Anomalies - review

Consider an action

$$S = \int L \, d^4x$$

which is invariant under a <u>global</u> $U(1)$ symmetry

$$\phi \to e^{i\theta}\phi$$

(and also $\phi^* \to e^{-i\theta}\phi^*$)

We could generalize this to n-fields but lets keep it simple for now.

Noethers theorem assures us that this symmetry induces a current

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu \phi)} i\phi + h.c.$$

such that

$$\partial_\mu J^\mu = 0$$

For example: $L = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi + m^2|\Psi|^2$

$\Rightarrow J^\mu = \bar{\Psi}\gamma^\mu\Psi$

Our main goal today is to check how this conservation equation manifests itself at a quantum level.

To this end consider

$$Z[A] = \int e^{iS + i\int A_\mu J^\mu d^4x} \, D\phi \, D\phi^*$$

with A_μ some arbitrary coupling. Note that A_μ is <u>not</u> dynamical. The symmetry is still a <u>global</u> symmetry. A_μ is usefull for us if we want to compute correlators of currents

$$\langle J^{\mu_1}(x_1) \ldots J^{\mu_n}(x_n) \rangle \Big|_{A=0} = \left(\frac{1}{i} \frac{\delta}{\delta A_{\mu_i}} \right)^n \ln Z \Big|_{A=0}$$

Let us ask what happens to $Z[A_\mu]$ under

$$\phi \to e^{i\theta}\phi \, , \quad A_\mu \to A_\mu + \partial_\mu \theta$$

for small θ. Note that we are not "gauging" θ. We are making an observation about the behavior of $Z[A_\mu]$

Let's consider

$$\phi \to (1+i\varepsilon)\phi \qquad A_\mu \to A_\mu - \partial_\mu \varepsilon$$

$$S \to S + \int \frac{\partial L}{\partial(\partial_\mu \phi)} i\partial_\mu(\varepsilon\phi) + \frac{\partial L}{\partial \phi} i\varepsilon\phi + h.c.\, d$$

$$= S + \int \frac{\partial L}{\partial(\partial_\mu \phi)} i\phi\partial_\mu \varepsilon + \underset{\substack{\uparrow \\ \text{constant} \\ \varepsilon \text{ is symmetry}}}{0} + h.c.$$

$$= S + \int J^\mu \partial_\mu \varepsilon \, d^4x$$

$$\int A_\mu J^\mu d^4x \to \int A_\mu J^\mu d^4x + \int (-\partial_\mu \varepsilon) J^\mu d^4x$$

$$+ \int A_\mu \frac{\delta J^\mu}{\delta \partial_\nu \varepsilon} \partial_\nu \varepsilon \; d^4x$$

So:

$$S + \int A\cdot J\, d^4x \to S + \int A\cdot J\, d^4x + \int A_\mu \frac{\delta J^\mu}{\delta \partial_\nu \varepsilon} \partial_\nu \varepsilon\, d^4x$$

For fermions $\dfrac{\delta J^\mu}{\delta \partial_\nu \varepsilon} = 0$ so the ~~action~~ ~~...~~ deformed action in Z is invariant.

For Bosons we could add a term quadratic in A so that $S + \int A\cdot J d^4x$ is invariant.
~~Exercise~~

Exercise:

$$\cancel{L = |\partial_\mu \phi|^2}$$

is invarian

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$$L = |\partial_\mu \phi|^2$$

What is J^μ? Find a deformed action

$$S = \int L\, d^4x + \int A_\mu J^\mu d^4x + \int A_\alpha A_\beta Q^{\alpha\beta} d$$

which is invariant under

$$\phi \to e^{i\theta}\phi \quad , \quad A_\mu \to A_\mu - \partial_\mu \theta .$$

For now we focus on fermions.

<u>If</u> $D\phi\, D\phi^*$ is invariant und $\phi \to e^{i\theta}\phi$

<u>then</u>

$$Z[A_\mu] = Z[A_\mu - \partial_\mu \varepsilon]$$

or for $\quad iW = \ln Z$

$$W[A_\mu] = W[A_\mu - \partial_\mu \varepsilon]$$

~~Let us suppose that $Z[A_\mu]$ is invariant under $A_\mu \to A_\mu - \partial_\mu \varepsilon$~~

Define

$$\delta_\varepsilon W \equiv W[A_\mu - \partial_\mu \varepsilon] - W[A_\mu]$$

$$= \int \frac{\delta W}{\delta A_\mu(x)} (-\partial_\mu \varepsilon(x))\, d^4x$$

$$= \int \varepsilon(x) \frac{\partial}{\partial x^\mu} \frac{\delta W}{\delta A_\mu(x)}\, d^4x$$

Thus $\delta_\varepsilon W = 0 \implies \langle \partial_\mu J^\mu \rangle = 0$

This is a big If. Suppose it is not ...

Regardless of $\delta_\varepsilon W = 0$, we should get that the commutator of Abelian gauge variations commute.

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = 0$$

This poses a non trivial restriction on $\delta_\varepsilon W \neq 0$.

Consider

$$\delta_\varepsilon W = -\int \varepsilon(x)\, G(x)\, d^4x$$

Then

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\, W = 0$$

implies

$$\cancel{-\int \varepsilon_1(x)\,\varepsilon_2(y)\,\frac{\partial}{\partial y^\alpha}\frac{\delta}{\delta A_\alpha(y^\alpha)}\,G(x)\,d^4x\,d^4y}$$

$$0 = \int \left[\varepsilon_2(x)\,\varepsilon_1(y)\,\frac{\partial}{\partial x^\alpha}\frac{\delta}{\delta A_\alpha(x^\alpha)}\,G(y) - \varepsilon_2(y)\,\varepsilon_1(x)\,\frac{\partial}{\partial y^\alpha}\frac{\delta}{\delta A_\alpha(y)}\,G(x)\right]_0$$

$G = 0$ is a solution. But are there others?

~~For convenience, let's define a volume form~~

$$\cancel{\int \varepsilon(x)\, G(x)\, d^4x = \int G_\varepsilon}$$

~~Let's also define our manifold as ∂M~~

$$\cancel{\delta_\varepsilon W = -\int_{\partial M} G_\varepsilon}$$

Suppose there exists a 5-form I such that $I[A]$ $A = A_\mu dx^\mu$ and

$A_\mu dx^\mu \xrightarrow[M\to\partial M]{} A_\mu dx^\mu$, and I is a polynomial which is gauge invariant up-to boundary terms: Define $\delta_\varepsilon I = dG_\varepsilon$

~~Define~~ Consider

$$\cancel{W[A]\,A} \int_M I =$$

$$\delta_\varepsilon \int_M I = \int \delta_\varepsilon I = \int I[A_\mu - \partial_\mu \varepsilon] - I[A_\mu]$$

Note that

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \int_{\mathcal{M}} I = \int [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] I = 0$$

since there is no integral

Thus

$$0 = \int_{\mathcal{M}} \delta_{\varepsilon_1} dG_{\varepsilon_2} - \delta_{\varepsilon_2} dG_{\varepsilon_1} \underset{\substack{\text{exercise} \\ (*)\ [d, \delta_\varepsilon] = 0}}{=} \int_{\mathcal{M}} d(\delta_{\varepsilon_1} G_{\varepsilon_2} - \delta_{\varepsilon_2} G_{\varepsilon_1})$$

$$= \int_{\partial\mathcal{M}} \delta_{\varepsilon_1} G_{\varepsilon_2} - \delta_{\varepsilon_2} G_{\varepsilon_1}$$

Now ~~[illegible]~~ G_ε is a 4-form linear in ε

$$\int G_\varepsilon = \int \varepsilon(x)\, G(x)\, d^4x$$

Then, for $\delta_\varepsilon W = -\int G_\varepsilon$ we have

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] W = 0$$

Proving (*)

$$\delta_\varepsilon Q = \int_{\partial M} \varepsilon\, d\frac{\delta Q}{\delta A} = -\int d\varepsilon \wedge \frac{\delta Q}{\delta A}$$

$$d\delta_\varepsilon Q = \int d\varepsilon \wedge d\frac{\delta Q}{\delta A} = -\int d\varepsilon \wedge \frac{\delta dQ}{\delta A}$$

$$= \int \varepsilon\, d\frac{\delta}{\delta A} dQ = \delta_\varepsilon dQ$$

Hence, existence of I such that

$$\delta_\varepsilon I = d G_\varepsilon \qquad (*)$$

implies a non trivial ~~on~~ $\delta_\varepsilon W$.
Taking d of (*)

$$d\delta_\varepsilon I = 0$$

$$\Rightarrow \delta_\varepsilon dI = 0$$

Define $P = dI$. We need to find a 6-form ~~strictly~~ which is exact and gauge invariant

Define

$$F = dA$$

Then

$$P = c\, F \wedge F \wedge F = c\, d(A \wedge F \wedge F)$$

$$\Rightarrow \quad I = c\, A \wedge F \wedge F$$

$$\Rightarrow \quad \delta I = c\, d\varepsilon \wedge F \wedge F = d(c\varepsilon\, F \wedge F)$$

$$\Rightarrow \quad G_\varepsilon = c\, \varepsilon\, F \wedge F$$

$$\Rightarrow \quad \partial_\mu J^\mu = -c\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Classification of gauge-invariant exact forms has been done by Mathematicians.

Chern-classes.