tilde{k}} \end{pmatrix}$$

$$\underline{a_{N-1\,i}} \underbrace{\left[\tilde{X}^{N-1}\ \tilde{X}^{i}\right]}_{i\sqrt{m}\,\tilde{X}^{i}_{\tilde{k}}}$$

- Vacuum counting by giving twisted mass

✓

- chiral ring / twisted chiral ring.

✓

$O_\pm(k)$, $SO(k)$, N funds $X_1, \cdots, X_N$, $W=0$

- regular (Coulomb ~~Coulomb branch~~) $\Longleftrightarrow$ $\theta_0 \equiv \pi(N-k+1)$.

- $N < k-1$: ~~SUSY~~

- $N = k-1 \xrightarrow{RG}$ Free theory of $(X_i X_j)$ $\quad \mathbb{C}^{\frac{N(N+1)}{2}}$

  $O_-(k), SO(k) \quad X_i^T X_j$

  $O_+(k)$: two copies $\mathbb{C}^{\frac{N(N+1)}{2}} \sqcup \mathbb{C}^{\frac{N(N+1)}{2}}$

- $N \geq k$ (regular) .... $\exists$ dual theory

$$\left.\begin{array}{l} O_+(k) \longleftrightarrow SO(\overbrace{N-k+1}^{\tilde{k}}) \\ SO(k) \longleftrightarrow O_+(\tilde{k}) \\ O_-(k) \longleftrightarrow O_-(\tilde{k}) \end{array}\right\}$$

N funds $\tilde{X}^1, \dots, \tilde{X}^N$

$\frac{N(N+1)}{2}$ singlets $(S_{\underbrace{ij}_{\text{symmetric}}})_{i,j=1}^{N}$

$$W = \sum_{i,j} S_{ij} (\tilde{X}^i \tilde{X}^j)$$

$$(X_i X_j) \longleftrightarrow S_{ij}$$

$SO(k) \longleftrightarrow O_+(\tilde{k})$:

$\mathbb{Z}_2 = O(k)/SO(k) \longleftrightarrow \left[O_+(\tilde{k})/SO(\tilde{k}) \cong \mathbb{Z}_2 \right]$ - Quantum symmetry

baryons $[X_{i_1} \cdots X_{i_k}] \longleftrightarrow$ twist operators.

Evidences :

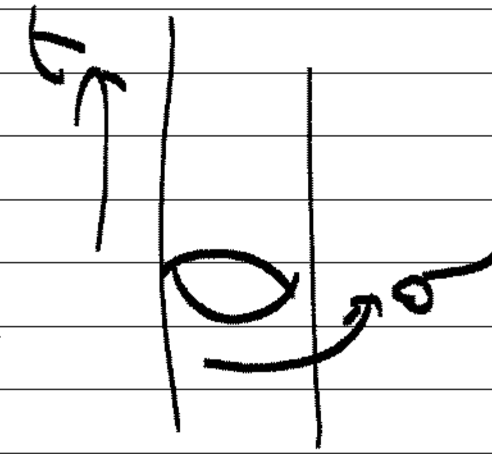
- $\hat{c} = kN - \frac{k(k-1)}{2} \longleftrightarrow \hat{c} = \frac{N(N+1)}{2} - \frac{\tilde{k}(\tilde{k}-1)}{2}$
- Flow by mass
- vacuum counting by twisted mass, "chiral/twisted ch ring"

Remark on "Quantum Sym in Orbifold":

$\mathcal{C}$: QFT with a discrete sym Γ

Abelian

$\rightsquigarrow$ orbifold $\mathcal{C}/\Gamma$



$\gamma \in \Gamma$, $\mathcal{C}/\Gamma$ has <u>twisted sector</u>

$$\gamma \cdot \phi(t, \sigma + 2\pi) = \phi(t, \sigma)$$

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$$

$\chi : \Gamma \to U(1)$ a character

$g_\chi : \mathcal{H} \to \mathcal{H}$; $v \in \mathcal{H}_\gamma \mapsto \chi(\gamma) \cdot v \in \mathcal{H}_\gamma$

"quantum sym assoc. to χ"

$\Gamma = \mathbb{Z}_2$, $\mathrm{Hom}(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$

$\{1, \gamma\}$

$\chi(\gamma) = 1 \longmapsto 1$

$\chi(\gamma) = -1 \longmapsto -1$

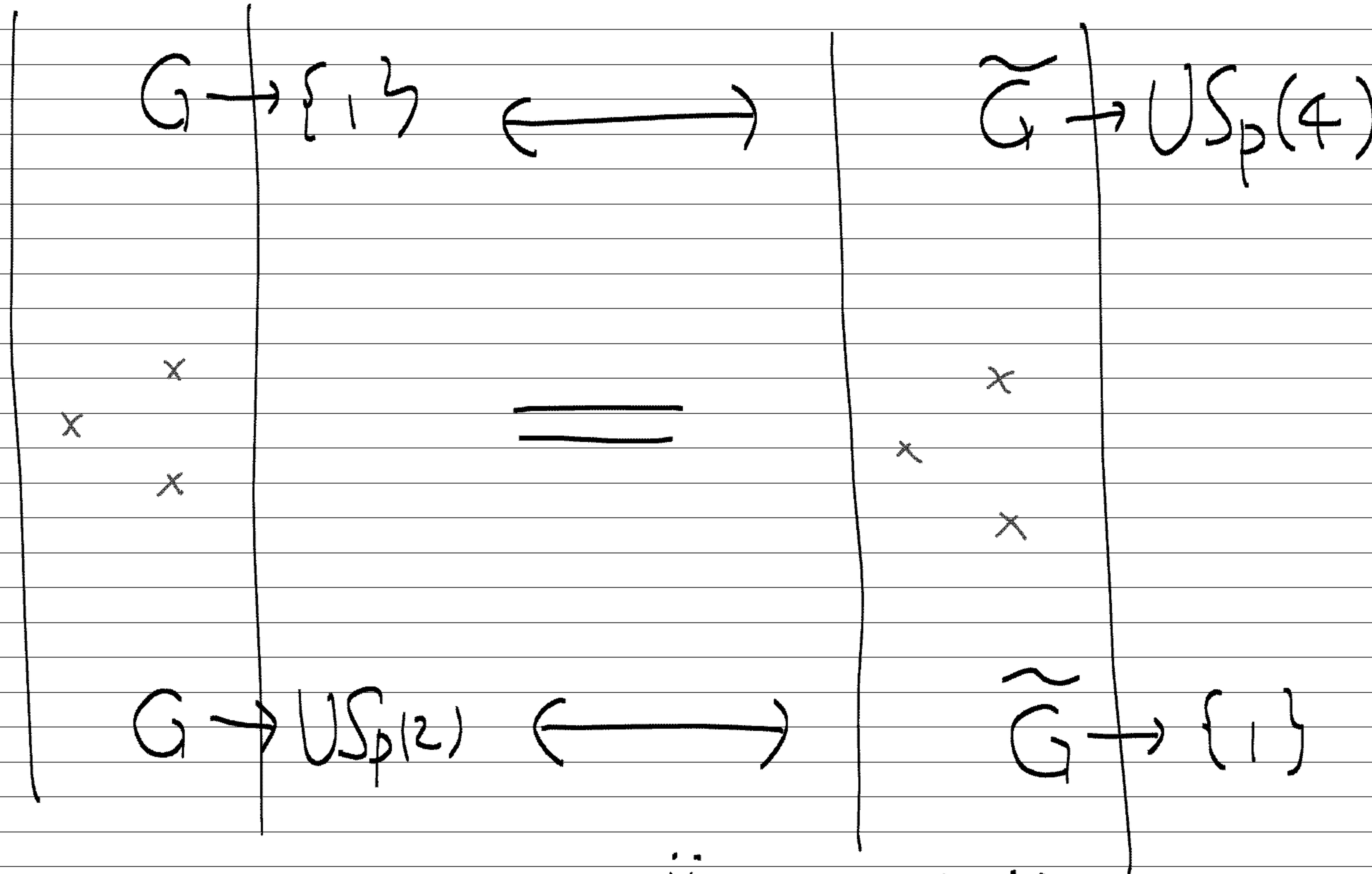
Apply this to the 2nd main ex.

$$G = U(2) = \frac{U(1)\times USp(2)}{\{(\pm 1, \pm \mathbb{1}_2)\}}, \quad \underbrace{p^1 \cdots p^7}_{(-2,\mathbb{1})}, \quad \underbrace{X_1 \cdots X_7}_{(1,\mathbb{2})}, \quad W = \sum A^{ij}(p)[X_i X_j]$$

$$\longrightarrow \tilde{G} = \frac{U(1)\times USp(4)}{\{(\pm 1, \pm \mathbb{1}_4)\}}, \quad \underbrace{p^1 \cdots p^7}_{(-2,\mathbb{1})}, \quad \underbrace{\tilde{X}^1 \cdots \tilde{X}^7}_{(-1,\mathbb{4})}, \quad \underset{(2,\mathbb{1})}{(a_{ij})}$$

$$W = \sum A^{ij}(p)\, a_{ij} + \sum a_{ij} [\tilde{X}^i \tilde{X}^j]$$

$$a_{ij} = \tilde{A}^{\ell}_{ij}\, \tilde{p}_\ell \quad \leftarrow \left(\sum_{ij} A^{ij}_k a_{ij} = 0 \ \forall k \right)$$



$$A^{ij}(p) + [\tilde{X}^i \tilde{X}^j] = 0$$

$\boxed{\text{rank } 4}$

Generalization

$\tilde{n} = N - k - 1 \qquad N = \text{odd}$

$$G = \frac{U(1) \times USp(k)}{\{(\pm 1, \pm \mathbb{1}_k)\}} \qquad \underbrace{p^1 \cdots p^M}_{(-2,\,\mathbb{1})} \qquad \underbrace{X_1 \cdots X_N}_{(1,\,\mathbb{k})} . \quad W = \sum A^{ij}_k p^k [X_i X_j]$$

$$\Leftrightarrow \tilde{G} = \frac{U(1) \times USp(\tilde{k})}{\{(\pm 1, \pm \mathbb{1}_{\tilde{k}})\}} \qquad \underbrace{\tilde{p}_1 \cdots \tilde{p}_{\tilde{M}}}_{(-2,\,\mathbb{1})} , \quad \underbrace{\tilde{X}^1 \cdots \tilde{X}^N}_{(-1,\,\tilde{\mathbb{k}})}$$

$$W = \sum \tilde{A}^{\ell}_{ij} \tilde{p}_\ell [\tilde{X}^i \tilde{X}^j] \qquad \left[\begin{array}{l} \sum_{ij} A^{ij}_k \tilde{A}^{\ell}_{ij} = 0 \\ \tilde{M} = \frac{N(N-1)}{2} - M \end{array} \right.$$

- $kN = 2M$:

$$Y_{\tilde A} = \{ \tilde p \in \mathbb{CP}^{\tilde M - 1} \mid \operatorname{rk} \tilde A(\tilde p) \le k \}$$

$$\binom{\frac{N-1}{2}}{k/2} =: n(k,N)$$

$$Y_A = \{ p \in \mathbb{CP}^{M-1} \mid \operatorname{rk} A(p) \le \tilde k \}$$

- $kN > 2M$: $\quad Y_{\tilde A} \longrightarrow Y_A + \frac{1}{2}(kN - 2M) \cdot n(k,N)$

- $kN < 2M$: $\quad Y_A \longrightarrow Y_{\tilde A} + \frac{1}{2}(2M - kN) \cdot n(k,N)$

Applications :

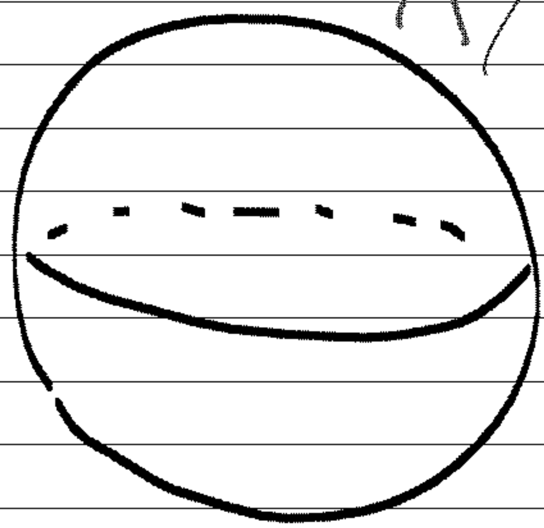
. Abelian LSM : can be used to find mirror dual

→ T-dualize the arg. of chiral.

H - Vafa 2000

· Exact results

A/B



$$= e^{-K}$$

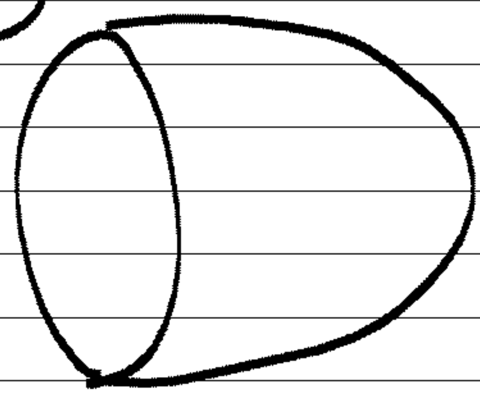
Kähler potential on $\mathcal{M}_{\text{twisted chiral}}$

chiral

$\mathcal{B}$

B

A



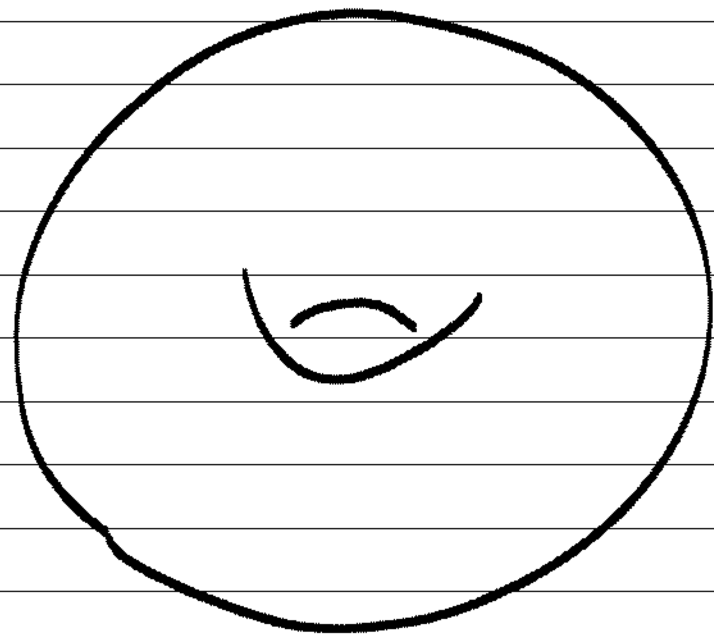
A

B

$$= Z(\mathcal{B})$$

$$\int_X \hat{\Gamma}_X \, \mathrm{ch}(\mathcal{B}) \, e^{B+i\omega} + \mathcal{O}(e^{-\omega})$$

$$\int_L \Omega$$



$$= \mathrm{Tr}_{\mathcal{H}_{RR}} (-1)^{F_L} y^{F_R} q^{H_R} \bar{q}^{H_L}$$

elliptic genus