Solving Conformal Theories with the Bootstrap
Overview and Recent Results

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(soon: LPTHE, Jussieu)

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Why return to the (conformal) bootstrap?

1. Conformal symmetry very powerful tool that goes largely unused in $D > 2$.
2. Completely non-perturbative tool to study field theories
   - Does not require SUSY, large $N$, or weak coupling.
3. In $D = 2$ conformal symmetry enhanced to Virasoro symmetry
   - Allows us to completely solve some CFTs ($c < 1$).
4. Long term hope: generalize this to $D > 2$?

Approach

- Use only “global” conformal group, valid in all $D$.
- Study crossing symmetry of a single scalar correlator $\langle \sigma\sigma\sigma\sigma \rangle$.*
  (* More recently extended to 4-pt functions of two scalars.)

Results

- Universal constraints (bounds) on spectrum/couplings in $\sigma \times \sigma$ OPE.
- Ising, $O(N)$ & some susy models seem to saturate these bounds.
- When bounds saturated crossing symmetry fixes full OPE.
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Tentative Outline

1. Overview & Philosophy
2. CFT basics
3. Conformal Blocks
   4. Crossing symmetry
   5. Geometry of the solution space
   6. Linear Programming
   7. “Solving” the 3d Ising
   8. Other results
10. Bootstrapping Theories with Four Supercharges (in $d = 2 - 4$)
11. Our (Modified) Simplex Algorithm
12. How to get Started Bootstrapping...
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Overview & Philosophy
What are CFTs and why are they interesting?

- Second order phase transition at the end of a line of first order transitions.
- Same CFT describes many disparate experimental systems.
- e.g. Ising model CFT is universal description of phase transitions with $\mathbb{Z}_2$ symmetry.
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![Phase transition diagram](from a talk by Stephanov'2004)
Very different “UV” theories can share the same IR behaviour.

**Example: Ising universality class**

- Lattice theory with nearest neighbor interactions

\[ H = -J \sum_{\langle i,j \rangle} s_is_j \]

with \( s_i = \pm 1 \) (only discrete translation or rotational symmetry).

- Has symmetry broken phase \( \langle s_i \rangle = \pm 1 \) and symmetric phase \( \langle s_i \rangle = 0 \).
- Ising model CFT describes theory at critical temperature \( T_c \) between two phases.
CFTs: A Field Theorist’s Perspective
Overview & Philosophy

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Example: **Ising universality class**

- **Scalar QFT (and \( \sigma(x) \in \mathbb{R} \))**

\[
S = \int d^D x \left[ (\nabla \sigma(x))^2 + t \sigma(x)^2 + \lambda_4 \sigma(x)^4 + \lambda_6 \sigma(x)^6 + \ldots \right]
\]

- **\( \mathbb{Z}_2 \) symmetry: \( \sigma \to -\sigma \)**
- **Theory has full rotational/translational invariance.**
- **Also has symmetric and symmetric broken phase (\( \langle \sigma \rangle \neq 0 \)).**
- **Mass is related to reduced temperature \( t \sim T - T_c \).**
- **In the IR flows to same CFT as lattice model!**
CFTs are **fixed points** of RG flow*

- Couplings $\lambda_i$ flow under rescalings $x \rightarrow \Lambda x$.
- Flow described by $\beta_i(\lambda_i)$ functions.

$$\beta_i(\lambda_i) = \frac{\partial \lambda_i}{\partial \log \Lambda}$$

- CFTs correspond to fixed points

$$\beta_i(\lambda_i^*) = 0 \quad (1)$$

- Eqns (1) strongly constrain couplings

$\Rightarrow$ CFTs very non-generic

(* More generally fixed points have *scale invariance* but generically this leads to *conformal invariance*.)
Correlation functions & Observables
Overview & Philosophy

QFT

▶ In a QFT we may have (asymptotic) observables, $\mathcal{O} \sim \phi^{\vec{k}}$.
▶ We compute/observer scattering amplitudes:

$$\langle \mathcal{O}_{\vec{k}_1} \mathcal{O}_{\vec{k}_2} \mathcal{O}_{\vec{k}_3} \mathcal{O}_{\vec{k}_4} \rangle \sim f(t, \lambda_i, \vec{k}_a, \Lambda)$$

▶ Observables depend on many continuous parameters.

CFT

▶ Observables are not asymptotic $\mathcal{O} \sim \phi(x), : \phi(x)^2 :, T_{\mu\nu}$.
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▶ Couplings generically fixed by $\beta$-functions.
▶ No dependence on dimensional-full scale $\Lambda$
   $\Rightarrow$ far fewer parameters!
▶ Correlators also strongly constrained by conformal invariance (much more than scale invariance).
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The Conformal Bootstrap
Overview & Philosophy

Question

CFTs are:

▶ Universal: realized as RG limits of many disparate theories.
▶ Strongly constrained by symmetry.

Can we describe their physics intrinsically (i.e. without picking a particular UV realization)?

(Partial) Answer

Yes!

▶ In $d = 2$ infinite classes of CFT can be solved using symmetry alone.
  $\Rightarrow$ solved means compute correlators of all local operators.
▶ In $d > 2$ no full solution but powerful numerical methods:
  2. Method is fully non-perturbative: works for strongly coupled theories!
▶ Method is conformal bootstrap and will be focus of these lectures.
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Example: the (3d) Ising Model

Intermezzo

\( \mathcal{E} \)-expansion

Wilson-Fisher set \( D = 4 - \mathcal{E} \) and study critical point of \( \sigma^4 \) perturbatively. Setting \( \mathcal{E} = 1 \) can compute anomolous dimensions in \( D = 3 \):

\[
[\sigma] = 0.5 \rightarrow 0.518 \ldots \\
[\epsilon] := [\sigma^2] = 1 \rightarrow 1.41 \ldots \\
[\epsilon'] := [\sigma^4] = 2 \rightarrow 3.8 \ldots
\]

Using \( \mathcal{E} \)-expansion, Monte Carlo and other techniques find partial spectrum:

<table>
<thead>
<tr>
<th>Field:</th>
<th>( \sigma )</th>
<th>( \epsilon )</th>
<th>( \epsilon' )</th>
<th>( T_{\mu\nu} )</th>
<th>( C_{\mu\nu\rho\lambda} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dim (( \Delta )): 0.518135(50)</td>
<td>1.41275(25)</td>
<td>3.832(6)</td>
<td>3</td>
<td>5.0208(12)</td>
<td></td>
</tr>
<tr>
<td>Spin (l): 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Only 5 operators and no OPE coefficients known for 3d Ising... Lots of room for improvement!
At fixed point **conformal symmetry** emerges:

- Strongly constrains data of theory.
- Can we use symmetry to fix e.g. $[\sigma], [\epsilon], [\epsilon'], \ldots$?
- Can we also fix interactions this way?
Our first goal: a completely general exclusion plot for $\Delta_\sigma$ & $\Delta_\epsilon$.

This exclusion bound applies to any conformal theories.

- It is completely non-perturbative.
- In generating it we use no Lagrangian or any data specifying a particular theory.
- Exclusion plots “knows” about Ising model!
- Using bootstrap can compute spectrum & interactions of many operators for any theory on the boundary of exclusion bound.

Generating such a plot and using it to (partially) “solve” conformal theories will be the main focus of this lecture.
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Allowed values of $\sigma - \epsilon$

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CFT Basics
Conformal Symmetry

Useful (incomplete) References

▶ Slava Rychkov’s $d > 2$ CFT lecture notes:
  https://sites.google.com/site/slavarychkov/CFT_LECTURES_Rychkov.pdf

▶ Alessandro Vichi’s PhD thesis:
  infosciences.epfl.ch/record/167898/files/EPFL_TH5116.pdf

▶ Numerical bootstrap:
  http://arxiv.org/abs/0807.0004

▶ OPE and conformal blocks:
  http://arxiv.org/abs/1208.6449
Conformal Symmetry

CFT Basics

Conformal transformations are angle preserving:

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x)$$

They generalizes (constant) scale transformations: $x \rightarrow \lambda x$. 
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Conformal Algebra

Extension of Poincaré:

\[ SO(1, D - 1) \times \mathbb{R}^{1, D-1} + D \text{ (Dilatations)} + K_\mu \text{ (Special conformal)} \]

Momentum \((P_\mu)\) and special conformal \((K_\mu)\) raise/lower level:

\[ [D, P_\mu] = i P_\mu, \quad [D, K_\mu] = -i K_\mu \]

Act like ladder operators to generate new states in a representation.
In a CFT should organize operators into representations of Conformal Group.

**Operators**

Operators should have **definite scaling dimension** $\Delta$ and spin $l$

\[ [D, \mathcal{O}] = i \Delta \mathcal{O}, \quad [M_{\mu\nu}, \mathcal{O}] = i R_M \cdot \mathcal{O} \]

with $R_M$ a spin-$l$ representation of rotation $M_{\mu\nu}$.

**Primary Operators**

Highest weight states, $\mathcal{O}$, are called primary operators. Satisfy:

Primary operators: \[ [K_\mu, \mathcal{O}] = 0 \]

From a highest weight state can construct an infinite number of descendants:

Descendents: \[ \mathcal{O}_n := P_{\mu_1} \ldots P_{\mu_n} \mathcal{O} \]

Correlators of descendants fixed by conformal symmetry (in terms of primaries).
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From definition primary operator $O$ transforms under $x \rightarrow \lambda x$:

$$O(\lambda x) = \lambda^{-\Delta}O(x)$$

Two Point Functions
Requiring correlators to be invariant under conf group gives:

$$\langle O_i(x)O_j(0) \rangle \xrightarrow{\text{scale}} \frac{a}{x^{\Delta_i + \Delta_j}} \xrightarrow{\text{special}} \frac{a \delta_{ij}}{x^{2\Delta_i}}$$

Set $a = 1$ as to fix normalization.

Three Point Functions

$$\langle O_i(x_i)O_j(x_j)O_k(x_k) \rangle \xrightarrow{\text{scale}} \sum_{a+b+c=\Delta_1+\Delta_2+\Delta_3} \frac{C_{ijk}}{x_i^a x_j^b x_k^c} \xrightarrow{\text{special}} \frac{C_{ijk}}{x_{ij}^{\delta_{ij}} x_{ik}^{\delta_{ik}} x_{jk}^{\delta_{jk}}}$$

with $\delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ and $x_{ij} = |\vec{x}_{ij}|$.

Correlators of descendents, $\partial_{\mu_1} \ldots \partial_{\mu_n} O$, computed by taking derivs.
Conformal Correlators
CFT Basics

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A more physical way to think about what a primary operator is via RG:

1. CFT should be invariant under RG step: \( x \rightarrow x' = \lambda x \)
2. In scale/conf invariant theory Hamiltonian is invariant: \( \hat{H} \rightarrow \hat{H}' = \hat{H} \).
3. Kinetic term in Lagrangian transforms like:

\[
d^d x' (\partial_{x'} \phi'(x'))^2 \rightarrow d^d x (\lambda^{d-2}) (\partial_{x} \phi'(x'))^2
\]

so invariant if we identify \( \phi'(x') = \lambda^{-\frac{d-2}{2}} \phi(x) \) (i.e. \( \Delta \phi = \frac{d-2}{2} \)).
4. Quantum corrections can modify this giving

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\] (2)

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7. Primaries are fields with nice transformations like eqn (2) and descendent transformation follows by taking derivs.
Conformal Correlators & RG

CFT Basics

A more physical way to think about what a primary operator is via RG:

1. CFT should be invariant under RG step: $x \rightarrow x' = \lambda x$
2. In scale/conf invariant theory Hamiltonian is invariant: $\hat{H} \rightarrow \hat{H}' = \hat{H}$.
3. Kinetic term in Lagrangian transforms like:

$$d^d x' \left( \partial_{x'} \phi'(x') \right)^2 \rightarrow d^d x \left( \lambda^{d-2} \right) \left( \partial_x \phi'(x') \right)^2$$

so invariant if we identify $\phi'(x') = \lambda^{-\frac{d-2}{2}} \phi(x)$ (i.e. $\Delta \phi = \frac{d-2}{2}$).

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In CFT it is natural to use **radial quantization**.

“Hamiltonian” is $\hat{D}$: $\vec{x} \rightarrow \lambda \vec{x}$

Operators defined on radial slices.

### Operator-State Correspondence

Each operator inserted at origin defines a state:

$$|\mathcal{O}\rangle := \mathcal{O}(0)|0\rangle$$

Can check $\hat{D}|\mathcal{O}\rangle = \Delta_{\mathcal{O}}|\mathcal{O}\rangle$.

Complete basis of states spanned by primaries + descendents inserted at origin.

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Operator Product Expansion (OPE)

CFT Basics

Operator Product Expansion

- Acting on primary state $|\mathcal{O}_j\rangle$ with primary operator $\mathcal{O}_i(x)$ gives new state that can be decomposed in conformal reps:

$$\mathcal{O}_i(x)|\mathcal{O}_j\rangle = \sum_{\alpha} c_{ij\alpha} |\Psi_{\alpha}\rangle$$

$\alpha$ runs over all eigenstates of $\hat{D}$.

- Contribution of descendents fixed by symmetry.
- Using Operator-State correspondence this gives operator product expansion.
- Repackage using diff op $D(x, \partial)$ (fixed by conformal symmetry).
- $C_{ijk}$ are dynamical data of theory (along with $\Delta_i, l_i$).
- OPE only holds if no operator inserted at $|y| < |x|$.
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Act with operator $O_i(x)|O_j\rangle$
Operator Product Expansion (OPE)

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How could we determine $D(x, \delta_x)$?

Use OPE to reduce 3-pt function to sum over 2-pt function

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \sum_k C_{12k} D(x_{12} \partial_2) \langle \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle$$

$$\frac{C_{123}}{x_{12}^{\delta_{12}} x_{13}^{\delta_{13}} x_{23}^{\delta_{23}}} = C_{123} D(x_{12} \partial_2) \langle \mathcal{O}_3(x_2) \mathcal{O}_3(x_3) \rangle$$

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Form of $D(x, \partial)$ can be fixed by related 2- and 3-pt function.

Now we know everything about OPE and 2/3-pt functions.
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Now we know everything about OPE and 2/3-pt functions.

What about higher point functions?
Correlation functions from OPE

CFT Basics

Any \(n\)-pt function can be reduced to \textit{sums} of 2- & 3-pt functions via OPE:

\[
\langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}_{\sum_k C_{12}^k \mathcal{D}(x_{12}, \partial x_2) \mathcal{O}_k(x_2)} \underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}_{\sum_l C_{34}^l \mathcal{D}(x_{34}, \partial x_4) \mathcal{O}_l(x_4)} \rangle = \sum_{k,l} C_{12}^k C_{34}^l \mathcal{D}(x_{12}, x_{34}, \partial x_2, \partial x_4) \langle \mathcal{O}_k(x_2)\mathcal{O}_l(x_4) \rangle
\]

- For 4-pt function get single sum of two \(\mathcal{D}\)'s acting on 2-pt function.
- Conformal symmetry fixes (for \(\mathcal{O}\) scalar primary of dim \(\Delta\)):

\[
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} g(u, v)
\]

with unknown function \(g\) of

\[
\begin{align*}
u &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \cdot
\end{align*}
\]

- \(u, v\) conformally invariant so form of \(g\) not fixed.
- If operators non-scalar or \(\Delta\)'s not equal technically more complicated (but conceptually the same).
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Any $n$-pt function can be reduced to *sums* of 2- & 3-pt functions via OPE:

$$\langle O_1(x_1)O_2(x_2) \quad O_3(x_3)O_4(x_4) \rangle = \sum_k C^k_{12} C^k_{34} D(x_{12}, x_{34}, \partial_{x_2}, \partial_{x_4}) \left( \frac{1}{x_{24}^{2\Delta_k}} \right)$$

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Axiomatic Formulation of a CFT

CFT Basics

CFT defined by specifying:

- Spectrum $S = \{\mathcal{O}_i\}$ of primary operators dimensions, spins: $(\Delta_i, l_i)$
- Operator Product Expansion (OPE)

$$\mathcal{O}_i(x) \cdot \mathcal{O}_j(0) \sim \sum_k C^k_{ij} D(x, \partial_x) \mathcal{O}_k(0)$$

$\mathcal{O}_i$ are primaries. Diff operator $D(x, \partial_x)$ encodes descendent contributions.

This data fixes all correlators in the CFT:

- 2-pt & 3-pt fixed:

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \frac{\delta_{ij}}{x^{2\Delta_i}}, \quad \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \frac{C_{ijk}}{x_{ij} \delta_{ik} \delta_{jk}}$$

- Higher pt functions contain no new dynamical info:

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 conformal block
What about central charge?

- In $d = 2$ the **central charge**, $c$, is a defining property of CFT.
- In $d > 2$ no canonical definition of central charge.
- Candidates:

  $$\langle T(z)T(0) \rangle \sim \frac{c_1}{z^4}, \quad \langle T \rangle \sim c_2 R, \quad S = 2\pi \sqrt{c_3 \Delta}$$

- In $d = 2$ all equal: $c_1 = c_2 = c_3$.
- Definitely **not** true for $d > 2$.
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- Appears in $\phi \times \phi$ OPE:

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We will restrict attention to “unitary” theories.

Generally we consider Euclidean theories so actually mean reflection positivity.

In reflection-positive theory norms of states must be positive.

Combined with conformal algebra unitarity gives constraints on $\Delta$.

\[
||P^\mu P_\mu |\mathcal{O}\rangle|| = \langle \mathcal{O}|K_\nu K^\nu P^\mu P_\mu |\mathcal{O}\rangle 
\propto \Delta \left( \Delta - \frac{d - 2}{2} \right)
\]

Here we use $K^\dagger = P$ (in radial quantization) and assumed $\mathcal{O}$ is a scalar. Similar arguments give:

\[
\Delta \geq \frac{d - 2}{2} \quad (l = 0)
\]

\[
\Delta \geq d + l - 2 \quad (l \geq 0)
\]

The $\Delta = 0$ solution is conformally invariant vacuum $|0\rangle$.  


We will restrict attention to “unitary” theories.

Generally we consider Euclidean theories so actually mean reflection positivity.

In reflection-positive theory norms of states must be positive.

Combined with conformal algebra unitarity gives constraints on $\Delta$.

\[
||P^\mu P_\mu |\mathcal{O}\rangle|| = \langle \mathcal{O} | K_\nu K^\nu P^\mu P_\mu |\mathcal{O}\rangle 
\propto \Delta \left( \Delta - \frac{d - 2}{2} \right)
\]

Here we use $K^\dagger = P$ (in radial quantization) and assumed $\mathcal{O}$ is a scalar.

Similar arguments give:

\[
\Delta \geq \frac{d - 2}{2} \quad (l = 0)
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\Delta \geq d + l - 2 \quad (l \geq 0)
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In reflection-positive theory norms of states must be positive.

Combined with conformal algebra unitarity gives constraints on $\Delta$.

\[ \langle \mathcal{O} | K^{\nu} K^{\nu} P^\mu P_\mu | \mathcal{O} \rangle \propto \Delta \left( \Delta - \frac{d - 2}{2} \right) \]

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Conformal Blocks
Definition

Conformal Blocks

Recall 4-pt function can be expressed in terms of $g(u, v)$:

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} g(u, v)$$

Can also use OPE to decompose 4-pt into sum:

$$\langle \underbrace{\mathcal{O}_1(x_1) \mathcal{O}_2(x_2)}_{\sum_k c_{12}^k} \underbrace{\mathcal{O}_3(x_3) \mathcal{O}_4(x_4)}_{\sum_l c_{34}^l} \rangle = \sum_k c_{12}^k c_{34}^k D(x_{12}, x_{34}, \partial x_2, \partial x_4) \left( \frac{1}{x_{24}^{2\Delta_k}} \right)$$

- The $D$'s acting on 2-pt function is called Conformal Block.
- Convenient to pull out trivial pre-factor and then express $g(u, v)$ in terms of CB decomposition.
- The conformal block depends on how we take the OPE (CB above is in the $(1 - 2),(3 - 4)$ channel).
Definition
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Recall 4-pt function can be expressed in terms of \( g(u, v) \):

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\langle \sum_k C_{12}^k D(x_{12}, \partial x_2) \mathcal{O}_k(x_2) \sum_l C_{34}^l D(x_{34}, \partial x_4) \mathcal{O}_l(x_4) \rangle = \sum_k C_{12}^k C_{34}^k \tilde{G}_{\Delta_k, l_k}(x_i)
\]

- The \( D \)'s acting on 2-pt function is called \textbf{Conformal Block}.
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$$\left\langle \sum_k c_{12}^k D(x_{12}, \partial x_2) O_k(x_2) \sum_l c_{34}^l D(x_{34}, \partial x_4)(x_3) O_l(x_4) \right\rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \sum_k c_{12}^k c_{34}^l G_{\Delta_k, l_k}(u, v)$$

1. The $D$’s acting on 2-pt function is called **Conformal Block**.
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1. The $D$’s acting on 2-pt function is called Conformal Block.
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Conformal blocks depend on two variables $u, v$.

It turns out a more natural coordinate system is $z, \bar{z}$ defined via

$$u = z \bar{z}, \quad v = (1 - z)(1 - \bar{z})$$

In $d = 2$ these are just standard complex coords.

In all $d$ conf symm can fix $x_1, \ldots, x_4$ to be co-planar.

- Use conf symm to fix $\vec{x}_1 = 0, \vec{x}_3 = 1, \vec{x}_4 = \infty$.
- $z, \bar{z}$ complex coords on plane spanned by unfixed $\vec{x}_2$.
- From definition follows

$$z, \bar{z} \to 0 \implies x_{12}, x_{34} \to 0$$
$$z, \bar{z} \to 1 \implies x_{14}, x_{23} \to 0$$
Analytic Properties
Conformal Blocks

1. From OPE convergence might not expect 
   \((1 - 2), (3 - 4)\) CB to converge if \(|z| > 1\).

2. Actually conf blocks more convergent than 
   OPE:
   - Treat e.g. \(x_1 \leftrightarrow x_2\) symmetrically.
   - Reflects freedom to choose origin in radial 
     quantization.

3. Conf blocks \(G(z, \bar{z})\) can be extended to full \(z, \bar{z}\) 
   plane by analytic continuation.

4. Analytic continuation has branch cut for \(z > 1\) 
   on real axis.

5. Explicit expressions exist in \(d = 2, 4\).

Example: conf blocks in \(d = 4\) (for equal external dimensions):

\[
G(z, \bar{z}) = \frac{z \bar{z}}{z - \bar{z}} \left[ k_{\Delta + l}(z)k_{\Delta - l - 2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right]
\]

\[
k_{\beta}(x) = x^{\beta/2} \, _2F_1(\beta/2, \beta/2, \beta; x)
\]
Crossing Symmetry
Definition
Crossing Symmetry

OPE decomposition of 4-pt function into CBs is not unique:

\[
\langle O_1 O_2 O_3 O_4 \rangle
\]

\[
\sum_k C_{12}^k C_{34}^k G_{\Delta_k, l_k}^{12;34} (u, v) = \sum_k C_{14}^k C_{23}^k G_{\Delta_k, l_k}^{14;23} (u, v)
\]

Consistency requires equivalence of two different contractions

When operators in correlator identical \( G^{12;34} \) and \( G^{14;23} \) simply related:

- Crossing means exchanging \( x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4 \) or equivalently \( u \leftrightarrow v \) implying

\[
G^{12;34} (u, v) = G^{14;23} (v, u)
\]

- Crossing symmetry give non-perturbative constraints on \( (\Delta_k, C_{ij}^k) \).
How constraining is crossing symmetry?

Intermezzo: Applications

Is crossing symmetry consistent with a gap?

$\sigma$-OPE: $\sigma \times \sigma \sim 1 + \epsilon + \ldots$

Crossing symmetric values of $\sigma - \epsilon$

<table>
<thead>
<tr>
<th>$\Delta_\epsilon$</th>
<th>$\Delta_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>1.00</td>
</tr>
<tr>
<td>0.55</td>
<td>1.20</td>
</tr>
<tr>
<td>0.60</td>
<td>1.40</td>
</tr>
<tr>
<td>0.65</td>
<td>1.60</td>
</tr>
<tr>
<td>0.70</td>
<td>1.80</td>
</tr>
<tr>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>0.80</td>
<td></td>
</tr>
</tbody>
</table>

- Assuming above OPE study crossing symmetry of:
  $$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$$

- Certain values of $\Delta_\sigma, \Delta_\epsilon$ inconsistent with crossing symmetry.

- Solutions to crossing:
  1. white region $\Rightarrow$ 0 solutions.
  2. blue region $\Rightarrow$ $\infty$ solutions.
  3. boundary $\Rightarrow$ 1 solution (unique)!

- Exclusion plots “knows” about Ising model!

Blue = solution may exists.
White = No solution exists.
A (More) Practical Formulation

Crossing Symmetry

So how do we check crossing symmetry in practice?

Consider four identical scalars: \( \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \quad \text{dim}(\phi) = \Delta_\phi \)

Recall crossing symmetry constraint:

\[
\frac{1}{2\Delta_\phi x_{12} 2\Delta_\phi x_{34}} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 G_{\Delta_k, l_k}^{12;34} (u, v) = \frac{1}{2\Delta_\phi x_{14} 2\Delta_\phi x_{23}} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 G_{\Delta_k, l_k}^{14;23} (u, v)
\]

\[
\sum_k \begin{array}{c}
1 \\
2 \\
\ \ | \ \\
\ \ | \ \\
\ \ | \ \\
3 \\
4
\end{array} = \sum_k \begin{array}{c}
1 \\
2 \\
\ \ | \ \\
\ \ | \ \\
\ \ | \ \\
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4
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\]
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So how do we check crossing symmetry \textit{in practice}?

Consider four \textit{identical} scalars:  
\[ \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \quad \text{dim}(\phi) = \Delta_{\phi} \]

Express everything in terms of \( u, v \): 
\[
\left( \frac{v}{u} \right)^{\Delta_{\phi}} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 G_{\Delta_k, l_k}^{12;34}(u, v) = \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 G_{\Delta_k, l_k}^{14;23}(u, v) 
\]

\[
\sum_k^{1234} = \sum_k^{1423} 
\]
So how do we check crossing symmetry in practice?

Consider four **identical** scalars: \( \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \quad \text{dim}(\phi) = \Delta_\phi \)

Move everything to LHS:

\[
v^{\Delta_\phi} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 \, G^{12;34}_{\Delta_k,l_k}(u, v) - u^{\Delta_\phi} \sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 \, G^{14;23}_{\Delta_k,l_k}(u, v) = 0
\]
So how do we check crossing symmetry in practice?

Consider four *identical* scalars: \[ \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle \quad \text{dim}(\phi) = \Delta \phi \]

Express as sum of functions with positive coefficients:

\[
\sum_{\mathcal{O}_k} \left( C_{\phi \phi}^k \right)^2 \left[ \sum_{p_k} \left( C_{\phi \phi}^k \right)^2 \left[ v^\Delta \phi G_{\Delta k, l_k} (u, v) - u^\Delta \phi G_{\Delta k, l_l} (v, u) \right] = 0 \right] = 0
\]

\[
\sum_k = \sum_k
\]

\[
\sum_k
\]
A (More) Practical Formulation
Crossing Symmetry

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Consider four **identical** scalars:

\[ \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \quad \text{dim}(\phi) = \Delta_\phi \]

\[
\sum_{O_k} (C_{\phi\phi}^k)^2 \left[ p_k^{\Delta_\phi G_{\Delta_k,l_k}(u,v)} - u^{\Delta_\phi G_{\Delta_k,l_l}(v,u)} \right] = 0
\]

Functions \( F_k(u,v) \) are formally infinite dimensional vectors.

\[
p_1 \left( F_1, F_1', F_1'', \ldots \right) + p_2 \left( F_2, F_2', F_2'', \ldots \right) + p_3 \left( F_3, F_3', F_3'', \ldots \right) + \cdots = \vec{0}
\]

1. Each component is a deriv at a the same point: e.g. \( F' := F^{(2,0)}(\bar{z}^*, \bar{z}^*) \).
2. Each vector \( \vec{v}_k \) represents the contribution of an operator \( O_k \).
3. Labels \( k = (\Delta, l) \) are continuous (because of \( \Delta \)).
4. If \( \{ \vec{v}_1, \vec{v}_2, \ldots \} \) span a **positive cone** there is no solution.
5. Efficient numerical methods to check if set of vectors \( \{ \vec{v}_k \} \) span a cone.
A (More) Practical Formulation

Crossing Symmetry

So how do we check crossing symmetry in practice?

Consider four identical scalars:

\[ \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle \quad \text{dim}(\phi) = \Delta \phi \]

\[
\sum_{O_k} \left( \frac{C_{\phi \phi}^k}{p_k} \right)^2 \left[ \nu^\Delta \phi \, G_{\Delta, l_k}(u, v) - u^\Delta \phi \, G_{\Delta, l_l}(v, u) \right] = 0
\]

Functions \( F_k(u, v) \) are formally infinite dimensional vectors.

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Convergence
Crossing Symmetry

Our goal will be to study/constrain spectrum $S$ of CFTs using sum rule.

Before continuing check: how quickly does conf. block expansion converge?

▶ Consider free scalar
  \[ \phi \times \phi \sim 1 + \phi^2 + T_{\mu\nu} + \ldots. \]

▶ Move contribution of identity, $I$, to LHS:
  \[ F_I = -\sum_{\mathcal{O}_k} (C_{\phi\phi}^k)^2 F_k(u, v) \]
  (can normalize so $F_I = 1$)

▶ Plot how quickly sum converges for free theory (along $z = \bar{z}$).

▶ **Note:** convergence best around $z = \bar{z} = \frac{1}{2}$ so choose this point for Taylor expansion.

Can prove that asymptotically tail of sum rule cut off at $\Delta^*$ is bounded by $e^{-\Delta^*}.$
Geometry of the Solution Space
To get some insight lets consider a very truncated problem.

We truncate to $\Delta < \Delta^*, l < l^*$ (this is a justifiable approximation).

Each order in Taylor expansion (around $z = \bar{z} = \frac{1}{2}$) of sum rule gives a necessary condition for crossing.

We consider only two Taylor coefficients $F^{(1,1)}$ & $F^{(3,0)}$ (this is not an approximation)!

Truncated sum rule becomes:

$$\sum_{\Delta<\Delta^*, l<l^*} p_{\Delta,l} \begin{pmatrix} F^{(3,0)}_{\Delta,l} \\ F^{(1,1)}_{\Delta,l} \end{pmatrix} \bar{v}_{\Delta,l} = 0 \tag{3}$$

Recall $p_k = (C_{\phi\psi}^k)^2 > 0$ so if $\{\bar{v}_{\Delta,l}\}$ form a positive cone we’re doomed (i.e. can’t solve eqn(3)).
The “Landscape” of CFTs
Geometry of the Solution Space

Constraining the spectrum

Figure: $S$: a putative spectrum in $D = 3$

- Unitarity implies:
  \[ \Delta \geq \frac{D - 2}{2} \quad (l = 0), \]
  \[ \Delta \geq l + D - 2 \quad (l \geq 0) \]

- “Carve” landscape of CFTs by imposing gap in scalar sector.
- Fix lightest scalar: $\sigma$.
- Vary next scalar: $\epsilon$.
- Spectrum otherwise unconstrained: allow any other operators.
Constraining Spectrum using Crossing Symmetry

Geometry of Solution Space

Is (truncated) crossing symmetry consistent with a gap?

\[ \langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle \]

Crossing symmetric values of \( \sigma - \epsilon \)

Blue = solution may exists.
White = No solution exists.
Two-derivative truncation

- Consider $\langle \sigma \sigma \sigma \sigma \rangle$.
- Fix $\Delta(\sigma) = 0.515$.
- We plot e.g. $(F^{(1,1)}, F^{(3,0)})$.

- Consider putative spectrum $\{\Delta_k, l_k\}$

\[
\Delta = \Delta_{\text{unitarity}} \\
l = 0 \text{ to } 10
\]

- Vectors represent operators.
- All vectors lie inside cone $\Rightarrow$ Inconsistent spectrum!
Cones in Derivative Space
Geometry of Solution Space

Derivatives $\iff$ Putative Spectrum

$L = 0$
$L = 2$
$L = 4$
$L = 6$
$L = 8$
$L = 10$

-0.6 -0.4 -0.2 0.2 0.4 0.6 0.8

$F^{1, 1}$

Unitarity Bound

$F^{1, 3, 0}$
Cones in Derivative Space
Geometry of Solution Space

- Allow even more operators in putative spectrum.
- **Scalar channel** plays essential role.
  - vectors span plane.
  - In particular can find \( p_k \geq 0 \)
  \[
  \sum_k p_k F_{\Delta_k, l_k} = 0
  \]
  - crossing sym. can be satisfied.

**Why does this work?**

- Cone boundary defined by low-lying operators.
- Higher \( \Delta, l \) operators less important.
- Follows from convergence of CB expansion.
Cones in Derivative Space

Geometry of Solution Space

Derivatives $\iff$ Putative Spectrum

$L = 0$
$D_0 = 0.76$

$\Delta_0 = 2.091$
$\Delta_0 = 0.76$

$F_{1, 1}^{\Delta, 0}$

$F_{1, 0}^{\Delta, 0}$

Unitarity Bound
Cones in Derivative Space
Geometry of Solution Space

Carving the Landscape of CFTs

1. Plot imposes **necessary** conditions.
2. Carve “landscape” via exclusion.

Any CFT in $D = 3$ with $\dim(\sigma) = 0.515$ must have another scalar with $0.76 \leq \Delta \leq 2.091$. 
The “Extremal Solution”
Geometry of Solution Space

Uniqueness of “Boundary Solution”
– Consider $\Delta_0 < 0.76$
  ▶ No combination of vecs give a zero.
  \[ \sum_i p_i \vec{F}_i \neq 0 \text{ for } p_i > 0 \]
– Consider $\Delta_0 > 0.76$
  ▶ Many ways to form vecs to give zero.
  ▶ Families of possible \{p_i\}.
  ▶ Neither spectrum nor OPE fixed.
– Consider $\Delta_0 = 0.76$
  ▶ Only one way to form zero.
  ▶ Non-zero $p_i$ fixed $\Rightarrow$ unique spectrum.
  ▶ Value of $p_i := (C_{ii}^k)^2$ fixed $\Rightarrow$ unique OPE.
  ▶ Non-zero $p_i$: $\Delta \sim 0.76, \quad L = 0$
  ▶ $\Delta = 3, \quad L = 2$

– NOTE: Num operators $\sim$ num components of $\vec{F}_i$
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Lessons Learned

Geometry of Solution Space

General properties of the approach:

1. We are never proving that a CFT exists.
   - We only check a *subset* of the constraints coming from a single correlator.
   - In $d > 2$ we do not even know what a sufficient criteria is for CFT to exist!

2. On the other hand we can prove that CFTs cannot exist with certain properties.

3. By adding more vectors (i.e. allowing more operators in the spectrum) we can transition from having no solutions to having many possible solutions.

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5. This “visual” proof was nice for intuition but does not generalize well when we consider many ($\gg 2$) constraints/Taylor coefficients at once.

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   - In $d > 2$ we do not even know what a sufficient criteria is for CFT to exist!

2. On the other hand we can prove that CFTs *cannot* exist with certain properties.

3. By adding more vectors (i.e. allowing more operators in the spectrum) we can transition from having no solutions to having many possible solutions.

4. In the boundary between these regions we get a unique solution.

5. This “visual” proof was nice for intuition but does not generalize well when we consider many ($\gg 2$) constraints/Taylor coefficients at once.

The next step is to formulate the bootstrap in a more abstract way and apply standard numerical algorithms (e.g. Linear Programming).
Lessons Learned
Geometry of Solution Space

General properties of the approach:

1. We are never proving that a CFT exists.
   - We only check a *subset* of the constraints coming from a single correlator.
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The next step is to formulate the bootstrap in a more abstract way and apply standard numerical algorithms (e.g. Linear Programming).
Linear Programming
The problem we want to solve is:

$$\sum_{\Delta, l} p_{\Delta, l} F_{\Delta, l}(z, \bar{z}) = 0$$

Taylor expanding around $z = \bar{z} = 1/2$ and requiring each order to vanish gives a matrix:

$$\begin{pmatrix}
    F_{1}^{(0,0)} & F_{2}^{(0,0)} & F_{3}^{(0,0)} \\
    F_{1}^{(2,0)} & F_{2}^{(2,0)} & F_{3}^{(2,0)} \\
    F_{1}^{(0,2)} & F_{2}^{(0,2)} & F_{3}^{(0,2)} \\
    \downarrow \partial & \vdots & \vdots \\
\end{pmatrix} \begin{pmatrix}
    p_{1} \\
    p_{2} \\
    p_{3} \\
    \vdots \\
\end{pmatrix} \Rightarrow \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    \vdots \\
\end{pmatrix}$$

which we must solve subject to $p_{i} \geq 0$.

- Rows of matrix $M_S$ are Taylor coefficeints (labelled by derivaties: $\partial^{m}_{z} \partial^{n}_{\bar{z}}$).
- Columns are operators $\mathcal{O}_k$ allowed in spectrum (continuous label $k = \{\Delta, l\} \in \mathcal{S}$)
- "Matrix" $M_S$ depends on the choice of allowed spectrum.
- "Vector" $\vec{p} = ((C_{\phi\phi}^{1})^{2}, (C_{\phi\phi}^{2})^{2}, \ldots)$ will have mostly zeros: non-zero $p_{i} \Leftrightarrow$ operator $(\Delta_{i}, l_{i})$ in the $\phi \times \phi$ OPE.
Direct Method
Bootstrap Formulation

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Methodology
Linear Programming

Can rephrase our problem as a linear optimization problem:

Minimize: \( \vec{c} \cdot \vec{p}, \)
subject to: \( M_S \cdot \vec{p} \sim \vec{b}, \quad \vec{p} \geq 0 \)

Where \( \sim \) can mean either \( =, \geq, \) or \( \leq. \)

- \( M_S \) is a matrix with columns \( \vec{v}_\alpha \), the derivs of the \( F_\alpha. \)
- \( \vec{p} = \{p_0, p_1, \ldots \} \), the squared coupling constants \( C_{\phi\phi}^\alpha. \)
- In simplest case \( \vec{c} = 0, \vec{b} = 0 \) and we take \( M_S \cdot \vec{p} = 0. \)

Some issues with this:
- \( \infty \) number of derivs \( \Rightarrow \) truncate to \( n. \)
- Couplings \( p_\Delta,l \) have a continuous label \( \Delta. \)
- For \( n \) derivs \( M_S \) is an \( n \times \infty \) matrix and \( \vec{p} \) is \( \infty \)-vector.

Approach: use Linear Programming (LP)

Modified Simplex Algorithm for (semi-) continuous variables.
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Bounding Operator Dimensions
Linear Programming

How do we produce a plot like this?

1. Fix $\Delta_\sigma$ to some value (e.g. 0.6).
2. Fix gap $\Delta_\epsilon = \Delta_1$ (e.g. = 1.6).
3. This defines a putative spectrum

$$S_1 = \{ \Delta_{l=0} \geq \Delta_1; \quad \Delta_{l>0} \geq \Delta_{\text{unitarity}} \}$$

4. Use LP to find $\vec{p}$ with $M = M_{S_1}$.
5. If LP find non-zero $\vec{p}$:
   $\Rightarrow$ CFT can exist
   increase $\Delta_1$ and try again.
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7. Bisecting in $\Delta_1$ until we find max value $\Delta_\epsilon^{\text{max}}$ (to some resolution).
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Figure: a putative spectrum $S$
Bounding Operator Dimensions
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![Figure: a putative spectrum $S$](image-url)
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Bounding OPE Coefficients
Linear Programming

What else can we bound?

Bootstrap allows us to:

- Consider arbitrary CFT data $\mathcal{S} = \{ (\Delta_i, \ell_i), C_{ijk} \}$.
- Check if this $\mathcal{S}$ is consistent with crossing sym of $\langle \sigma \sigma \sigma \sigma \rangle$.

Any time a bound is saturated can compute full OPE.

Kinds of bounds we can place on $\mathcal{S}$:

- Maximize a gap $\Delta_\epsilon$ in $\sigma \times \sigma$ OPE.
- Maximize coefficient of operator

\[
\sigma \times \sigma \sim 1 + \lambda^\epsilon_{\sigma \sigma} \epsilon + \cdots + \lambda^T_{\sigma \sigma} T_{\mu \nu} + \cdots
\]

Can bound dimension of first scalar on $\Delta_\epsilon$ (or any $\ell$).
Bounding OPE Coefficients

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Can bound (maximize) OPE coefficient of any operator.
Bounding OPE Coefficients

Linear Programming

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- Maximize coefficient of operator

\[
\sigma \times \sigma \sim 1 + \lambda_{\sigma\sigma}^\epsilon \epsilon + \cdots + \left( \frac{\Delta_\sigma}{\sqrt{c}} \right) T_{\mu\nu} + \ldots
\]

If operator e.g. \( T_{\mu\nu} \) get lower bound on \( c \).
Bounding OPE Coefficients

Linear Programming

- Instead of bounding $\Delta \epsilon$ we can formulate a different LP and bound OPE coefficients.

- Recall LP formulation:

  Minimize: $\vec{c} \cdot \vec{p}$,

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- E.g. if we take $c_{\Delta, l} = -1$ then LP will maximize OPE coefficient of $O_{\Delta, l}$.

When is this useful?

- Maximizing $T_{\mu \nu}$ OPE coeff.

- $p_{T_{\mu \nu}} = p_{D, l} = \left( \frac{\Delta^2}{c} \right)$ so equiv to $c$-minimization.
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“Solving” the 3d Ising Model?
Previous “State-of-the-Art”

“Solving” the 3d Ising Model?

3d Ising model

Using $\mathcal{E}$-expansion, Monte Carlo and other techniques find partial spectrum:

<table>
<thead>
<tr>
<th>Field:</th>
<th>$\sigma$</th>
<th>$\epsilon$</th>
<th>$\epsilon'$</th>
<th>$T_{\mu\nu}$</th>
<th>$C_{\mu\nu\rho\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dim ($\Delta$):</td>
<td>0.518135(50)</td>
<td>1.41275(25)</td>
<td>3.832(6)</td>
<td>3</td>
<td>5.0208(12)</td>
</tr>
<tr>
<td>Spin (l):</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Only 5 operators and no OPE coefficients known for 3d Ising!

Lots of room for improvement!

Our Goal

Compute these anomolous dimensions (and many more) and OPE coefficients using the bootstrap applied along the boundary curve (i.e. the EFM).
Spectrum of the 3d Ising Model

“Solving” the 3d Ising Model?

A first problem: what point on the boundary? what is correct value of $\sigma$?

1. “Kink” is not so sharp when we zoom in.
2. Gets sharper as we add more constraints.

Is there a better way to compute $\Delta_\sigma$ for 3d Ising?
“Solving” the 3d Ising Model?

Our first bound plot was made by maximizing $\Delta_\epsilon$.

$$\sigma \times \sigma \sim 1 + \lambda_{\sigma\sigma}^{\epsilon} \epsilon + \cdots + \lambda_{\sigma\sigma}^{T} T_{\mu\nu} + \cdots$$

- Look for solutions to crossing that maximize e.g. $C_{\sigma\sigma}^{T}$.
- OPE coefficient of stress tensor fixed by conformal symmetry (in terms of $c$).
- $c$ canonical stress-tensor normalization: $\langle T_{\mu\nu} T_{\rho\sigma} \rangle \sim c$.
- $T_{\mu\nu}$ OPE max $\Rightarrow c$ minimization.
Another way to find an extremal solution is to maximize an OPE coefficient.

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- \( c \) canonical stress-tensor normalization: \( \langle T_{\mu \nu} T_{\rho \sigma} \rangle \sim c \).
- \( T_{\mu \nu} \) OPE max \( \Rightarrow c \) minimization.

\[ D=3 \]

![Graph showing the ratio of \( C_T/C_{T,\text{free}} \) as a function of \( \Delta_{\sigma} \)]
"Solving" the 3d Ising Model?

Another way to find an extremal solution is to maximize an OPE coefficient.

\[ \sigma \times \sigma \sim 1 + \lambda_{\sigma \sigma} \epsilon + \cdots + \left( \frac{\Delta_{\sigma}}{\sqrt{c}} \right) T_{\mu \nu} + \cdots \]

In both \( d = 2, 3 \) Ising model minimizes \( c \).

Reproduces \( c = \frac{1}{2} \) in \( d = 2 \) to high precision!
3d Critical Exponents from $c$-minimization

“Solving” the 3d Ising Model?

Why minimize $c$?

- $c$-minimization also maximizes $\Delta_\epsilon \Rightarrow$ equivalent approaches.
- Location of “minimum” well defined while “kink” is somewhat ambiguous.
- $c$-minimization is more numerically stable and philosophically palatable.

Fixing $\Delta_\sigma$ to be at min of $c$ we compute exponents using extremizing solution:

Our estimates are 2-3× better than nearest competition.
- Get OPE coefficients to same precision.
- Can also compute higher spin/dimension fields in principle.
- In practice technical difficulties due to approximately conserved currents.
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Multi-correlator Bootstrap [Kos, Poland, Simmons-Duffin]

“Solving” the 3d Ising Model?

Similar to old bootstrap bounds but now consider $\langle \sigma \sigma \sigma \sigma \rangle$, $\langle \sigma \epsilon \sigma \epsilon \rangle$, and $\langle \epsilon \epsilon \epsilon \epsilon \rangle$.

1. Provides access to $\mathbb{Z}_2$ odd spectrum.
2. Only assumption: $\sigma$ is only relevant $\mathbb{Z}_2$ scalar.
3. 3d Ising model now approx $\text{only}$ solution for small $\sigma$!
Origin of the Kink?

(wild speculation)
Spectrum near the kink undergoes rapid re-arrangement.

Plots for next Scalar and Spin 2 Field

1. “Kink” in \((\epsilon, \sigma)\) plot due to rapid rearrangement of *higher dim spectrum*.
2. This is why its important to determine \(\sigma\) to high precision.
3. Does re-arrangement hint at some analytic structure we can use?
Can we find a nice explanation of the kink in $d = 2$?

- In $d = 2$ Virasoro strongly constrains spectrum.
- Minimal models ($c < 1$) have few (Virasoro) primaries in short representations of Virasoro.
- Ising model has only two Virasoro primaries: $|\sigma\rangle$ and $|\epsilon\rangle$.
- Virasoro decendent
  \[ T' = (L_{-2} + \eta L_{-1}^2) |\epsilon\rangle \]
  is a spin 2 SL(2,R) primary for certain values of $\eta$.
- Correct value of $\eta$ depends on $c$.
- Norm of $T'$ fixed by Virasoro.
- $T'$ becomes null at $c = 1/2$ (or $\Delta_\sigma = 1/8$)
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Origin of the Kink


Spin 2 Bound in $d=3$

Spin 2 Spectrum in $d=2$

Is $d=3$ kink also related to a null state decoupling?
Interpolating the Kink

Origin of the Kink

How can we understand 3d structures in terms of 2d?

Fractional spacetime dimension

- Conformal blocks analytic function of spacetime dimension $d$.
- Special structures emerge in $d = 1, 2, 4$

⇒ lessons for $d = 3$?

1. Follow “breaking” of Virasoro from $d = 2$?
2. Compare with $\mathcal{E}$-expansion.
3. Find kinks for all $1 < d < 4$
   ⇒ easier to extract kinematics

Track spectrum from $d = 2$ to $d = 3$?
Ising Model in Fractional Dimensions

Origin of the Kink

Upper bound on first scalar operator dimension

\[ \Delta \epsilon \approx 2 \Delta \sigma \]

\[ D = 2 \]

\[ D = 2.5 \]

\[ D = 3 \]

\[ D = 3.2 \]

\[ D = 3.5 \]

\[ D = 3.7 \]

\[ D = 3.8 \]

\[ D = 3.9 \]

\[ D = 4 \]
Other Ideas/Speculation

Origin of the Kink

Constraints from Higher Spin symmetry

1. Anomalous dimension of higher spin currents bounded

\[ \delta \Delta \leq 2(\Delta_\sigma - \Delta_{\text{free}}) \sim 0.037 \]

E.g. spin 4 has dim 5.02 (conserved current is \( \Delta = 5 \)).

[Nachtmann, Komargodski & Zhiboedov]

2. Higher spin symmetry only weakly-broken \( \Rightarrow \) approx conserved currents.
   \( \Rightarrow \) Origin of numerical difficulties for \( L > 2 \) spectrum

3. In \( d = 2, 4 \) higher spin fields conserved \( \Rightarrow \) is \( d = 3 \) maximal breaking?

“Hidden” Virasoro-like Symmetry in \( d = 3 \)

1. Ising model has infinite-dim symmetry in \( d = 2, 4 \).

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Other Applications
**\( \mathcal{N} = 4 \) bounds**

Other Applications [Liendo, Rastelli, van Rees]

Setup \( \mathcal{N} = 4 \) bootstrap. Bounds on leading twist dimension \( \Delta_\ell \) for \( \ell = 0, 2, 4 \).

(holds for all values of \( g_{YM} \))

1. Conjecture “corner” corresponds to a self-dual point.
2. Matches some checks computed by resumming perturbative results.
3. Non-perturbative bounds/results at \( g \sim \mathcal{O}(1) \).
4. [Alday, Bissi] also conjectured analytic results for OPE.
Analytic results/conjectures for 6d $\mathcal{N} = (0, 2)$ theories

Other Applications [Liendo, Rastelli, van Rees]

Bootstrap in $\mathcal{N} = (0, 2)$ theories requires first solving “mini-bootstrap” in a BPS subsector.

1. BPS subsector defined via cohomology of some charge.
3. Cohomology be projected onto 2d subspace of $\mathbb{R}^6$.
4. Operators in cohomology are chiral, depend only on $z \in \mathbb{C}$.
5. This sector has a 2d $\mathcal{W}$-algebra symmetry.
6. $\mathcal{W}$-algebra fixes three point functions of operators in cohomology.
7. By conjecturing algebra for $A_n$ theories get 3-pt functions for all valus of $N$!
Liberation at large spin

Other Applications [Fitzpatrick, Kaplan, Poland, Simmons-Duffin], [Komargodski, Zhiboedov]

In limit $|u| \ll |v| < 1$ leading twist field dominate bootstrap equations.

**Bootstrap equation in $u \to 0$ limit**

For every scalar $\phi$ of dimension $\Delta_\phi$

- CFT must contain an infinite tower of operators with dimension $\tau \to 2\Delta_\phi + \ell$.
- Asymptotic estimates holds in $\ell \to \infty$ limit.
- Morally these operators are $\phi(\partial^2)^n\partial_{\mu_1} \ldots \partial_{\mu_\ell} \phi$.

Another interesting constraint, suggested by Nachtmann (’70s), but derived more thoroughly by [Komargodski, Zhiboedov]:

**Convexity (Nachtmann’s theorem)**

Twist, $\tau_\ell = \Delta - \ell$, of leading twist operator

- Is an increasing, convex function of $\ell$.
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New Methods
Pros/Cons of our Method

Methods explained so far have several good properties:

1. Can make rigorous statements about non-existence of CFTs for various spectra, $S$.
2. We have control over our sources of error so can provide systematic error bounds.

But there are several un-features (i.e. bad things):

1. The method is a “blunt tool”: cannot easily “pick” which theory to study.
2. If theory we’re interested in is not at boundary of solution can’t get unique spectrum.
3. Computationally intensive: depending on desired accuracy might require computer cluster.
4. Only applies to unitary CFTs.

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2. Can give spectrum/OPE even for theories not on our boundaries.
3. Computationally much lighter (runs on a laptop).
4. Adding more correlators does not make it much harder.

But also here there are several problems:

1. Method is very non-systematic.
2. Requires much stronger assumption and some input from other methods.
3. Very little control over the error made from approximations.
4. Does not rigorously prove anything.

If could cure some of these problems (i.e. 1 & 3) with the method might be a much better way to proceed than our bootstrap.
The Competition: Gliozzi Method

New Methods

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If could cure some of these problems (i.e. 1 & 3) with the method might be a much better way to proceed than our bootstrap.
Recall in our method we make the following sort of assumptions about the spectrum:

Figure: a putative spectrum $S$
The Competition: Gliozzi Method

New Methods

This yields the following equation we solve (with $M_S$ depending on $S$):

$$
\begin{pmatrix}
F_1^{(0,0)} & F_2^{(0,0)} & F_3^{(0,0)} \\
F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} \\
F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} \\
\downarrow \partial & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
\Delta \\
\vdots
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
\vdots
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots
\end{pmatrix}
$$

Note:

- The columns of the matrix are labeled by $(\Delta, l)$ and there is a continuous infinity of them.
- This system is generally very under-determined.
- Generically there are $\infty$-many solutions to this equation (i.e. for $S$ deep in allowed region).
- This is because we made very weak assumptions about spectrum (e.g. just gap $\Delta_\epsilon$).
The Competition: Gliozzi Method

New Methods

This yields the following equation we solve (with $M_S$ depending on $S$):

$$\begin{bmatrix}
F_1^{(0,0)} & F_2^{(0,0)} & F_3^{(0,0)} \\
F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} \\
F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} \downarrow \\
\vdots & \vdots & \vdots \uparrow
\end{bmatrix}
\begin{bmatrix}
\Delta \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \downarrow \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \downarrow \\
\vdots
\end{bmatrix}$$

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\begin{pmatrix}
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F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} \\
F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} \\
\downarrow \partial & \vdots & \vdots \\
\end{pmatrix}_M
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
\vdots \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
\end{pmatrix}
$$

Note:

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- This system is generally very under-determined.
- Generically there are $\infty$-many solutions to this equation (i.e. for $S$ deep in allowed region).
- This is because we made very weak assumptions about spectrum (e.g. just gap $\Delta_e$).
The Competition: Gliozzi Method

New Methods

This yields the following equation we solve (with $M_S$ depending on $S$):

$$
\begin{pmatrix}
F_1^{(0,0)} & F_2^{(0,0)} & F_3^{(0,0)} \\
F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} \\
F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} \\
\downarrow \partial & \vdots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
\Delta \\
\end{pmatrix}
M_S
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
\vdots \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
\end{pmatrix}
$$

Note:

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The Competition: Gliozzi Method
New Methods

This yields the following equation we solve (with $M_S$ depending on $S$):

$$
\begin{bmatrix}
F_1^{(0,0)} & F_2^{(0,0)} & F_3^{(0,0)} & \Delta \\
F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} & \vdots \\
F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} & \vdots \\
\downarrow \partial & \vdots & \vdots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
\vdots \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
\end{bmatrix}
$$

$M_S$

Note:

- The columns of the matrix are labeled by $(\Delta, l)$ and there is a continuous infinity of them.
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The Competition: Gliozzi Method
New Methods

Gliozzi makes the following (well motivated?) assumptions:

1. Generically a CFT has a discrete (even sparse) spectrum.
2. Also we might expect that generically there should be unique solution to crossing (if we know enough about the spectrum).

So he proposes the following:

**Our assumptions**

\[ \sigma \times \sigma \sim 1 + \epsilon + \ldots \]

And we try to maximize \( \Delta \epsilon \) while allowing anything in ’…’.

Gliozzi wants to make the following much stronger assumptions:

**Gliozzi’s assumptions**

\[ \sigma \times \sigma \sim 1 + \epsilon + \epsilon' + T_{\mu\nu} + C_{\mu\nu\rho\lambda} + \ldots \]

Solve for \( \Delta \epsilon, \Delta \epsilon', \Delta C_{\mu\nu\rho\lambda} \) by requiring uniqueness and dropping ’…’.
The Competition: Gliozzi Method

New Methods

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Solve for \( \Delta \epsilon, \Delta \epsilon', \Delta C_{\mu\nu\rho\lambda} \) by requiring **uniqueness** and **dropping ’…’**.
Gliozzi proposes to solve the following simplified problem:

\[
\begin{pmatrix}
F_{e}^{(0,0)} & F_{e'}^{(0,0)} & F_{T_{\mu\nu}}^{(0,0)} & F_{C_4}^{(0,0)} \\
F_{e}^{(2,0)} & F_{e'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\
F_{e}^{(0,2)} & F_{e'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\
F_{e}^{(2,2)} & F_{e'}^{(2,2)} & F_{T_{\mu\nu}}^{(2,2)} & F_{C_4}^{(2,2)} \\
F_{e}^{(4,0)} & F_{e'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)}
\end{pmatrix}
\begin{pmatrix}
p_e \\
p_{e'} \\
p_{T_{\mu\nu}} \\
p_{C_4}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\mathcal{M}_{\sigma, e, e', C}
\]

- He truncates the spectrum to \( N \) discrete operators (here \( N = 4 \)).
- He truncates derivates to some \( M \geq N \) (here \( M = 5 \)).
- He keeps \( n < N \) free parameters: \( \Delta_{\sigma}, \Delta_{e}, \Delta_{e'}, \Delta_{C_4} \) (so here \( n = 4 \)).
- He calls the CFT “truncatable” if system above admits unique solutions.

**Intuition:** at a given derivative order, \( M \), there should be a unique (approx) solution by keeping only first \( N \) operators (hence: “truncatable”).
Gliozzi proposes to solve the following simplified problem:

\[
\begin{pmatrix}
F_{\epsilon}^{(0,0)} & F_{\epsilon'}^{(0,0)} & F_{T_{\mu\nu}}^{(0,0)} & F_{C_4}^{(0,0)} \\
F_{\epsilon}^{(2,0)} & F_{\epsilon'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\
F_{\epsilon}^{(0,2)} & F_{\epsilon'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\
F_{\epsilon}^{(2,2)} & F_{\epsilon'}^{(2,2)} & F_{T_{\mu\nu}}^{(2,2)} & F_{C_4}^{(2,2)} \\
F_{\epsilon}^{(4,0)} & F_{\epsilon'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)}
\end{pmatrix}
\begin{pmatrix}
p_{\epsilon} \\
p_{\epsilon'} \\
p_{T_{\mu\nu}} \\
p_{C_4}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

He truncates the spectrum to \( N \) discrete operators (here \( N = 4 \)).

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F_{\epsilon}^{(2,0)} & F_{\epsilon'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\
F_{\epsilon}^{(0,2)} & F_{\epsilon'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\
F_{\epsilon}^{(2,2)} & F_{\epsilon'}^{(2,2)} & F_{T_{\mu\nu}}^{(2,2)} & F_{C_4}^{(2,2)} \\
F_{\epsilon}^{(4,0)} & F_{\epsilon'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)}
\end{pmatrix}
\begin{pmatrix}
p_{\epsilon} \\
p_{\epsilon'} \\
p_{T_{\mu\nu}} \\
p_{C_4}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
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\[
\begin{pmatrix}
F^{(0,0)}_\epsilon & F^{(0,0)}_{\epsilon'} & F^{(0,0)}_{T_{\mu\nu}} & F^{(0,0)}_{C_4} \\
F^{(2,0)}_\epsilon & F^{(2,0)}_{\epsilon'} & F^{(2,0)}_{T_{\mu\nu}} & F^{(2,0)}_{C_4} \\
F^{(0,2)}_\epsilon & F^{(0,2)}_{\epsilon'} & F^{(0,2)}_{T_{\mu\nu}} & F^{(0,2)}_{C_4} \\
F^{(2,2)}_\epsilon & F^{(2,2)}_{\epsilon'} & F^{(2,2)}_{T_{\mu\nu}} & F^{(2,2)}_{C_4} \\
F^{(4,0)}_\epsilon & F^{(4,0)}_{\epsilon'} & F^{(4,0)}_{T_{\mu\nu}} & F^{(4,0)}_{C_4}
\end{pmatrix}
\begin{pmatrix}
p_\epsilon \\
p_{\epsilon'} \\
p_{T_{\mu\nu}} \\
p_{C_4}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[\mathcal{M}_{\sigma, \epsilon, \epsilon', C}\]

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\[
\begin{pmatrix}
F_{\epsilon}^{(0,0)} & F_{\epsilon'}^{(0,0)} & F_{T_{\mu\nu}}^{(0,0)} & F_{C_4}^{(0,0)} \\
F_{\epsilon}^{(2,0)} & F_{\epsilon'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\
F_{\epsilon}^{(0,2)} & F_{\epsilon'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\
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F_{\epsilon}^{(4,0)} & F_{\epsilon'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)} \\
\end{pmatrix}
\begin{pmatrix}
p_{\epsilon} \\
p_{\epsilon'} \\
p_{T_{\mu\nu}} \\
p_{C_4} \\
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0 \\
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The Competition: Gliozzi Method

New Methods

Gliozzi proposes to solve the following simplified problem:

\[
\begin{pmatrix}
F_{(0,0)}^{0} & F_{(0,0)}^{0} & F_{(0,0)}^{T_{\mu\nu}} & F_{C_{4}}^{0} \\
F_{(2,0)}^{0} & F_{(2,0)}^{0} & F_{(2,0)}^{T_{\mu\nu}} & F_{C_{4}}^{0} \\
F_{(0,2)}^{0} & F_{(0,2)}^{0} & F_{(0,2)}^{T_{\mu\nu}} & F_{C_{4}}^{0} \\
F_{(2,2)}^{0} & F_{(2,2)}^{0} & F_{(2,2)}^{T_{\mu\nu}} & F_{C_{4}}^{0} \\
F_{(4,0)}^{0} & F_{(4,0)}^{0} & F_{(4,0)}^{T_{\mu\nu}} & F_{C_{4}}^{0}
\end{pmatrix}
\begin{pmatrix}
p_{e} \\
p_{e'} \\
p_{T_{\mu\nu}} \\
p_{C_{4}}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\mathcal{M}_{\sigma,\epsilon,\epsilon',C}
\]

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The Competition: Gliozzi Method

New Methods

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\[
\begin{pmatrix}
F^{(0,0)}_\epsilon & F^{(0,0)}_\epsilon' & F^{(0,0)}_T & F^{(0,0)}_{C_4} \\
F^{(2,0)}_\epsilon & F^{(2,0)}_\epsilon' & F^{(2,0)}_T & F^{(2,0)}_{C_4} \\
F^{(0,2)}_\epsilon & F^{(0,2)}_\epsilon' & F^{(0,2)}_T & F^{(0,2)}_{C_4} \\
F^{(2,2)}_\epsilon & F^{(2,2)}_\epsilon' & F^{(2,2)}_T & F^{(2,2)}_{C_4} \\
F^{(4,0)}_\epsilon & F^{(4,0)}_\epsilon' & F^{(4,0)}_T & F^{(4,0)}_{C_4}
\end{pmatrix}
\begin{pmatrix}
p_\epsilon \\
p_\epsilon' \\
p_{T_{\mu\nu}} \\
p_{C_4}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[\mathcal{M}_{\sigma,\epsilon,\epsilon',c}\]

How do we solve this system?

1. Gliozzi claims crossing symmetry should have a unique solution.
2. If \(M = N\) (so \(\mathcal{M}\) square) uniqueness means \(\det(\mathcal{M}) = 0\).
3. More generally take \(M > N\) and require \(\det(\mathcal{M}_i) = 0\) with \(\mathcal{M}_i\) square sub-matrices.
4. Example: \(M = N + 1\).
   - \(\mathcal{M}_i\) are \(M\) square sub-matrices given by removing \(i\)'th row.
Gliozzi proposes to solve the following simplified problem:

\[
\begin{pmatrix}
F_{\varepsilon}^{(0,0)} & F_{\varepsilon'}^{(0,0)} & F_{T_{\mu\nu}}^{(0,0)} & F_{C_4}^{(0,0)} \\
F_{\varepsilon}^{(2,0)} & F_{\varepsilon'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\
F_{\varepsilon}^{(0,2)} & F_{\varepsilon'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\
F_{\varepsilon}^{(4,0)} & F_{\varepsilon'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)}
\end{pmatrix}
\begin{pmatrix}
p_{\varepsilon} \\
p_{\varepsilon'} \\
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The Competition: Gliozzi Method

New Methods

So e.g. if we take $M = N + 1$ get a system of $M(= 5$ here) equations:

$$\det(M_i) = 0 \quad i = 1, \ldots, 5$$

for $n = 4$ unknowns (because the $\det(M_i)$ depends on $\Delta_\sigma, \Delta_\epsilon, \Delta_\epsilon', \Delta_{C_4}$).

- $M$ eqns for $n$ unknowns is generally over-constrained.
- Gliozzi proposes to solve $n$ eqns at a time (to get unique solutions) and compute “spread” of solutions.
- E.g. for 3d ising model fixed $C_4 = 5.022$ (as input) and computes $\Delta_\sigma, \Delta_\epsilon$:

![Graph showing the relationship between $\delta(\Delta_\sigma)$ and $\delta(\Delta_\epsilon)$]
The Competition: Gliozzi Method

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![Graph showing the relationship between $\delta(\Delta_\sigma)$ and $\delta(\Delta_\epsilon)$ with data points.](image)
The Competition: Gliozzi Method

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So e.g. if we take $M = N + 1$ get a system of $M(= 5$ here) equations:

$$\det(M_i) = 0 \quad i = 1, \ldots, 5$$

for $n = 4$ unknowns (because the $\det(M_i)$ depends on $\Delta_\sigma, \Delta_\epsilon, \Delta_\epsilon', \Delta_{C_4}$).

- $M$ eqns for $n$ unknowns is generally over-constrained.
- Gliozzi proposes to solve $n$ eqns at a time (to get unique solutions) and compute “spread” of solutions.
- E.g. for 3d Ising model fixed $C_4 = 5.022$ (as input) and computes $\Delta_\sigma, \Delta_\epsilon$: 

\[\begin{align*}
\delta(\Delta_\sigma) &\quad 1 \times 10^{-7} \\
5 \times 10^{-8} &\quad -5 \times 10^{-8} \\
-2 \times 10^{-9} &\quad 2 \times 10^{-9}
\end{align*}\]
The Competition: Gliozzi Method
New Methods

1. Using this approach computes $\Delta_\sigma$, $\Delta_\epsilon$, $\Delta_{\epsilon'}$, $\Delta_{\epsilon''}$, etc... in 3d Ising.

2. His solution has one free parameter which he fixes using $\Delta_{C_4} = 5.022$ (input from Monte Carlo).

3. Getting these quantities requires much less computation than our method.

Problem: by throwing away high-dim operators he’s making an error $O(e^{-\Delta_\ast})$ with $\Delta_\ast \sim 10$. So real equation is:

$$
\begin{pmatrix}
F_{\epsilon}^{(0,0)} & F_{\epsilon'}^{(0,0)} & F_{T_{\mu\nu}}^{(0,0)} & F_{C_4}^{(0,0)} \\
F_{\epsilon}^{(2,0)} & F_{\epsilon'}^{(2,0)} & F_{T_{\mu\nu}}^{(2,0)} & F_{C_4}^{(2,0)} \\
F_{\epsilon}^{(0,2)} & F_{\epsilon'}^{(0,2)} & F_{T_{\mu\nu}}^{(0,2)} & F_{C_4}^{(0,2)} \\
F_{\epsilon}^{(2,2)} & F_{\epsilon'}^{(2,2)} & F_{T_{\mu\nu}}^{(2,2)} & F_{C_4}^{(2,2)} \\
F_{\epsilon}^{(4,0)} & F_{\epsilon'}^{(4,0)} & F_{T_{\mu\nu}}^{(4,0)} & F_{C_4}^{(4,0)} \\
\end{pmatrix}
\begin{pmatrix}
p_{\epsilon} \\
p_{\epsilon'} \\
p_{T_{\mu\nu}} \\
p_{C_4} \\
\end{pmatrix} =
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\end{pmatrix}
$$

with $||\vec{\alpha}|| < e^{-10}$.
We have no idea how varying $\vec{\alpha}$ affects values of $\Delta_\epsilon$, $\Delta_{\epsilon'}$, etc.
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The Competition: Gliozzi Method
New Methods

1. In Gliozzi method must guess number of operators, \( N \), with \( \Delta < \Delta^* \).
2. Wrong guess may give no solutions or too many solutions, etc.
3. Need \( \Delta^* \) to be small else would have too many parameters.
4. In our approach \( \Delta^* \gg 1 \) (can be 50, 100, etc.).
5. So there are many choices to make in Gliozzi method \( \Rightarrow \) more of an art.
6. Also no control over the (relatively) larger error.
7. Solving \( \det M_i = 0 \) non-trivial as we increase \( M, N \) and \( n \).

But
1. Can be done with Mathematica on a laptop.
2. Doesn’t assume unitarity at all (he checks it in non-unitary theories).
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Bootstrapping Theories with Four Supercharges (in $d = 2 - 4$)
How can we include SUSY constraints in bootstrap?

1. SUSY relates OPE coefficients of components of SUSY multiplets in some correlators.
2. This yields superconformal blocks for whole SUSY multiplet:
   \[ G_{\Delta,l} = G_{\Delta,l} + c_1 G_{\Delta+1,l+1} + c_2 G_{\Delta+1,l-1} + c_3 G_{\Delta+2,l} \]
   with \( c_1, c_2, c_3 \) fixed by SUSY.
3. SUSY fixes dimensions of protected operators by e.g. \( a \)-maximization.
4. SUSY imposes stronger unitarity bounds in terms of R-charge:
   \[ \Delta \geq \frac{d - 1}{2} R \]
   (for scalars in theories with 4 supercharges)

Can now try to bootstrap using these additional constraints.
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The Benefits of SUSY
Bootstrapping Theories with Four Supercharges

In theories with 4 supercharges dimensions of some operators constrained by SUSY.

- **Chiral operator** is annihilated by half supercharges.
- SUSY algebra contains $U(1)$ R-charge and $\Delta$ of chiral operator depends on its R-charge:
  \[
  \Delta = \left( \frac{d - 1}{2} \right) R
  \]
- When only one kind of field superpotential can fix R-charge uniquely since superpotential $W$ must have $R$-charge 2. E.g.:
  \[
  W = X^3
  \]
  implies superfield $X$ has R-charge 2/3.
- If more than one field (e.g. $XY^2$) can use $a$-maximization to compute R-charge.
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SUSY Constraints

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Let's consider theories with \textbf{four} supercharges in \( d = 2 - 4 \).

- Let \( \Phi \) be complex chiral scalar field so \( \Delta = \left( \frac{d-1}{2} \right) R \).
- To get strong constraints we consider

\[
\langle \Phi \Phi \Phi \Phi \rangle, \quad \langle \Phi \bar{\Phi} \Phi \bar{\Phi} \rangle
\]

- Although multiple correlators there’s lots of overlap so actually not so hard.
- \( \Phi \) carries \( R \)-charge so can decompose OPE in reps of \( R \)-charge:

\[
\Phi \times \bar{\Phi} \sim \text{singlets}
\]

\[
\Phi \times \Phi \sim \text{R-charge 2}
\]

- CB decomposition gives different constraints in each \( R \)-charge channel.
- Because \( \Phi, \bar{\Phi} \) not identical can get odd-spin contributions in \( \langle \Phi \bar{\Phi} \Phi \bar{\Phi} \rangle \).
- SUSY (+R-charge) fixes dims of some contributions in \( \Phi \times \Phi \) OPE:

\[
\Phi \times \Phi \sim 1 + \Psi_{d-2\Delta\Phi,0} + \Phi^2 + \ldots
\]

with \( \Delta \geq |2\Delta\Phi - (d-1)| + l + (d-1) \) for ’…”.
SUSY Constraints
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- Because $\Phi, \bar{\Phi}$ not identical can get odd-spin contributions in $\langle \Phi\bar{\Phi}\Phi\bar{\Phi} \rangle$.
- SUSY (+R-charge) fixes dims of some contributions in $\Phi \times \Phi$ OPE:

$$\Phi \times \Phi \sim 1 + \Psi_{d-2\Delta\Phi,0} + \Phi^2 + \ldots$$

with $\Delta \geq |2\Delta\Phi - (d - 1)| + l + (d - 1)$ for ‘…”.
SUSY Constraints

Bootstrapping Theories with Four Supercharges

Let's consider theories with four supercharges in $d = 2 - 4$.

- Let $\Phi$ be complex chiral scalar field so $\Delta = \left(\frac{d-1}{2}\right) R$.
- To get strong constraints we consider

$$\langle \Phi \Phi \Phi \Phi \rangle, \quad \langle \Phi \bar{\Phi} \Phi \bar{\Phi} \rangle$$

- Although multiple correlators there's lots of overlap so actually not so hard.
- $\Phi$ carries $R$-charge so can decompose OPE in reps of $R$-charge:

$$\Phi \times \bar{\Phi} \sim \text{singlets}$$
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Results
Bootstrapping Theories with Four Supercharges

So what kind of bounds to we get?

- Bounds for $d = 2 - 4$.
- Multiple kinks!!
- Horizontal dashed line: $\Delta \Phi$ in Wess-Zumino model
- This is SUSY version of $\phi^4$ theory!
- $\Delta \Phi$ fixed because superpotential $W = \Phi^3$

has $R = 2$ so $\Delta \Phi = \frac{d-1}{3}$.

- So we found SUSY Ising for $d = 2 - 4$.
- But also two more kinks!
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$\Delta [\Phi \bar{\Phi}]$ Bound (zoomed)
Results
Bootstrapping Theories with Four Supercharges

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Results/Checks

Bootstrapping Theories with Four Supercharges

SUSY also imposes interesting dynamical constraints on theory

- Consider Wess-Zumino model: chiral superfield $X = \Phi + \ldots$ with cubic superpotential:

  $W = X^3$

- Implies superfield $X$ has R-charge 2/3 and (in $d = 3$) $\Delta \Phi = R$.

- The fact that in e.g. WZ model can compute $\Delta \Phi$ exactly means can extract spectrum easily.

- SUSY eqns $\frac{\partial W}{\partial X} = 0$ implies $\Phi^2$ should decouple in theory.
New Structure
Bootstrapping Theories with Four Supercharges

In non-SUSY 3d Ising found interesting (surprising) kinematical structure.

What about SUSY case?

Bounds on Spin 1*

(*Because of susy $T_{\mu\nu}$ and $T'_{\mu\nu}$ are actually SUSY descendents in spin 1 multiplet.)

SUSY analog of 3d null states!!
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SUSY analog of 3d null states!!
Our (Modified) Simplex Algorithm
The Standard Simplex Algorithm

Our (Modified) Simplex Algorithm

Recall the original problem we want to solve

Minimize: \( \vec{c}^T \cdot \vec{x} \),

subject to: \( M \cdot \vec{x} = \vec{b}, \quad \vec{x} \geq 0 \)

Trivial solution is \( \vec{x} = 0 \) and \( \vec{y} = A^{-1}\vec{b} \) but now must reduce cost:

1. Turning on component \( x_\alpha \) reduces cost via

\[
\vec{c}_y^T \cdot \vec{y} = \vec{c}_y^T A^{-1}(\vec{b} - x_\alpha \vec{v}_\alpha)
\]

2. Choose component \( \alpha \) by maximizing \( \vec{c}_y^T \cdot A^{-1}\vec{v}_\alpha \) (contribution to cost).

3. Increase \( x_\alpha \) until some component of \( \vec{y} \) becomes some zero. e.g.

\[
x_\alpha \vec{v}_\alpha^i = b^i \quad \Rightarrow \quad y^i = 0
\]

4. “Swap in” \( \vec{v}_\alpha \) into \( i \)’th column of \( A \).

5. Repeat with new \( A \) until all components of \( \vec{y} \) set to zero.

6. This yields “feasable” \( \vec{x} \); can then turn on \( c_x \) and continue.
The Standard Simplex Algorithm

Our (Modified) Simplex Algorithm

Introduce $n$ “slack” variables $y$ with costs $\bar{c}_y^T = (1, \ldots, 1)$ and solve:

Minimize: $\bar{c}_y^T \cdot \bar{y}$,
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Minimize: \( \tilde{c}_y^T \cdot \tilde{y}, \)

subject to: \( M \cdot \tilde{x} + A \cdot \tilde{y} = \tilde{b}, \quad \tilde{x}, \tilde{y} \geq 0 \)

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Introduce $n$ “slack” variables $y$ with costs $c^T_y = (1, \ldots, 1)$ and solve:

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Modifying the Simplex Algorithm

Our (Modified) Simplex Algorithm

Modifications for \( x \) continuously labelled, \( \alpha \rightarrow \Delta \):

- \( \vec{y} \) is \( n \)-vector and \( A \) is \( n \times n \) matrix so remains discrete.
- \( x_\Delta \) only appears in minimzation stage so we need to maximize:
  \[
  \rho(\Delta) = c_y^T A^{-1} \vec{v}_\Delta
  \]

- Various approaches:
  1. Branch & bound on \( \rho(\Delta) \).
  2. Local quadratic or cubic approximation.

- Technical Issues:
  1. \( \rho(\Delta) \) approximated by very high-order polynomial \( O(\Delta^{60}) \).
  2. Matrix \( A \) becomes ill-conditioned in most physically interesting cases.
  3. Numerical precision insufficient \( \Rightarrow \) MPFR/GMP
How to get Started Bootstrapping...
Some Resources (code)
How to get Started Bootstrapping…

So here are some ways to start:

Conformal Blocks

1. In 2d/4d can use exact expressions (see e.g. arXiv:0807.0004).
2. In general \(d\) (or just faster):
   - See JuliaBootS (below).

Bootstrapping

1. Mathematica’s LinearProgramming function (good luck!)
2. Mathematica + IBM’s CPLEX LP (email me for mathematica plugin)
3. Mathematica + SDPA (probably the most standard solution now)
5. Our python implementation (to be released soon ?!?!)
6. Roll your own…
Thanks
Dual Method
Bootstrap Formulation

- We can also formulate the problem in a “dual” way.
- Instead of solving for $\lambda$ we can look for a diff op $\alpha$ such that:

$$\alpha(F_i) > 0 \quad \forall F_i \quad \left( \alpha = \sum_{n,m} \alpha_{n,m} \partial^n_z \partial^m_{\bar{z}} \right)$$

- Taylor expanding around $z = \bar{z} = 1/2$ and requiring each order to vanish gives a matrix:

$$
\begin{pmatrix}
F_1^{(0,0)} & F_2^{(0,0)} & F_3^{(0,0)} \\
F_1^{(2,0)} & F_2^{(2,0)} & F_3^{(2,0)} \\
F_1^{(0,2)} & F_2^{(0,2)} & F_3^{(0,2)} \\
\downarrow \partial & \vdots & \vdots \\
\end{pmatrix} \xrightarrow{\Delta} 
\begin{pmatrix}
\lambda_1^2 \\
\lambda_2^2 \\
\lambda_3^2 \\
\vdots \\
\lambda \end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
\end{pmatrix}
$$

which we must solve subject to $\lambda_i^2 \geq 0$.

- Solving for $\lambda_i^2$ is called the **Direct Problem**.