The uses of entanglement entropy in QFT and holography

Tatsuma Nishioka (University of Tokyo)

The 9th Asian Winter School, Busan, Korea

Introduction

Entanglement
Entropy

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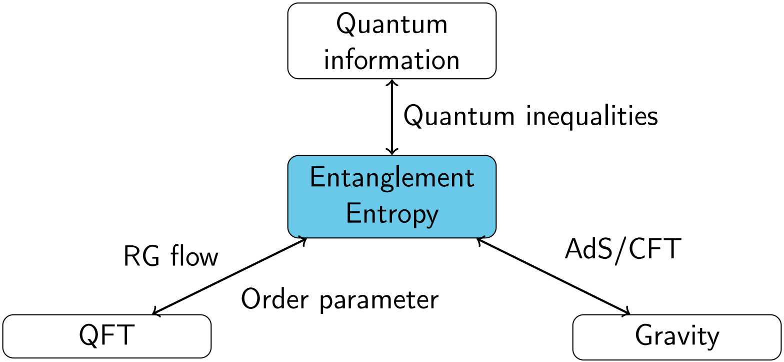

What's the role of entanglement entropy in QFT?

- Entanglement entropy as a measure of degrees of freedom

- Construct a monotonic function $c(\text{Energy})$ of the energy scale
  - Entropic c-theorem in two dimensions
  - F-theorem in three dimensions

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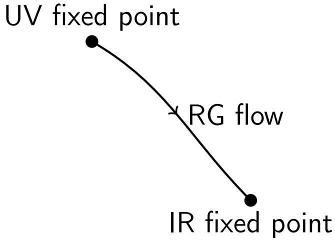

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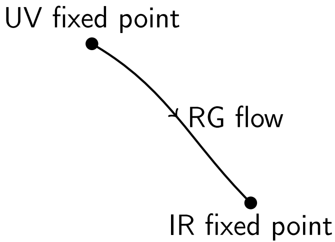

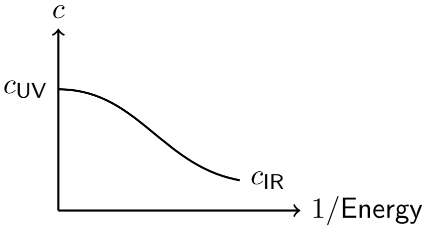

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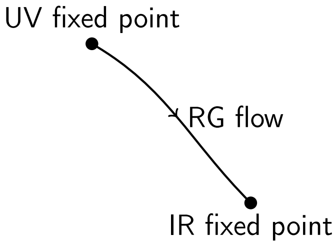

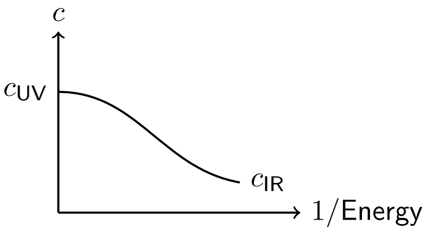

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  - Similarity between MERA and AdS space
  - 1st law of entanglement and linearized Einstein equation of GR

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Holography geometrizes the renormalization group (RG) flow

$$[\mathsf{R}, \mathsf{G}] = \mathsf{RG} - \mathsf{GR} = 0$$

Outline

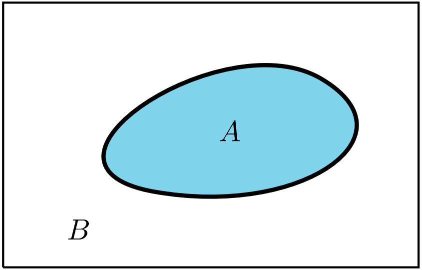
References

- Nielsen-Chuang, “Quantum Computation and Quantum Information", Cambridge university press, 2010
- Calabrese-Cardy, arXiv:0905.4013
- Casini-Huerta, arXiv:0903.5284
- Solodukhin, arXiv:1104.3712
- TN-Ryu-Takayanagi, arXiv:0905.0932
- Takayanagi, arXiv:1204.2450

Outline

Definition of entanglement entropy

- Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$



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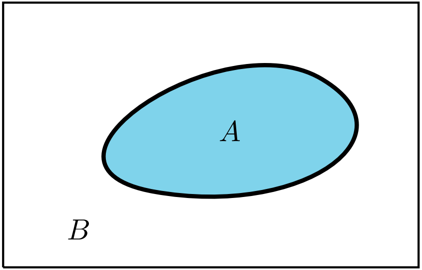

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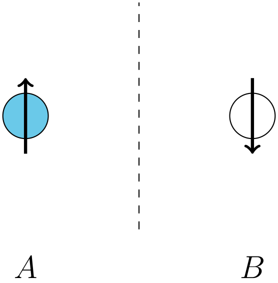
- Reduced density matrix:

$$\rho_A = \mathrm{tr}_B \rho_{tot} = \sum_i \langle\psi_B^i|\rho_{tot}|\psi_B^i\rangle$$

$\mathcal{H}_B = \{|\psi_B^1\rangle, |\psi_B^2\rangle, \cdots\}$ orthonormal basis

Example: two spin system



- Hilbert spaces: $\mathcal{H}_A = \{|\uparrow\rangle_A, |\downarrow\rangle_A\}$, $\mathcal{H}_B = \{|\uparrow\rangle_B, |\downarrow\rangle_B\}$

Example: two spin system

- Given a ground state ($\langle\Psi|\Psi\rangle = 1$):

$$|\Psi\rangle = \cos\theta\,|\uparrow\rangle_A|\downarrow\rangle_B + \sin\theta\,|\downarrow\rangle_A|\uparrow\rangle_B$$

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- Matrix notation:

$$\rho_A = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

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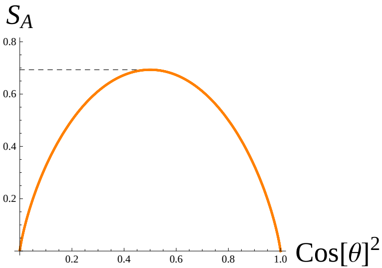
  - $\cos^2\theta = \frac{1}{2}$: Maximally entangled, $S_A = \log 2$
  - $\cos^2\theta = 0, 1$: No entanglement , $S_A = 0$



Quantum mechanical system

Suppose $|\Psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |\psi_A^i\rangle |\psi_B^j\rangle , \qquad d_{A,B} \equiv \mathrm{dim}\mathcal{H}_{A,B}$

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$c_{ij} = c_i^A c_j^B$: pure product state

$$|\Psi\rangle = |\Psi_A\rangle|\Psi_B\rangle \ , \qquad |\Psi_{A,B}\rangle \equiv \sum_i c_i^{A,B} |\psi_{A,B}^i\rangle \ ,$$

$$\rho_A = |\Psi_A\rangle\langle\Psi_A| \quad \Rightarrow \quad S_A = 0$$

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$$c_{ij} = U_{ik} \lambda_k V_{kj} \ , \qquad U, V \ : \text{unitary} \ ,$$

$$|\Psi\rangle = \sum_{k=1}^{\min(d_A, d_B)} \lambda_k |\tilde{\psi}_A^k\rangle |\tilde{\psi}_B^k\rangle \ , \qquad \lambda_k \geq 0 \ , \sum_k \lambda_k^2 = 1 \ ,$$

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Maximally entangled state

$$\text{For } \lambda_1 = \lambda_2 = \cdots = 1/\sqrt{\min(d_A, d_B)} \ ,$$

$$S_A = \log \min(d_A, d_B)$$

Properties of entanglement entropy

For a pure ground state

$$S_A = S_{\bar{A}}$$

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Strong subadditivity

$$S_{A\cup B\cup C} + S_B \le S_{A\cup B} + S_{B\cup C}$$
$$S_A + S_C \le S_{A\cup B} + S_{B\cup C}$$

for any three disjoint regions A, B and C

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Mutual information

$$I(A,B) \equiv S_A + S_B - S_{A\cup B} \geq 0$$

for any disjoint two regions A and B

n-th Rényi entropy

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Inequalities

$$\begin{aligned} \partial_n S_n &\leq 0 \\ \partial_n \left(\frac{n-1}{n} S_n \right) &\geq 0 \\ \partial_n \left((n-1) S_n \right) &\geq 0 \\ \partial_n^2 \left((n-1) S_n \right) &\leq 0 \end{aligned}$$

Relative entropy

For two states ρ and σ

$$S(\rho||\sigma) = \mathrm{tr}\left[\rho(\log\rho - \log\sigma)\right]$$

It measures the distance between the two states

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$$S(\rho||\sigma) \geq \frac{1}{2}||\rho - \sigma||^2 \qquad \text{Positivity}$$

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The strong subadditivity follows from the last inequality

Outline

QFTs and replica trick

- $\dim\mathcal{H} = \infty$ in QFT

- Useful trick:

$$S_A = -\partial_n \log \mathrm{tr}_A \rho_A^n \Big|_{n=1} \qquad (\mathrm{tr}_A \rho_A = 1)$$

- Z_n: partition function on n-covering space

$$\mathrm{tr}_A \rho_A^n = \frac{Z_n}{(Z_1)^n}$$

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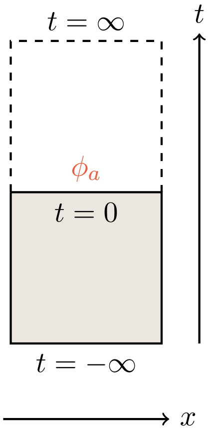
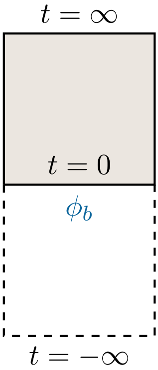
Path integral representation of the wave function

$\langle \phi_a | \Psi \rangle =$ 

$\langle \Psi | \phi_b \rangle =$ 

States $|\phi_{a,b}\rangle$ are the boundary conditions at $t = 0$

Replica trick and covering space

$$[\rho_A]_{ab} = \frac{1}{Z_1}\int [\mathcal{D}\phi^B(t=0,\vec{x}\in B)]\ \left(\langle\phi_a^A|\langle\phi^B|\right)|\Psi\rangle\langle\Psi|\left(|\phi_b^A\rangle|\phi^B\rangle\right)\ ,$$

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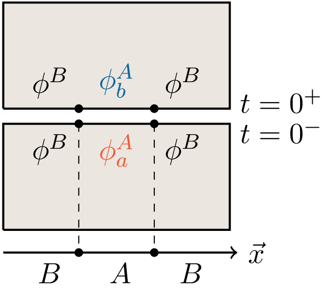

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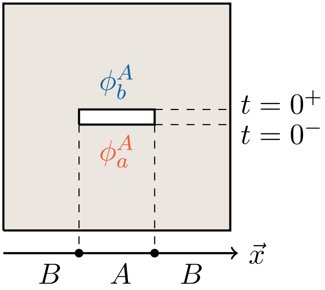
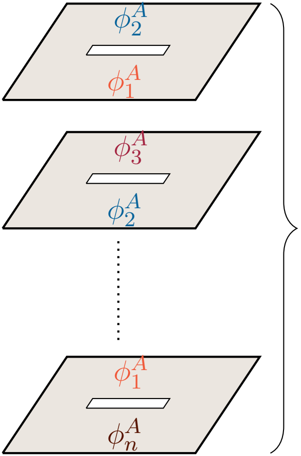


Replica trick and covering space

$$\mathrm{tr}_A \rho_A^n = \frac{1}{(Z_1)^n}$$



n copies $\equiv Z_n$

$$= \frac{Z_n}{(Z_1)^n}$$

Replica trick and covering space

Entanglement entropy

$$S_A = -\left(\partial_n - 1\right) \log Z_n \Big|_{n=1}$$

All we need to know is the partition function Z_n on the n-fold cover $\mathcal{M}_n$!

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Comment

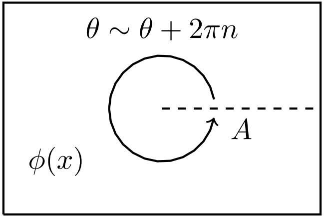
Regarding $\beta = 2\pi n$ as an inverse temperature

$$S_A = \left(\beta \partial_\beta - 1\right) \left(\beta F\right) \Big|_{\beta = 2\pi}$$

where $\beta F(\beta) = -\log Z_n$

- Suppose $A = \{x > 0, t = 0\}$ on $\mathcal{M} = \mathbb{R}^2$
- $\mathcal{M}_n : ds^2 = dr^2 + r^2 d\theta^2$
  with $r \geq 0$, $\theta \sim \theta + 2\pi n$
- $\log Z_n = -\frac{1}{2} \log\det(-\nabla^2 + m^2)|_{\mathcal{M}_n}$
- $S_A = -\frac{1}{12}\log(m^2\epsilon^2)$

  $\epsilon \ll 1$: UV cutoff



$$I = \frac{1}{2}\int d^2x \left[(\partial_\mu \phi)^2 + m^2\phi^2\right]$$

Example: A half space

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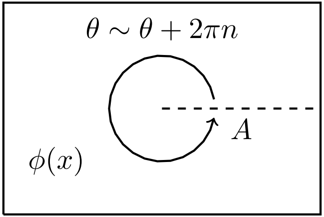

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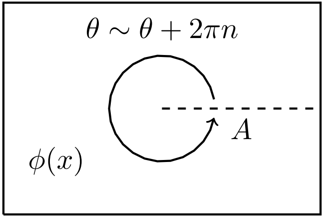

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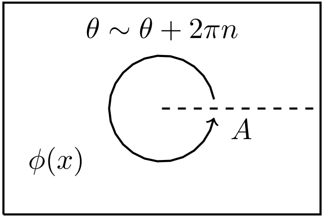

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UV structures of the partition function

- The partition function has UV divergences

$$\log Z_n[g_{\mu\nu}] = C_d \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \Lambda^d + C_{d-2} \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \Lambda^{d-2} \mathcal{R} + C_{d-4} \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \Lambda^{d-4} \mathcal{R}^2 + \cdots$$

where $\Lambda \gg 1$ is a UV cutoff scale, $\mathcal{R}$ is a Ricci scalar

- The n-fold cover $\mathcal{M}_n$ differs from $\mathcal{M} \equiv \mathcal{M}_1$ near the entangling surface $\Sigma \equiv \partial A$

$$\int_{\mathcal{M}_n} \mathcal{R}^i - n \int_{\mathcal{M}} \mathcal{R}^i \sim \int_{\Sigma} \#$$

UV structures of the partition function

- The partition function has UV divergences

$$\log Z_n[g_{\mu\nu}] = C_d \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \Lambda^d + C_{d-2} \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \Lambda^{d-2} \mathcal{R} + C_{d-4} \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \Lambda^{d-4} \mathcal{R}^2 + \cdots$$

where $\Lambda \gg 1$ is a UV cutoff scale, $\mathcal{R}$ is a Ricci scalar

- The n-fold cover $\mathcal{M}_n$ differs from $\mathcal{M} \equiv \mathcal{M}_1$ near the entangling surface $\Sigma \equiv \partial A$

$$\int_{\mathcal{M}_n} \mathcal{R}^i - n \int_{\mathcal{M}} \mathcal{R}^i \sim \int_{\Sigma} \#$$

UV structure of entanglement entropy

- The entropy has UV divergences coming from the correlation near Σ

UV structure of entanglement entropy

$$S_A = c_{d-2}\Lambda^{d-2} + c_{d-4}\Lambda^{d-4} + \cdots \ ,$$

with coefficients schematically written as

$$c_{d-2i} = \sum_{l+m=i-1} \int_\Sigma \mathcal{R}^l \mathcal{K}^{2m} \ ,$$

$\mathcal{K}$: the extrinsic curvature

- It starts from the area law divergence

$$c_{d-2} \propto \mathrm{Vol}(\Sigma)$$

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UV finiteness of the mutual information

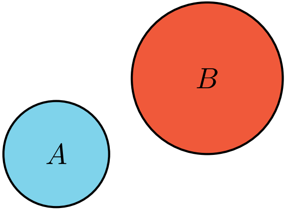
- For two disjoint regions A and B the mutual information

$$I(A,B) = S_A + S_B - S_{A\cup B}$$

- The UV divergences cancel out!

$$\int_{\Sigma(A)} + \int_{\Sigma(B)} - \int_{\Sigma(A\cup B)} = 0$$

- The mutual information is scheme independent



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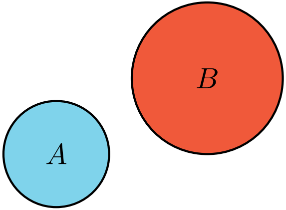

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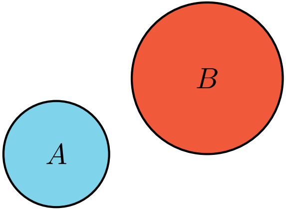

Heat kernel expansion

- Consider a free massive scalar whose effective action is

$$\log Z_n = -\frac{1}{2} \log \det(-\nabla^2 + m^2)$$

- The heat kernel coefficients $a_i(\mathcal{M}_n)$ depends on the geometry $\mathcal{M}_n$ and known for a smooth manifold

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$$\begin{aligned}\log Z_n &= -\frac{1}{2}\log\det(-\nabla^2 + m^2) \\ &= \frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\,\mathrm{tr}\,K_{\mathcal{M}_n}(s)\,e^{-m^2 s}\end{aligned}$$

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The expansion of the heat kernel $K_{\mathcal{M}_n}(s) \equiv e^{s\nabla^2}$

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Heat kernel expansion of entanglement entropy

- a_i decompose to bulk and surface parts in $n \to 1$ limit

$$a_i = a_i^{\mathsf{bulk}} + (1-n)a_i^{\Sigma} + O\left((1-n)^2\right)$$

where the bulk part satisfies

$$a_i^{\mathsf{bulk}}(\mathcal{M}_n) = n\, a_i^{\mathsf{bulk}}(\mathcal{M}_1)$$

- The entropy is determined by only the surface part

$$S_A = \frac{1}{(4\pi)^{\frac{d}{2}}}\left[\frac{a_0^{\Sigma}}{d}\frac{1}{\epsilon^d} + \frac{a_1^{\Sigma} - m^2 a_0^{\Sigma}}{d-2}\frac{1}{\epsilon^{d-2}} + \cdots\right]$$

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On a smooth manifold $\mathcal{M}$ [E.g. Vassilevich, hep-th/0306138]

$$a_0^{\text{bulk}} = \int_{\mathcal{M}} 1 \,, \qquad a_1^{\text{bulk}} = \frac{1}{6}\int_{\mathcal{M}} \mathcal{R}$$

- Apply it to a regularized geometry $\widetilde{\mathcal{M}}_n$

$$ds^2_{\widetilde{\mathcal{M}}_n} = f_\epsilon(r)dr^2 + r^2 d\theta^2 + \cdots$$

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$$f_\epsilon(r \to 0) = n^2 \,, \qquad f_\epsilon(r > \epsilon) = 1 \,, \qquad \epsilon \ll 1$$

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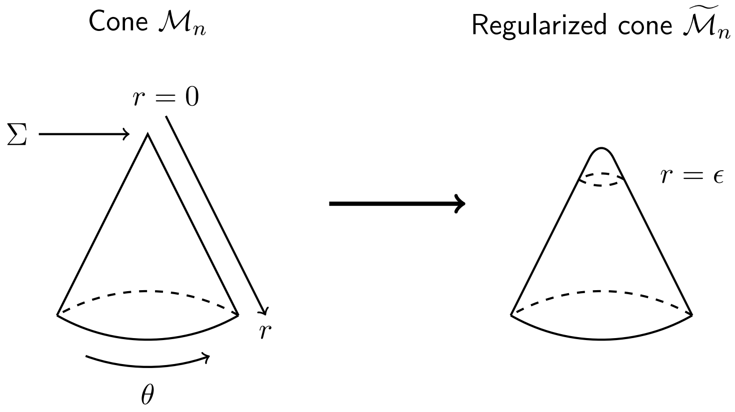

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On the regularized geometry $\widetilde{\mathcal{M}}_n$ [Fursaev-Patrushev-Solodukhin 13]

$$\int_{\widetilde{\mathcal{M}}_n} 1 = n \int_{\widetilde{\mathcal{M}}_1} 1$$

$$\int_{\widetilde{\mathcal{M}}_n} \mathcal{R} = n \int_{\mathcal{M}_1} \mathcal{R} + 4\pi(1-n) \int_\Sigma 1 + O\left((1-n)^2\right)$$

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Again we obtain the area law divergence

$$S_A = \frac{1}{6(d-2)(4\pi)^{d/2-1}} \frac{\mathrm{Vol}(\Sigma)}{\epsilon^{d-2}} + \cdots$$

(the subleading terms are similarly obtained)

Outline

- Under the conformal transformation

$$\bar{g}_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \ ,$$

CFT is invariant for some Δ

$$I[\bar{g}_{\mu\nu}, \bar{\phi}] = I[g_{\mu\nu}, \phi] \ , \qquad \bar{\phi}(x) = \Omega^{-\Delta}(x)\phi(x)$$

- Example: A conformally coupled scalar field with $\Delta = d/2 - 1$ on a curved space

$$I[g_{\mu\nu}, \phi] = \frac{1}{2}\int d^d x \sqrt{g}\, \left[\partial_\mu \phi \partial^\mu \phi + \frac{d-2}{4(d-1)} \mathcal{R}\, \phi^2\right]$$

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Conformal anomaly

- For CFT, the variation of the action is zero for $\delta g_{\mu\nu} = 2\delta\Omega g_{\mu\nu}$

$$0 = \delta I[g_{\mu\nu}] = \int d^d x \, \delta g_{\mu\nu} \frac{I[g_{\mu\nu}]}{\delta g_{\mu\nu}} = - \int d^d x \sqrt{g} \, T_\mu{}^\mu \, \delta\Omega(x) \ ,$$

- The trace of the stress-energy tensor should vanish classically

$$T_\mu{}^\mu = g^{\mu\nu} \frac{2}{\sqrt{g}} \frac{\delta I}{\delta g^{\mu\nu}} = 0$$

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Conformal anomaly in entanglement entropy

- A scaling of length $l \to e^\sigma l$ is equivalent to $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$

$$l \frac{d}{dl} \log Z_n = \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \langle T_\mu{}^\mu \rangle$$

- The entanglement entropy satisfies

$$l \frac{d}{dl} S_A = \int_{\mathcal{M}_1} d^d x \sqrt{g}\, \langle T_\mu{}^\mu \rangle - \lim_{n\to 1} \partial_n \int_{\mathcal{M}_n} d^d x \sqrt{g}\, \langle T_\mu{}^\mu \rangle \equiv c_0$$

- If the rhs does not vanish (it can happen in even dimensions), EE has a logarithmic divergence

$$S_A \supset c_0 \log(l/\epsilon)$$

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CFT in two dimensions

- In two dimensions, only $E_2 = \mathcal{R}/(4\pi)$ exists and choosing $A = c/3$

$$c_0 = \frac{c}{24\pi}\left[\int_{\mathcal{M}_1} \mathcal{R} - \lim_{n\to 1} \partial_n \int_{\mathcal{M}_n} \mathcal{R}\right]$$

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$$\int_{\widetilde{\mathcal{M}}_n} \mathcal{R} = n \int_{\mathcal{M}_1} \mathcal{R} + 4\pi(1-n) \int_{\Sigma} 1 + O\left((1-n)^2\right)$$

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EE of an interval of width l in CFT$_2$

$$S_A = \frac{c}{3}\log(l/\epsilon) + (\text{finite})$$

CFT in four dimensions

- There are one Euler density and one Weyl invariant

$$E_4 = \frac{1}{32\pi^2}\left(\mathcal{R}^2_{\mu\nu\rho\sigma} - 4\mathcal{R}^2_{\mu\nu} + \mathcal{R}^2\right)$$

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- There are general formulae for the Riemann tensors on the regularized manifold $\widetilde{\mathcal{M}}_n$

CFT$_4$ with central charges $A = a$, $B = c$

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Summary of UV divergences

In even dimensions

$$S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \frac{c_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{c_2}{\epsilon^2} + c_0 \log \frac{l}{\epsilon} + \cdots$$

c_0 : depends on the central charges

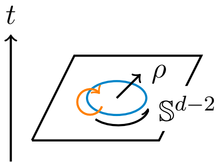
In odd dimensions

$$S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \frac{c_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{c_1}{\epsilon} + (-1)^{\frac{d-1}{2}} F$$

F : scheme independent constant

Conformal map for spherical entangling surface

- Let A be a ball $\{\rho \le R, t = 0\}$ in $\mathbb{R}^d$



$$\mathbb{R}^d$$

$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2$$

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- The coordinate transformation [Casini-Huerta-Myers 11]

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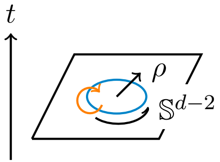

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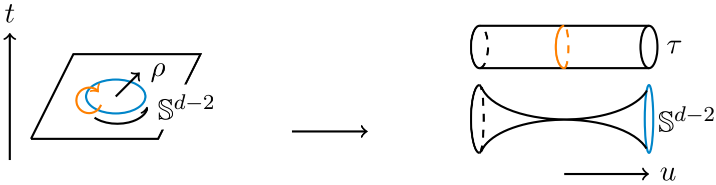

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$\mathbb{S}^1 \times \mathbb{H}^{d-1}$

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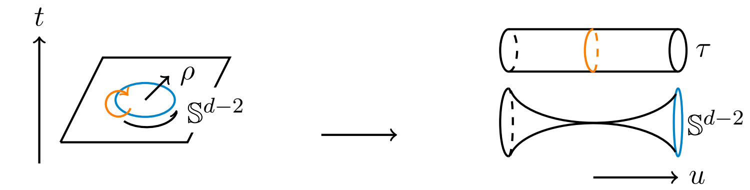
- For CFT, the partition function is invariant

$$Z_n[\mathbb{R}^d] = Z[\mathbb{S}^1 \times \mathbb{H}^{d-1}]|_{\tau \sim \tau + 2\pi n} = \mathrm{tr}(e^{-\beta H})|_{\beta = 2\pi n}$$



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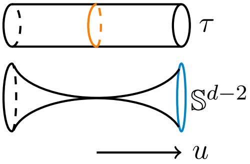
Further map to sphere

- A coordinate transformation $\sinh u = \cot\theta$ yields a map to $\mathbb{S}^d$



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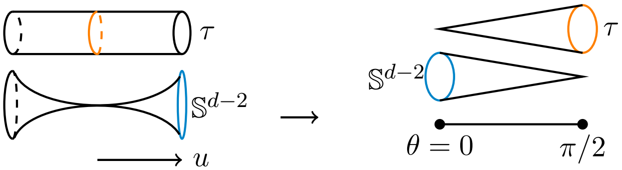

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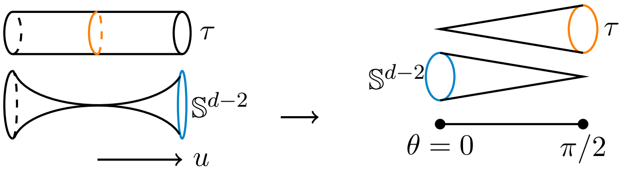

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Replica partition function

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Relation to a sphere partition function

- After the conformal transformation, the entropy is mapped to a sphere partition function

For CFT and spherical entangling surface

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Outline

The AdS geometries

- Consider the flat $(d+2)$-dimensional pseudo Euclidean space defined by

$$ds^2 = -dy_{-1}^2 - dy_0^2 + dy_1^2 + \cdots + dy_d^2$$

- The AdS_{d+1} space with the radius L is defined as a submanifold satisfying

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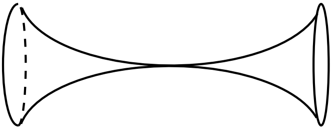

Poincaré coordinates

- The coordinate transformations

$$y_{-1} = \frac{L^2 - t^2 + z^2 + \sum_{i=1}^{d-1} x_i^2}{2z} \,, \qquad y_d = \frac{-L^2 - t^2 + z^2 + \sum_{i=1}^{d-1} x_i^2}{2z}$$

$$y_0 = Lt/z \,, \qquad y_i = Lx_i/z \,, \qquad (i = 1, \cdots d-1)$$

- The metric becomes

$$ds^2 = L^2 \left[\frac{dr^2}{r^2} + r^2 \left(-dt^2 + \sum_{i=1}^{d-1} dx_i^2 \right) \right]$$

- These coordinates cover half of the whole AdS_{d+1} space and the Euclidean boundary at $r = \infty$ is $\mathbb{R}^d$

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GKP-W relation

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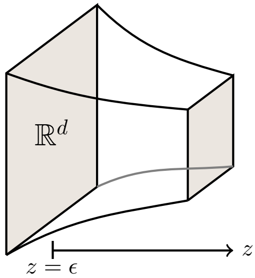
- Consider the Einstein-Hilbert action

$$I_{\mathsf{bulk}}[\mathcal{B}] = -\frac{1}{16\pi G_N}\int_{\mathcal{B}} d^{d+1}x\,\sqrt{g}\,\left(\mathcal{R}+\frac{d(d-1)}{L^2}\right)$$

AdS$_{d+1}$ space



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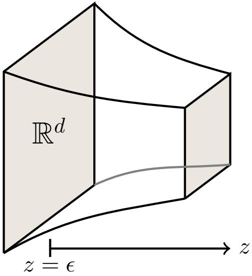

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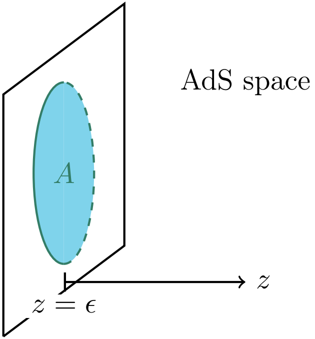

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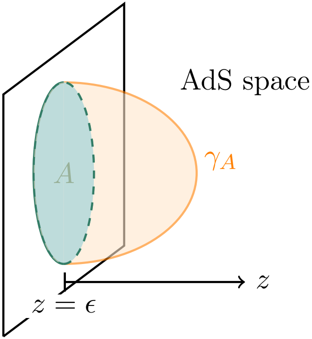

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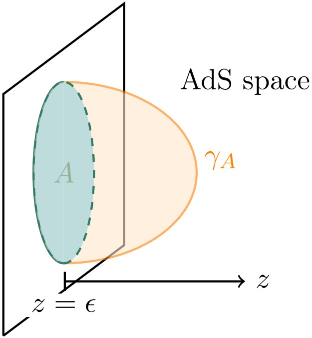

Reproduce the area law divergence

$$S_A = \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + \cdots$$

ϵ : UV cutoff at $z = \epsilon$

Holographic proof of strong subadditivity

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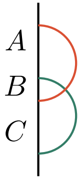

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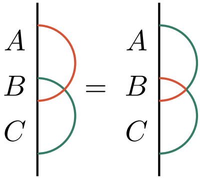

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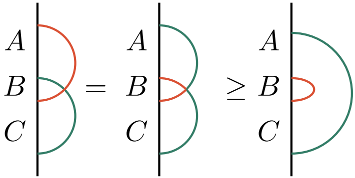

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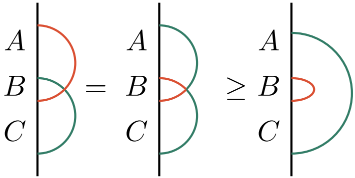

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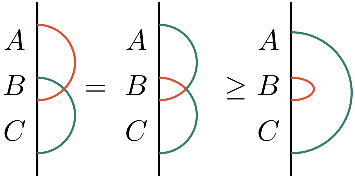

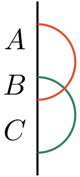

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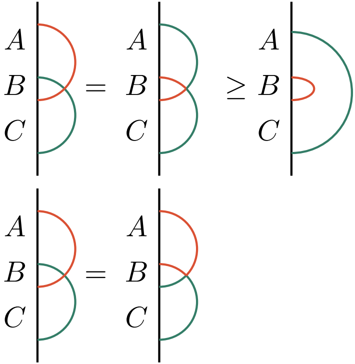

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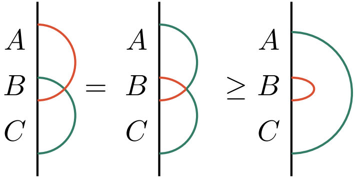

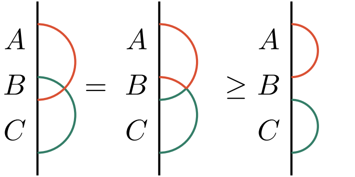

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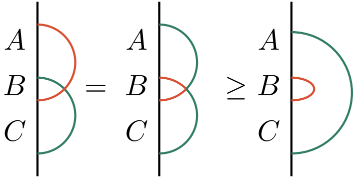
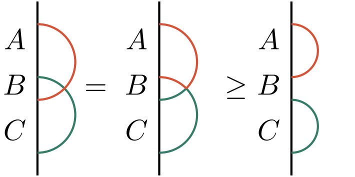


$$S_A + S_C \le S_{A\cup B} + S_{B\cup C}$$



Spherical entangling surface in CFT$_d$

- In the Poincaré patch,
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$$ds^2 = L^2 \frac{dz^2 + dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2}{z^2}$$

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$$\text{Area}(\gamma_A) = L^{d-1}\text{Vol}(\mathbb{S}^{d-2}) \int_0^R d\rho\, \frac{\rho^{d-2}}{z^{d-1}(\rho)} \sqrt{1 + (\partial_\rho z)^2}$$

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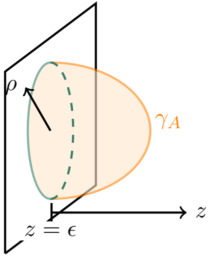

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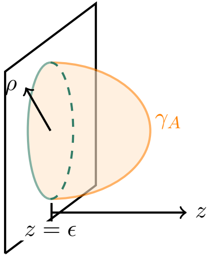

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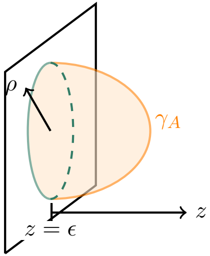

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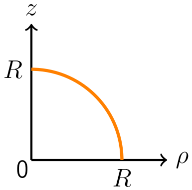

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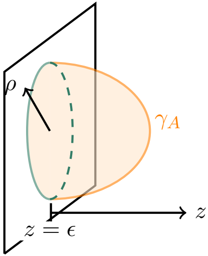

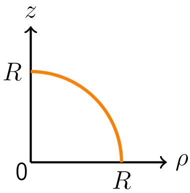

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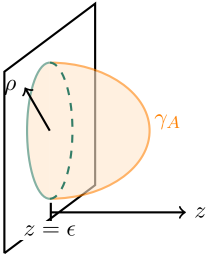

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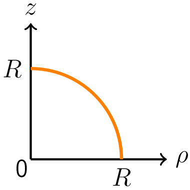

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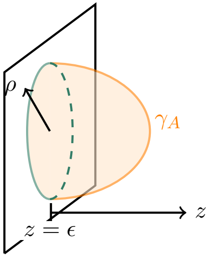

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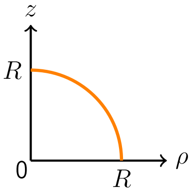


In even dimensions

$$c_0 = (-1)^{\frac{d}{2}+1}A = (-1)^{\frac{d}{2}+1}\frac{L^{d-2}}{2G_N}\frac{\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}$$

Viewpoint from the hyperbolic coordinates

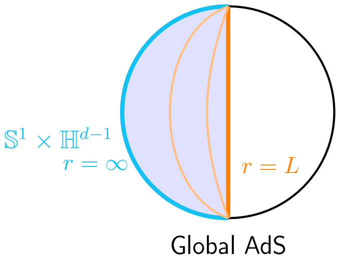
- The entangling surface Σ is at the spatial infinity in the hyperbolic coordinates

- The minimal surface is anchored on Σ

- It coincides with the BH horizon!

Holographic EE for spherical entangling surface

$$S_A(R) = S_{\rm BH}(T) = S_{\rm therm}(T)$$



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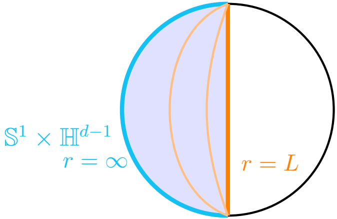

Global AdS

- The entangling surface Σ is at the spatial infinity in the hyperbolic coordinates

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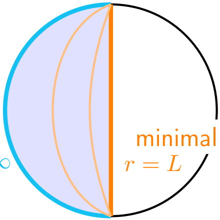

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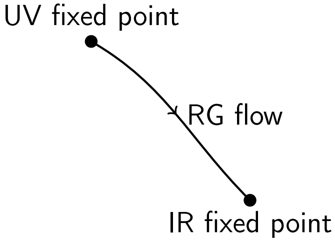
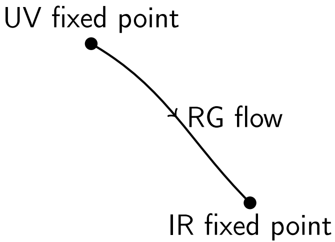
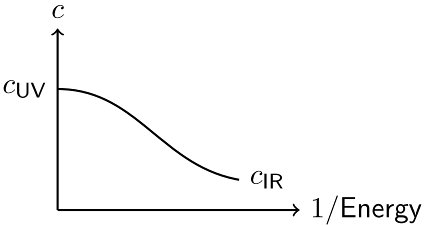
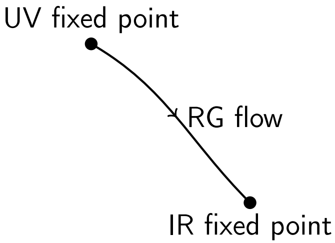
Outline

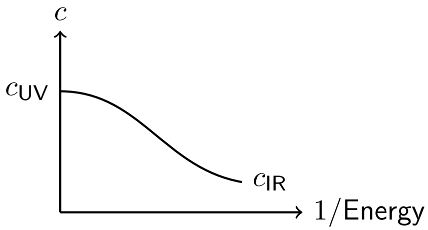

UV fixed point
RG flow
IR fixed point

RG flow and c-function









c-function can be a measure of degrees of freedom!

[Zamolodchikov 86, Cardy 88, Komargodski-Shwimmer 11]

Entropic c-theorem

- 2d entropic c-function:

$$c(r) \equiv 3r \frac{dS_A(r)}{dr}$$

- Interpolate two fixed points

$$c(r) \to c_{UV} \quad (r \to 0) \, , \qquad c(r) \to c_{IR} \quad (r \to \infty)$$

- SSA + Lorentz invariance $\Rightarrow$ monotonicity [Casini-Huerta 04]

$$c'(r) \le 0$$

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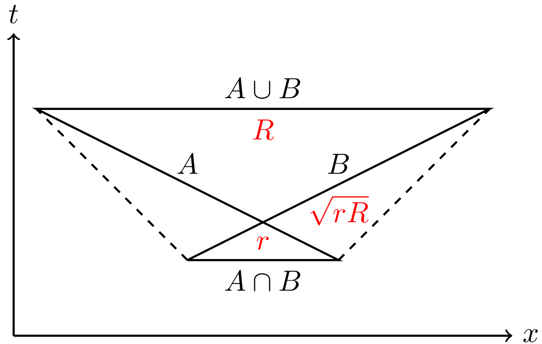

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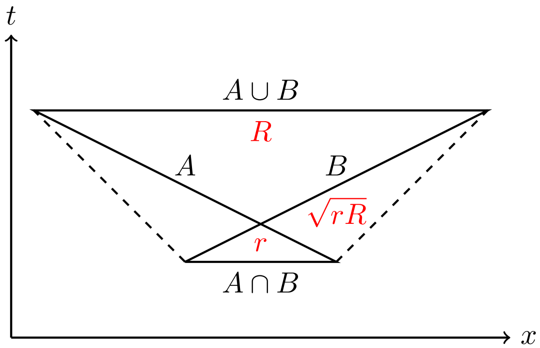
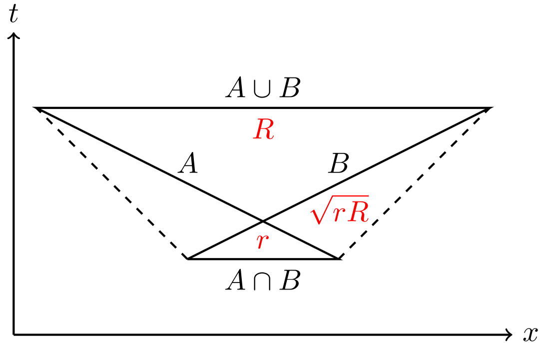


SSA

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

Proof of entropic c-theorem



SSA

$$2S(\sqrt{rR}) \geq S(R) + S(r) \quad \Rightarrow \quad c'(r) \leq 0$$

Entanglement and c-theorem in 2d

- Entropic c-function (not stationary at a fixed point)

$$c(t) = c \quad \text{for CFT} \ , \qquad c'(t) \le 0$$

- Zamolodchikov's c-function (stationary at a fixed point)

$$c'(t) = -\frac{3}{2} G_{ij} \beta^i \beta^j \le 0 \ , \qquad \frac{\partial c}{\partial g^i} = G_{ij} \beta^j$$

- Thermal c-function

$$F_{\text{Therm}} \sim c\, T^2$$

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- C_T-theorem: [Petkou 94]

$$C_T|_{\rm UV} \geq C_T|_{\rm IR} \ , \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle_{\rm CFT} = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

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EE in CFT$_3$ and F-theorem

We use a renormalized partition function in the F-theorem

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Proof of the F-theorem by using entanglement entropy?

Renormalized entanglement entropy

- Interpolating function between $F_{\rm UV}$ and $F_{\rm IR}$

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- Proof of monotonicity [Casini-Huerta 12]

SSA + Lorentz invariance $\qquad \Rightarrow \qquad \mathcal{F}'(R) = R\, S''(R) \le 0$

EE in gapped phase

- Large m expansion: [cf. Grover-Turner-Vishwanath 11]

$$S_A(R) = \alpha \frac{\ell_\Sigma}{\epsilon} + \beta\, m\, \ell_\Sigma - \gamma + \sum_{l=0}^{\infty} \frac{c^{\Sigma}_{-1-2l}}{m^{2l+1}}$$

  - ℓ_Σ: length of $\Sigma = \partial A$
  - γ: topological entanglement entropy [Kitaev-Preskill 05, Levin-Wen 05]

- $c^{\Sigma}_{-1-2l} = \int_\Sigma f(\kappa, \partial_s \kappa, \partial_s^2 \kappa, \cdots)$

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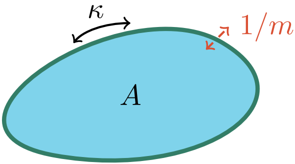
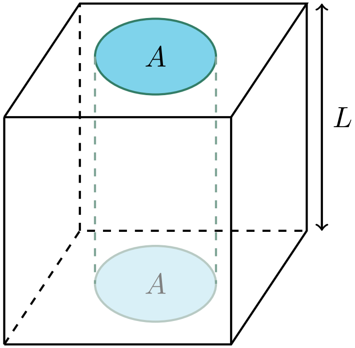


- Dimensional reduction:
  $\mathbb{R}^{2,1} \times \mathbb{S}^1 \to \mathbb{R}^{2,1}$
  [Huerta 11, Klebanov-TN-Pufu-Safdi 12]

- Entangling surface: $\Sigma \times \mathbb{S}^1 \to \Sigma$

4d EE from 3d EE

$$S^{(3+1)}_{\Sigma\times\mathbb{S}^1} = \sum_{n\in\mathbb{Z}} S^{(2+1)}_{\Sigma}\left(m = \left|\frac{2\pi n}{L}\right|\right)$$



Dimensional reduction for free massless fields

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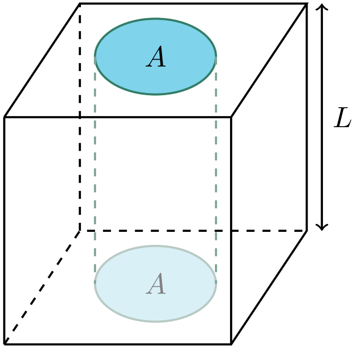

Shape dependence of EE in 3d
[Klebanov-TN-Pufu-Safdi 12]

- Log divergence in the large L limit

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REE of free massive scalar in IR region

- REE: $\Sigma =$ a circle of radius $R \quad \Rightarrow \quad \kappa = \frac{1}{R}$
- REE is monotonically decreasing to zero in IR!
- What happens in small mass region?

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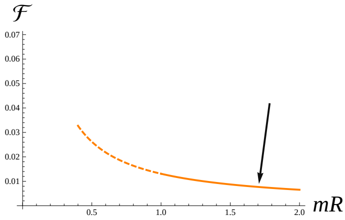

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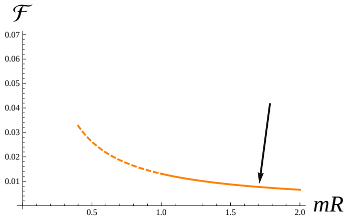

Numerical study of REE for free massive scalar

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- $\mathcal{F}$ is not stationary at UV fixed point!
  $(\partial_{(mR)^2}\mathcal{F}|_{(mR)^2=0} \sim \langle\phi^2\rangle \neq 0)$
  [Klebanov-TN-Pufu-Safdi 12, TN 14]

$$\mathcal{F} = F_{\rm UV} - 0.13(mR)^2$$

- IR divergence?

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  [Klebanov-TN-Pufu-Safdi 12, TN 14]

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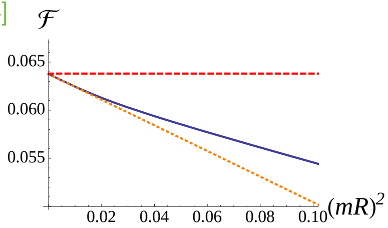

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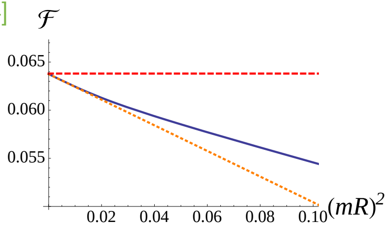

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Holographic RG flow

- Asymptotically AdS space

$$ds^2 = \frac{L^2}{z^2}\left[\frac{dz^2}{f(z)} - dt^2 + d\vec{x}^2_{d-1}\right]$$

if $f(z) \to 1$ as $z \to 0$

- Consider the Einstein gravity coupled to matters

$$I = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\left[\mathcal{R} + \mathcal{L}_{\text{matter}}\right]$$

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Null energy condition

$$T^{\text{matter}}_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \text{for any null vector} \quad (\xi_\mu\xi^\mu = 0)$$

- The null vector

$$\xi^z = \sqrt{f(z)}\ , \qquad \xi^t = 1\ , \qquad \xi^i = 0 \qquad (i \neq t, z)$$

- The Einstein equation

$$T^{\text{matter}}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} g_{\mu\nu}$$

- Applying the NEC

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A counterpart of SSA?

Domain wall RG flow

- A solution interpolating two AdS spaces of radius L and L_{IR}

$$f(z) = 1 + \eta\, g(z)\ , \qquad \eta = \frac{L^2}{L_{\text{IR}}^2} - 1 \ll 1$$

where $g'(z) \geq 0\ , \quad g(0) = 0\ , \quad g(\infty) = 1$

- Perturbatively calculate the entropy across $\mathbb{S}^{d-2}$ with $\rho_0(z) = \sqrt{R^2 - z^2}$

$$\rho(z) = \rho_0(z) + \eta\, \rho_1(z) + O(\eta^2)$$

which gives the expansion of the area functional

$$\text{Area} = A_0 + \eta\, A_1 + O(\eta^2)$$

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- The variation of the $\mathcal{F}$-function [Liu-Mezei 12]

$$\Delta\mathcal{F}(R) = -\eta\,\frac{\pi L^2}{2G_N}\int_0^1 dz\, g(zR)$$

- The difference between UV and IR

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HEE in confining geometry

- Cap off in IR of AdS space
  $\Leftrightarrow$ confining gauge theory
- Two minimal surfaces:
  - disk type
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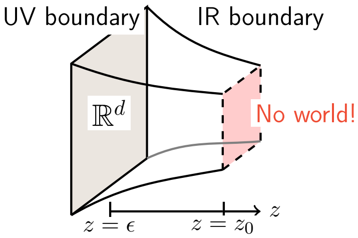

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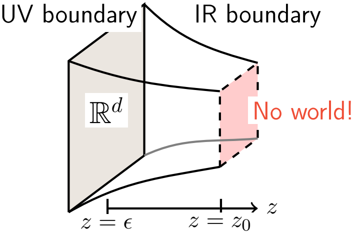

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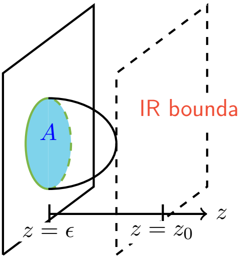

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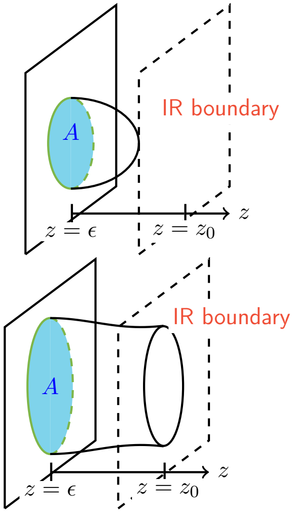

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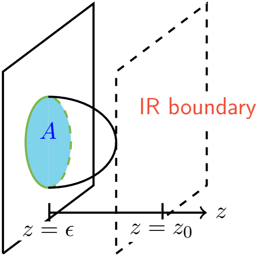

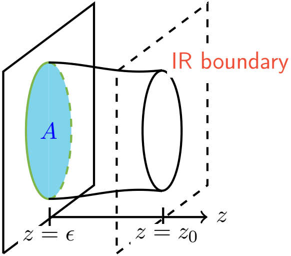

Holographic example: CGLP solution

[Klebanov-TN-Pufu-Safdi 12]

- Dual to a gapped $(2+1)$-dim QFT [Cvetic-Gibbons-Lu-Pope 00]
- Relevant deformation at UV: $S = S_{\rm UV} + g \int d^3x\, \mathcal{O}(x)$
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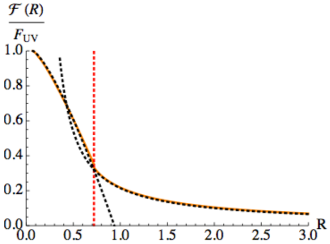

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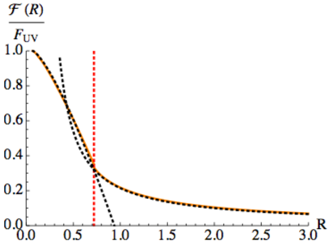

When is REE stationary for a relevant perturbation?

Outline

Relevant perturbation in AdS/CFT

- Perturbation of CFT$_d$: $S = S_{\rm CFT} + g \int d^d x\, \mathcal{O}$
- Holographically described by a free massive scalar Φ of mass M in AdS$_{d+1}$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{(M L_{AdS})^2 + \frac{d^2}{4}}$$

- Two boundary conditions near $z \to 0$ [Klebanov-Witten 99]

$$\Phi(z, \vec{x}) \to z^{\Delta_+}[A(\vec{x}) + \cdots] + z^{\Delta_-}[B(\vec{x}) + \cdots]$$

  - $\Delta = \Delta_+$: $A = \langle \mathcal{O} \rangle$, $B = g$ (standard quantization)
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Relevant perturbation of HEE [TN 14]

- The backreacted geometry by the scalar field

$$f(z) = 1 + \begin{cases} \# z^{2\Delta_-} + \cdots \ , & \Delta \neq d/2 \\ \# z^d (\log z)^2 + \cdots \ , & \Delta = d/2 \end{cases}$$

- HEE of a disk in the backreacted geometry ($t \equiv g R^{d-\Delta}$)

$$\frac{dS}{dt} = \begin{cases} -\# t^{2\Delta_-/(d-\Delta)-1} + \cdots \ , & \Delta \neq d/2 \\ -\# t \log^2 t + \cdots \ , & \Delta = d/2 \end{cases}$$

- Near the UV fixed point ($t = 0$)

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Classification of EE for a relevant preturbation

(1)	$d/2 < \Delta < d$:	stationary $\quad (dS/dt\|_{t=0} = 0)$
(2)	$d/3 < \Delta \le d/2$:	stationary, but the perturbation fails
(3)	$d/2 - 1 < \Delta \le d/3$:	neither stationary nor perturbative

Comparison to field theory results

- Free massive scalar in 3d: $\Delta = 1 \quad \Rightarrow (3)$

$$S(mR) = S(0) - \#(mR)^2 + \cdots$$

Consistent with the numerical computation of the $\mathcal{F}$-function! [Klebanov-TN-Pufu-Safdi 12, TN 14]

- Free massive fermion in 2d (an interval): $\Delta = 1 \quad \Rightarrow (2)$

$$S(mR) = S(0) - \#(mR)^2 \log^2(mR) + \cdots$$

Agree with the known results! [Casini-Fosco-Huerta 05, Herzog-TN 13]

Perturbation of EE

- Under the variation $\rho_A \to \rho_A + \delta\rho_A$

1st law of entanglement

$$\delta S_A = -\mathrm{tr}_A(\delta\rho_A \log \rho_A)$$

where $\delta\langle\mathcal{O}\rangle \equiv \mathrm{tr}_A(\delta\rho_A \mathcal{O})$

- The modular Hamiltonian $H_A \equiv -\log \rho_A$ is given by the stress-energy tensor

- For a spherical entangling surface in CFT [Casini-Huerta-Myers 11]

$$H_A = 2\pi \int_{r\le R} d^{d-1}x \frac{R^2 - r^2}{2R} T_{00}(x)$$

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- In the gravity theory with the holographic stress tensor $\delta\langle T_{00}(x)\rangle \propto h_{\mu\nu}(x, z\to 0)$

The variation of the metric $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$

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It yields the linearized Einstein equation $\delta E_{\mu\nu}[h] = 0$!

[Lashkari-McDermott-Raamsdonk 13,

Faulkner-Guica-Hartman-Myers-Raamsdonk 13]

Outline

Summary

- EE a useful measure of degrees of freedom defined in arbitrary dimensions
  - For even d, central charge dependence in the logarithmic term
  - For odd d, the finite part as an analogue of central charge

- The entropic c-theorem in two dimensions and the F-theorem in three dimensions

- REE not a c-function in the Zamolodchikov's sense (non-stationarity $\simeq$ IR divergence)

- 1st law of entanglement = the linearized Einstein equation through the holographic formula

Open problems

- Proof of the a- and F-theorem with SSA in higher dimensions? (The holographic c-theorem [Myers-Sinha 10, Freedman-Gubser-Pilch-Warner 99, $\cdots$])
- Perturbative computation of EE in QFT? IR divergence? [Rosenhaus-Smolkin 14]
- Holographic Rényi entropy formula? (For a spherical entangling surface, [Hung-Myers-Smolkin-Yale 11])

Open problems

- Is SSA equivalent to the Null energy condition?
  [Lashkari-Rabideau-Sabella-Garnier-Raamsdonk 14, Bhattacharya-Hubeny-Rangamani-Takayanagi 14]

- Einstein gravity from entanglement at non-linear level? (with MERA [Swingle 09, Raamsdonk 09, Nozaki-Ryu-Takayanagi 12, $\cdots$])

- $\cdots$ and more!

See the slides of the workshop
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http://www.maths.dur.ac.uk/ dma0mr/qiqg-ubc/

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