The uses of entanglement entropy in QFT and holography

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Introduction

- Quantum information
- Quantum inequalities
- Entanglement
- Entropy
- RG flow
- Order parameter
- QFT
- AdS/CFT
- Gravity
What’s the role of entanglement entropy in QFT?

- Entanglement entropy as a measure of degrees of freedom

- Construct a monotonic function $c(Energy)$ of the energy scale
  - Entropic $c$-theorem in two dimensions
  - $F$-theorem in three dimensions
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UV fixed point

RG flow

IR fixed point

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- \( F \)-theorem in three dimensions
What’s the role of entanglement entropy in QFT?

- Entanglement entropy as a **measure of degrees of freedom**

![Graph showing RG flow from UV fixed point to IR fixed point](image)

- **Construct a monotonic function** $c(\text{Energy})$ of the energy scale
  - Entropic $c$-theorem in two dimensions
  - $F$-theorem in three dimensions
What’s the role of entanglement entropy in QFT?

- **An order parameter for various phase transitions**
  - Confinement/deconfinement (like Polyakov loop)
  - Quantum phase transition (no symmetry breaking, no classical order parameter)

- **Reconstruction of bulk geometry from entanglement**
  - Similarity between MERA and AdS space
  - 1st law of entanglement and linearized Einstein equation of GR
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Holography geometrizes the renormalization group (RG) flow

\[ [R, G] = RG - GR = 0 \]
1 Basics of entanglement entropy
2 Field theoretic methods
3 Conformal field theory
4 Holographic method
5 Renormalization group flow
6 Perturbation
7 Summary
References

- Calabrese-Cardy, arXiv:0905.4013
- Casini-Huerta, arXiv:0903.5284
- Solodukhin, arXiv:1104.3712
- Takayanagi, arXiv:1204.2450
1. Basics of entanglement entropy
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Definition of entanglement entropy

Divide a system to $A$ and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$

Definition

$$S_A = -\text{tr}_A \rho_A \log \rho_A$$
Definition of entanglement entropy

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Definition of entanglement entropy

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- \( |\Psi\rangle\): wave function of a ground state

\[ \rho_{tot} = \frac{1}{\langle \Psi | \Psi \rangle} |\Psi\rangle \langle \Psi| \]
Definition of entanglement entropy

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- \(|\Psi\rangle\): wave function of a ground state

\[ \rho_{\text{tot}} = \frac{1}{\langle \Psi | \Psi \rangle} |\Psi\rangle \langle \Psi| \]

- Reduced density matrix:

\[ \rho_A = \text{tr}_B \rho_{\text{tot}} = \sum_i \langle \psi_i^B | \rho_{\text{tot}} | \psi_i^B \rangle \]

\[ \mathcal{H}_B = \{ |\psi_1^B\rangle, |\psi_2^B\rangle, \cdots \} \] orthonormal basis
Example: two spin system

Hilbert spaces: $\mathcal{H}_A = \{|\uparrow\rangle_A, |\downarrow\rangle_A\}$, $\mathcal{H}_B = \{|\uparrow\rangle_B, |\downarrow\rangle_B\}$
Example: two spin system

- Given a ground state ($\langle \Psi | \Psi \rangle = 1$):

  $$|\Psi\rangle = \cos \theta |\uparrow\rangle_A |\downarrow\rangle_B + \sin \theta |\downarrow\rangle_A |\uparrow\rangle_B$$
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- Reduce density matrix:

\[
\rho_A = B \langle \downarrow | \Psi \rangle \langle \Psi | \downarrow \rangle_B + B \langle \uparrow | \Psi \rangle \langle \Psi | \uparrow \rangle_B \\
= \cos^2 \theta |\uparrow\rangle_A \langle \uparrow | A + \sin^2 \theta |\downarrow\rangle_A \langle \downarrow |
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\]

\[
= \cos^2 \theta \mid \uparrow \rangle_A \langle \uparrow \mid + \sin^2 \theta \mid \downarrow \rangle_A \langle \downarrow \mid
\]

- Matrix notation:

\[
\rho_A = \begin{pmatrix}
\cos^2 \theta & 0 \\
0 & \sin^2 \theta
\end{pmatrix}
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Example: two spin system

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- EE as a function of \( \theta \):

\[
|\Psi\rangle = \cos \theta |\uparrow\rangle_A |\downarrow\rangle_B + \sin \theta |\downarrow\rangle_A |\uparrow\rangle_B
\]

\[
S_A = -\text{tr}_A \rho_A \log \rho_A
\]

\[
= - \cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta)
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- **EE as a function of** $\theta$: $|\Psi\rangle = \cos \theta |\uparrow\rangle_A |\downarrow\rangle_B + \sin \theta |\downarrow\rangle_A |\uparrow\rangle_B$

$$S_A = -\text{tr}_A \rho_A \log \rho_A$$

$$= - \cos^2 \theta \log(\cos^2 \theta) - \sin^2 \theta \log(\sin^2 \theta)$$

- $\cos^2 \theta = \frac{1}{2}$: Maximally entangled, $S_A = \log 2$
- $\cos^2 \theta = 0, 1$: No entanglement, $S_A = 0$
Suppose $|\Psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |\psi^i_A\rangle |\psi^j_B\rangle$, $d_{A,B} \equiv \dim \mathcal{H}_{A,B}$.
Suppose \( |\Psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |\psi^i_A\rangle |\psi^j_B\rangle \), \( d_{A,B} \equiv \dim \mathcal{H}_{A,B} \)

\[ c_{ij} = c^A_i c^B_j : \text{pure product state} \]

\[ |\Psi\rangle = |\Psi_A\rangle |\Psi_B\rangle , \quad |\Psi_{A,B}\rangle \equiv \sum_i c^{A,B}_i |\psi^i_{A,B}\rangle , \]

\[ \rho_A = |\Psi_A\rangle \langle \Psi_A| \quad \Rightarrow \quad S_A = 0 \]
Quantum mechanical system

Suppose \( |\Psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |\psi^i_A\rangle |\psi^j_B\rangle \), \( d_{A,B} \equiv \dim \mathcal{H}_{A,B} \)

\( c_{ij} \neq c^A_i c^B_j \) : entangled state

\[
c_{ij} = U_{ik} \lambda_k V_{kj} , \quad U, V : \text{unitary ,}
\]

\[
|\Psi\rangle = \sum_{k=1}^{\min(d_A,d_B)} \lambda_k |\tilde{\psi}^k_A\rangle |\tilde{\psi}^k_B\rangle , \quad \lambda_k \geq 0 , \sum_k \lambda_k^2 = 1 ,
\]

\[
\Rightarrow S_A = - \sum_k \lambda_k^2 \log \lambda_k^2
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$$\Rightarrow S_A = - \sum_k \lambda_k^2 \log \lambda_k^2 = S_B$$

Maximally entangled state

For $\lambda_1 = \lambda_2 = \cdots = 1/\sqrt{\min(d_A,d_B)}$ ,

$$S_A = \log \min(d_A, d_B)$$
Properties of entanglement entropy

For a pure ground state

\[ S_A = S_{\bar{A}} \]
Properties of entanglement entropy

For a pure ground state

\[ S_A = S_{\bar{A}} \]

Strong subadditivity

\[ S_{A \cup B \cup C} + S_B \leq S_{A \cup B} + S_{B \cup C} \]
\[ S_A + S_C \leq S_{A \cup B} + S_{B \cup C} \]

for any three disjoint regions \( A, B \) and \( C \)
Properties of entanglement entropy

For a pure ground state

\[ S_A = S_{\overline{A}} \]

Strong subadditivity

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for any three disjoint regions \( A, B \) and \( C \)

Mutual information

\[ I(A, B) \equiv S_A + S_B - S_{A \cup B} \geq 0 \]

for any disjoint two regions \( A \) and \( B \)
$n$-th Rényi entropy

$$S_n(A) = \frac{1}{1-n} \log \text{tr}_A \rho_A^n$$
$n$-th Rényi entropy

\[ S_n(A) = \frac{1}{1-n} \log tr_A \rho_A^n \]

It reduces to the entanglement entropy in $n \to 1$ limit

\[ S_A = \lim_{n \to 1} S_n(A) \]
Rényi entropies

\( n \)-th Rényi entropy

\[ S_n(A) = \frac{1}{1-n} \log \text{tr} A \rho_A^n \]

It reduces to the entanglement entropy in \( n \to 1 \) limit

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Inequalities

\[ \partial_n S_n \leq 0 \]

\[ \partial_n \left( \frac{n-1}{n} S_n \right) \geq 0 \]

\[ \partial_n ((n-1)S_n) \geq 0 \]

\[ \partial_n^2 ((n-1)S_n) \leq 0 \]
Relative entropy

For two states $\rho$ and $\sigma$

$$S(\rho||\sigma) = \text{tr} [\rho \log \rho - \log \sigma]$$

It measures the distance between the two states.
Relative entropy

For two states $\rho$ and $\sigma$

$$S(\rho||\sigma) = \text{tr} \left[ \rho (\log \rho - \log \sigma) \right]$$

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Properties

$$S(\rho||\rho) = 0$$

$$S(\rho||\sigma) \geq \frac{1}{2} ||\rho - \sigma||^2$$  \hspace{1cm} \text{Positivity}$$

$$S(\rho||\sigma) \geq S(\text{tr}_p \rho||\text{tr}_p \sigma)$$  \hspace{1cm} \text{Monotonicity}$$
Relative entropy

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$$S(\rho||\sigma) = \text{tr} \left[ \rho (\log \rho - \log \sigma) \right]$$

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Properties

- $S(\rho||\rho) = 0$
- $S(\rho||\sigma) \geq \frac{1}{2} ||\rho - \sigma||^2$ (Positivity)
- $S(\rho||\sigma) \geq S(\text{tr}_p \rho||\text{tr}_p \sigma)$ (Monotonicity)

The strong subadditivity follows from the last inequality
Outline

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\[
\text{dim} \mathcal{H} = \infty \text{ in QFT}
\]

- Useful trick:
  \[
  S_A = -\partial_n \log \text{tr}_A \rho^n_A \bigg|_{n=1} \quad (\text{tr}_A \rho_A = 1)
  \]

- \(Z_n\): partition function on \(n\)-covering space
  \[
  \text{tr}_A \rho^n_A = \frac{Z_n}{(Z_1)^n}
  \]
QFTs and replica trick

- \( \dim \mathcal{H} = \infty \) in QFT

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- $Z_n$: partition function on $n$-covering space

\[ \text{tr}_A \rho_A^n = \frac{Z_n}{(Z_1)^n} \]
Path integral representation of the wave function

\[ \langle \phi_a | \Psi \rangle = \langle \Psi | \phi_b \rangle \]

States \( |\phi_{a,b} \rangle \) are the boundary conditions at \( t = 0 \)
\[ [\rho_A]_{ab} = \frac{1}{Z_1} \int [\mathcal{D}\phi^B(t = 0, \vec{x} \in B)] \left( \langle \phi_a^A | \phi^B | \right) \langle \Psi | \langle \Psi | \left( | \phi_b^A \rangle | \phi^B \rangle \right), \]
Replica trick and covering space

\[
[rho_A]_{ab} = \frac{1}{Z_1} \int [D\phi^B(t = 0, \vec{x} \in B)] \left( \langle \phi_a^A | \phi^B | \Psi \rangle \langle \Psi | ( | \phi^A_b \rangle | \phi^B \rangle) \right),
\]

\[
= \frac{1}{Z_1} \int [D\phi^B(t = 0, \vec{x} \in B)]
\]
$[\rho_A]_{ab} = \frac{1}{Z_1}$
Replica trick and covering space

\[ \text{tr}_A \rho_A^n = \frac{1}{(Z_1)^n} \]

\[ = \frac{Z_n}{(Z_1)^n} \]

\[ n \text{ copies} \equiv Z_n \]
Replica trick and covering space

Entanglement entropy

\[ S_A = - (\partial_n - 1) \log Z_n \bigg|_{n=1} \]

All we need to know is the partition function \( Z_n \) on the \( n \)-fold cover \( \mathcal{M}_n \)!
Replica trick and covering space

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Comment

Regarding \( \beta = 2\pi n \) as an inverse temperature

\[ S_A = (\beta \partial_\beta - 1) (\beta F) \bigg|_{\beta=2\pi} \]

where \( \beta F(\beta) = - \log Z_n \)
Example: A half space

- Suppose \( A = \{x > 0, t = 0\} \) on \( \mathcal{M} = \mathbb{R}^2 \)

- \( \mathcal{M}_n : ds^2 = dr^2 + r^2 d\theta^2 \)
  with \( r \geq 0, \ \theta \sim \theta + 2\pi n \)

- \( \log Z_n = -\frac{1}{2} \log \det(-\nabla^2 + m^2)|_{\mathcal{M}_n} \)

- \( S_A = -\frac{1}{12} \log(m^2 \epsilon^2) \)
  \( \epsilon \ll 1 : \) UV cutoff

\[
I = \frac{1}{2} \int d^2 x \left[ (\partial_\mu \phi)^2 + m^2 \phi^2 \right]
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$\theta \sim \theta + 2\pi n$
Example: A half space

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  \( \epsilon \ll 1 : \) UV cutoff

\[
I = \frac{1}{2} \int d^2 x \left[ (\partial_\mu \phi)^2 + m^2 \phi^2 \right]
\]
The partition function has UV divergences

\[
\log Z_n[g_{\mu\nu}] = C_d \int_{M_n} d^d x \sqrt{g} \Lambda^d + C_{d-2} \int_{M_n} d^d x \sqrt{g} \Lambda^{d-2} R \\
+ C_{d-4} \int_{M_n} d^d x \sqrt{g} \Lambda^{d-4} R^2 + \ldots
\]

where \( \Lambda \gg 1 \) is a UV cutoff scale, \( R \) is a Ricci scalar

The \( n \)-fold cover \( M_n \) differs from \( M \equiv M_1 \) near the entangling surface \( \Sigma \equiv \partial A \)

\[
\int_{M_n} R^i - n \int_{M} R^i \sim \int_{\Sigma} \#
\]
The partition function has UV divergences

\[ \log Z_n[g_{\mu\nu}] = C_d \int_{\mathcal{M}_n} d^d x \sqrt{g} \Lambda^d + C_{d-2} \int_{\mathcal{M}_n} d^d x \sqrt{g} \Lambda^{d-2} \mathcal{R} + C_{d-4} \int_{\mathcal{M}_n} d^d x \sqrt{g} \Lambda^{d-4} \mathcal{R}^2 + \cdots \]

where \( \Lambda \gg 1 \) is a UV cutoff scale, \( \mathcal{R} \) is a Ricci scalar.

The \( n \)-fold cover \( \mathcal{M}_n \) differs from \( \mathcal{M} \equiv \mathcal{M}_1 \) near the entangling surface \( \Sigma \equiv \partial A \)

\[ \int_{\mathcal{M}_n} \mathcal{R}^i - n \int_{\mathcal{M}} \mathcal{R}^i \sim \int_{\Sigma} \# \]
The entropy has UV divergences coming from the correlation near $\Sigma$

$$S_A = c_{d-2} \Lambda^{d-2} + c_{d-4} \Lambda^{d-4} + \cdots ,$$

with coefficients schematically written as

$$c_{d-2i} = \sum_{l+m=i-1} \int_{\Sigma} R^l K^{2m} ,$$

$K$: the extrinsic curvature

It starts from the area law divergence

$$c_{d-2} \propto \text{Vol}(\Sigma)$$
The entropy has UV divergences coming from the correlation near $\Sigma$

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It starts from the area law divergence

\[ c_{d-2} \propto \text{Vol}(\Sigma) \]
For two disjoint regions $A$ and $B$ the mutual information

$$I(A, B) = S_A + S_B - S_{A \cup B}$$

The UV divergences cancel out!

$$\int_{\Sigma(A)} + \int_{\Sigma(B)} - \int_{\Sigma(A \cup B)} = 0$$

The mutual information is scheme independent.
For two disjoint regions $A$ and $B$, the mutual information

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The mutual information is scheme independent
Consider a free massive scalar whose effective action is

$$\log Z_n = -\frac{1}{2} \log \det (-\nabla^2 + m^2)$$

The heat kernel coefficients $a_i(\mathcal{M}_n)$ depends on the geometry $\mathcal{M}_n$ and known for a smooth manifold.
Consider a free massive scalar whose effective action is

$$\log Z_n = \frac{1}{2} \log \det (-\nabla^2 + m^2)$$

$$= \frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} \text{tr} K_{\mathcal{M}_n}(s) e^{-m^2 s}$$

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Consider a free massive scalar whose effective action is

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The expansion of the heat kernel \( K_{\mathcal{M}_n}(s) \equiv e^{s\nabla^2} \)

\[
\text{tr} K_{\mathcal{M}_n}(s) = \frac{1}{(4\pi s)^{d/2}} \sum_{i=0}^{\infty} a_i(\mathcal{M}_n) s^i
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The expansion of the heat kernel \( K_{\mathcal{M}_n}(s) \equiv e^{s\nabla^2} \)

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The heat kernel coefficients \( a_i(\mathcal{M}_n) \) depends on the geometry \( \mathcal{M}_n \) and known for a smooth manifold.
\( a_i \) decompose to bulk and surface parts in \( n \to 1 \) limit

\[
a_i = a_i^{\text{bulk}} + (1 - n)a_i^\Sigma + O((1 - n)^2)
\]

where the bulk part satisfies

\[
a_i^{\text{bulk}}(M_n) = n a_i^{\text{bulk}}(M_1)
\]

The entropy is determined by only the surface part

\[
S_A = \frac{1}{(4\pi)^{\frac{d}{2}}} \left[ \frac{a_0^\Sigma}{d} \frac{1}{\epsilon^d} + \frac{a_1^\Sigma - m^2 a_0^\Sigma}{d - 2} \frac{1}{\epsilon^{d-2}} + \ldots \right]
\]
\( a_i \) decompose to bulk and surface parts in \( n \to 1 \) limit

\[
a_i = a_i^{\text{bulk}} + (1 - n) a_i^\Sigma + O\left((1 - n)^2\right)
\]

where the bulk part satisfies

\[
a_i^{\text{bulk}}(\mathcal{M}_n) = n a_i^{\text{bulk}}(\mathcal{M}_1)
\]

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\]
Heat kernel coefficients on $\mathcal{M}_n$

On a smooth manifold $\mathcal{M}$ [E.g. Vassilevich, hep-th/0306138]

$$a_0^{\text{bulk}} = \int_{\mathcal{M}} 1 , \quad a_1^{\text{bulk}} = \frac{1}{6} \int_{\mathcal{M}} \mathcal{R}$$

- Apply it to a regularized geometry $\tilde{\mathcal{M}}_n$

$$ds^2_{\tilde{\mathcal{M}}_n} = f_\epsilon(r) dr^2 + r^2 d\theta^2 + \cdots$$

where $f_\epsilon(r)$ is a smooth function that behaves as

$$f_\epsilon(r \to 0) = n^2 , \quad f_\epsilon(r > \epsilon) = 1 , \quad \epsilon \ll 1$$
On a smooth manifold $\mathcal{M}$ [E.g. Vassilevich, hep-th/0306138]

$$
\begin{align*}
a^\text{bulk}_0 &= \int_{\mathcal{M}} 1, & a^\text{bulk}_1 &= \frac{1}{6} \int_{\mathcal{M}} \mathcal{R}
\end{align*}
$$

- Apply it to a regularized geometry $\mathcal{\widetilde{M}}_n$

$$
ds^2_{\mathcal{\widetilde{M}}_n} = f_\epsilon(r) dr^2 + r^2 d\theta^2 + \cdots
$$

where $f_\epsilon(r)$ is a smooth function that behaves as

$$
f_\epsilon(r \to 0) = n^2, \quad f_\epsilon(r > \epsilon) = 1, \quad \epsilon \ll 1
$$
Heat kernel coefficients on $\mathcal{M}_n$

Cone $\mathcal{M}_n$

Regularized cone $\tilde{\mathcal{M}}_n$

$r = 0$

$r = \epsilon$

$\Sigma$

$\theta$
On the regularized geometry $\tilde{M}_n$ [Fursaev-Patrushev-Solodukhin 13]

\[
\begin{align*}
\int_{\tilde{M}_n} 1 &= n \int_{\tilde{M}_1} 1 \\
\int_{\tilde{M}_n} \mathcal{R} &= n \int_{M_1} \mathcal{R} + 4\pi (1 - n) \int_{\Sigma} 1 + O \left((1 - n)^2\right)
\end{align*}
\]

which yields $a_0^\Sigma = 0$ and $a_1^\Sigma = 2\pi \text{Vol}(\Sigma)/3$
On the regularized geometry $\tilde{M}_n$ [Fursaev-Patrushev-Solodukhin 13]

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which yields $a^\Sigma_0 = 0$ and $a^\Sigma_1 = 2\pi \text{Vol}(\Sigma)/3$

Again we obtain the area law divergence

\[
S_A = \frac{1}{6(d - 2)(4\pi)^{d/2-1}} \frac{\text{Vol}(\Sigma)}{\epsilon^{d-2}} + \cdots
\]

(the subleading terms are similarly obtained)
1 Basics of entanglement entropy

2 Field theoretic methods

3 Conformal field theory

4 Holographic method

5 Renormalization group flow

6 Perturbation

7 Summary
Conformal field theory

- Under the conformal transformation

\[ \bar{g}_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) , \]

CFT is invariant for some \( \Delta \)

\[ I[\bar{g}_{\mu\nu}, \bar{\phi}] = I[g_{\mu\nu}, \phi] , \quad \bar{\phi}(x) = \Omega^{-\Delta}(x) \phi(x) \]

- Example: A conformally coupled scalar field with \( \Delta = d/2 - 1 \) on a curved space

\[ I[g_{\mu\nu}, \phi] = \frac{1}{2} \int d^d x \sqrt{g} \left[ \partial_\mu \phi \partial^\mu \phi + \frac{d - 2}{4(d - 1)} R \phi^2 \right] \]
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For CFT, the variation of the action is zero for $\delta g_{\mu\nu} = 2\delta \Omega g_{\mu\nu}$

$$0 = \delta I[g_{\mu\nu}] = \int d^d x \delta g_{\mu\nu} \frac{I[g_{\mu\nu}]}{\delta g_{\mu\nu}} = -\int d^d x \sqrt{g} T^\mu_\mu \delta \Omega(x),$$

The trace of the stress-energy tensor should vanish classically

$$T^\mu_\mu = g^\mu_\nu \frac{2}{\sqrt{g}} \frac{\delta I}{\delta g^\mu_\nu} = 0$$

Quantum mechanically, however, it does not for even $d$
Conformal anomaly

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Quantum mechanically, however, it does not for even $d$
\[
\langle T_\mu^\mu \rangle = \frac{(-1)^{\frac{d}{2}+1}}{2} A E_d + \sum_i B_i I_i
\]

- \( E_d \): the Euler density \( (\int_{S^d} E_d = 2) \)
- \( I_i \): the independent Weyl invariants in \( d \) dimensions

- The coefficients \( A \) and \( B_i \) are the central charges
Conformal anomaly

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Conformal anomaly in entanglement entropy

- A scaling of length $l \rightarrow e^\sigma l$ is equivalent to $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$

$$l \frac{d}{dl} \log Z_n = \int_{\mathcal{M}_n} d^d x \sqrt{g} \langle T_\mu^\mu \rangle$$

- The entanglement entropy satisfies

$$l \frac{d}{dl} S_A = \int_{\mathcal{M}_1} d^d x \sqrt{g} \langle T_\mu^\mu \rangle - \lim_{n \rightarrow 1} \partial_n \int_{\mathcal{M}_n} d^d x \sqrt{g} \langle T_\mu^\mu \rangle \equiv c_0$$

- If the rhs does not vanish (it can happen in even dimensions), EE has a logarithmic divergence

$$S_A \supset c_0 \log(l/\epsilon)$$
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$$S_A \supset c_0 \log(l/\epsilon)$$
CFT in two dimensions

- In two dimensions, only $E_2 = \mathcal{R}/(4\pi)$ exists and choosing $A = c/3$

\[
c_0 = \frac{c}{24\pi} \left[ \int_{\mathcal{M}_1} \mathcal{R} - \lim_{n \to 1} \partial_n \int_{\mathcal{M}_n} \mathcal{R} \right]
\]

- Applying the formula

\[
\int_{\tilde{\mathcal{M}}_n} \mathcal{R} = n \int_{\mathcal{M}_1} \mathcal{R} + 4\pi(1 - n) \int_{\Sigma} 1 + O((1 - n)^2)
\]
In two dimensions, only \( E_2 = \mathcal{R}/(4\pi) \) exists and choosing \( A = c/3 \)

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In two dimensions, only \( E_2 = \mathcal{R} / (4\pi) \) exists and choosing \( A = c/3 \)

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Applying the formula

\[
\int_{\tilde{\mathcal{M}}_n} \mathcal{R} = n \int_{\mathcal{M}_1} \mathcal{R} + 4\pi(1 - n) \int_{\Sigma} 1 + O((1 - n)^2)
\]

EE of an interval of width \( l \) in CFT\(_2\)

\[
S_A = \frac{c}{3} \log(l/\epsilon) + \text{(finite)}
\]
There are one Euler density and one Weyl invariant

\[
E_4 = \frac{1}{32\pi^2} \left( \mathcal{R}_{\mu\nu\rho\sigma}^2 - 4\mathcal{R}_{\mu\nu}^2 + \mathcal{R}^2 \right)
\]

\[
I_4 = \frac{1}{16\pi^2} \left( \mathcal{R}_{\mu\nu\rho\sigma}^2 - 2\mathcal{R}_{\mu\nu}^2 + \frac{1}{3} \mathcal{R}^2 \right)
\]

There are general formulae for the Riemann tensors on the regularized manifold \( \tilde{\mathcal{M}}_n \)

\[
CFT_4 \text{ with central charges } A = a, \ B = c
\]

\[
S_A = \frac{c_2}{\epsilon^2} + c_0 \log \frac{l}{\epsilon} + \text{(finite)}
\]

\[
c_0 = -\frac{a}{2} \int E_2 + \frac{c}{6\pi} \int \left( \mathcal{R}_\Sigma + 2\mathcal{R}_{abab} - \mathcal{R}_{aa} + \frac{1}{2} (\mathcal{K}^a_{\mu})^2 - 2\mathcal{K}^a_{\mu\nu} \mathcal{K}^a_{\mu\nu} \right)
\]
CFT in four dimensions

- There are one Euler density and one Weyl invariant

\[ E_4 = \frac{1}{32\pi^2} \left( R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2 \right) \]

\[ I_4 = \frac{1}{16\pi^2} \left( R_{\mu\nu\rho\sigma}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2 \right) \]

- There are general formulae for the Riemann tensors on the regularized manifold \( \tilde{M}_n \)

CFT\(_4\) with central charges \( A = a, B = c \)

\[ S_A = \frac{c_2}{\epsilon^2} + c_0 \log \frac{l}{\epsilon} + \text{(finite)} \]

\[ c_0 = -\frac{a}{2} \int \Sigma E_2 + \frac{c}{6\pi} \int \Sigma \left( R_{\Sigma} + 2R_{abab} - R_{aa} + \frac{1}{2}(K_{\mu}^a)^2 - 2K_{\mu \nu}^a K_{\mu \nu} \right) \]
In general even dimensions

- There are one Euler density and several Weyl invariants

- Using a formula

\[ \int_{\tilde{M}_n} E_d = n \int_{M_1} E_d + (1 - n) \int_{\Sigma} E_{d-2} \]

\[
CFT_d
\]

\[
S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \frac{c_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{c_2}{\epsilon^2} + c_0 \log \frac{l}{\epsilon} + \text{(finite)}
\]

\[
c_0 = \frac{(-1)^{\frac{d}{2}+1}}{2} A \int_{\Sigma} E_{d-2} + \cdots
\]
In general even dimensions

- There are one Euler density and several Weyl invariants

- Using a formula

\[
\int_{\tilde{\mathcal{M}}_n} E_d = n \int_{\mathcal{M}_1} E_d + (1 - n) \int_{\Sigma} E_{d-2}
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\]

\[
c_0 = \frac{(-1)^{\frac{d}{2}+1}}{2} A \int_{\Sigma} E_{d-2} + \cdots
\]
Summary of UV divergences

In even dimensions

\[ S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \frac{c_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{c_2}{\epsilon^2} + c_0 \log \frac{l}{\epsilon} + \cdots \]

\( c_0 \) : depends on the central charges

In odd dimensions

\[ S_A = \frac{c_{d-2}}{\epsilon^{d-2}} + \frac{c_{d-4}}{\epsilon^{d-4}} + \cdots + \frac{c_1}{\epsilon} + (-1)^{\frac{d-1}{2}} F \]

\( F \) : scheme independent constant
Let $A$ be a ball $\{\rho \leq R, t = 0\}$ in $\mathbb{R}^d$.

\[
ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2
\]
Let $A$ be a ball $\{\rho \leq R, t = 0\}$ in $\mathbb{R}^d$.

The coordinate transformation [Casini-Huerta-Myers 11]

\[
    t = R \frac{\sin \tau}{\cosh u + \cos \tau}, \quad \rho = R \frac{\sinh u}{\cosh u + \cos \tau},
\]

\[
ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega^2_{d-2}
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$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega^2_{d-2}$$

$$ds^2 = d\tau^2 + du^2 + \sinh^2 u d\Omega^2_{d-2}$$
Let $A$ be a ball $\{\rho \leq R, t = 0\}$ in $\mathbb{R}^d$.

For CFT, the partition function is invariant

$$Z_n[\mathbb{R}^d] = Z[\mathbb{S}^1 \times \mathbb{H}^{d-1}]|_{\tau \sim \tau + 2\pi n} = \text{tr}(e^{-\beta H})|_{\beta = 2\pi n}$$

\[
\begin{align*}
\mathbb{R}^d & \quad ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2 \\
\mathbb{S}^d & \quad ds^2 = d\tau^2 + du^2 + \sinh^2 u d\Omega_{d-2}^2
\end{align*}
\]
The conformal map to the hyperbolic coordinates leads to the equivalence of the EE across $S^{d-2}$ and the thermal entropy on $\mathbb{H}^{d-1}$ at $T = 1/(2\pi)$

$$S_A = S_{\text{therm}}[\mathbb{H}^{d-1}]|_{T=1/(2\pi)}$$

This relation will be derived from the holographic viewpoint.
The conformal map to the hyperbolic coordinates leads to the equivalence of the EE across $S^{d-2}$ and the thermal entropy on $\mathbb{H}^{d-1}$ at $T = 1/(2\pi)$

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This relation will be derived from the holographic viewpoint.
A coordinate transformation $\sinh u = \cot \theta$ yields a map to $S^d$

$$S^1 \times \mathbb{H}^{d-1}$$

$$ds^2 = d\tau^2 + du^2 + \sinh^2 u d\Omega_{d-2}^2$$
A coordinate transformation $\sinh u = \cot \theta$ yields a map to $S^d$

\[ ds^2 = d\tau^2 + du^2 + \sinh^2 u \, d\Omega_{d-2}^2 \]

\[ ds^2 = d\theta^2 + \sin^2 \theta d\tau^2 + \cos^2 \theta d\Omega_{d-2}^2 \]
A coordinate transformation \( \sinh u = \cot \theta \) yields a map to \( S^d \)

\[
S^1 \times H^{d-1}
\]
\[
ds^2 = d\tau^2 + du^2 + \sinh^2 u \, d\Omega_{d-2}^2
\]
\[
S^{d-2}
\]
\[
ds^2 = d\theta^2 + \sin^2 \theta \, d\tau^2 + \cos^2 \theta \, d\Omega_{d-2}^2
\]

**Replica partition function**

\[
Z_n[\mathbb{R}^d] = Z[S^1 \times H^{d-1}]_{\tau \sim \tau + 2\pi n} = Z[S^d]_{\tau \sim \tau + 2\pi n}
\]
Relation to a sphere partition function

After the conformal transformation, the entropy is mapped to a sphere partition function

For CFT and spherical entangling surface

\[ S_A = \log Z[S^d] \]

\[ S^d_n: \text{the } n\text{-fold cover of } S^d \]

\[ ds^2 = d\theta^2 + \sin^2 \theta d\tau^2 + \cos^2 \theta d\Omega_{d-2}^2, \quad \tau \sim \tau + 2\pi n \]

This relation allows us to calculate EE exactly for free field and SUSY gauge theories!
After the conformal transformation, the entropy is mapped to a sphere partition function.

For CFT and spherical entangling surface

\[ S_A = \log Z[S^d] \]

- \( S^d_n \): the \( n \)-fold cover of \( S^d \)

\[ ds^2 = d\theta^2 + \sin^2 \theta d\tau^2 + \cos^2 \theta d\Omega^2_{d-2} , \quad \tau \sim \tau + 2\pi n \]

- This relation allows us to calculate EE exactly for free field and SUSY gauge theories!
Consider the flat \((d + 2)\)-dimensional pseudo Euclidean space defined by

\[ ds^2 = -dy_{-1}^2 - dy_0^2 + dy_1^2 + \cdots + dy_d^2 \]

The \(\text{AdS}_{d+1}\) space with the radius \(L\) is defined as a submanifold satisfying

\[ -y_{-1}^2 - y_0^2 + y_1^2 + \cdots + y_d^2 = -L^2 \]
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\[-y_{-1}^2 - y_0^2 + y_1^2 + \cdots + y_d^2 = -L^2\]
Poincaré coordinates

The coordinate transformations

\[
y_{-1} = \frac{L^2 - t^2 + z^2 + \sum_{i=1}^{d-1} x_i^2}{2z}, \quad y_d = \frac{-L^2 - t^2 + z^2 + \sum_{i=1}^{d-1} x_i^2}{2z},
\]

\[y_0 = \frac{Lt}{z}, \quad y_i = \frac{Lx_i}{z}, \quad (i = 1, \cdots d-1)\]

The metric becomes

\[
ds^2 = L^2 \left[ \frac{dr^2}{r^2} + r^2 \left( -dt^2 + \sum_{i=1}^{d-1} dx_i^2 \right) \right]
\]

These coordinates cover half of the whole AdS\(_{d+1}\) space and the Euclidean boundary at \(r = \infty\) is \(\mathbb{R}^d\)
The coordinate transformations

\[
y_{-1} = \frac{L^2 - t^2 + z^2 + \sum_{i=1}^{d-1} x_i^2}{2z}, \quad y_d = \frac{-L^2 - t^2 + z^2 + \sum_{i=1}^{d-1} x_i^2}{2z}
\]

\[
y_0 = Lt/z, \quad y_i = Lx_i/z, \quad (i = 1, \cdots d - 1)
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The metric becomes

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The metric becomes

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These coordinates cover half of the whole AdS\(_{d+1}\) space and the Euclidean boundary at \( r = \infty \) is \( \mathbb{R}^d \).
Choose the coordinates as

\[ y_{-1} = L \cosh \rho \sin \tau , \quad y_0 = L \cosh \rho \cos \tau \]
\[ y_i = L \sinh \rho \ e^i , \quad (i = 1, \ldots, d) \]

where \( e^i \) satisfy \( \sum_{i=1}^{d} (e^i)^2 = 1 \)

The metric becomes

\[ ds^2 = L^2 \left[ - \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2 \right] \]

The Euclidean boundary is \( S^d \)
Choose the coordinates as

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The Euclidean boundary is \( S^d \)
Hyperbolic coordinates

- The coordinate transformations:

\[ y_{-1} = r \cosh u, \quad y_0 = \sqrt{r^2 - L^2} \sinh \frac{t}{L} \]

\[ y_d = \sqrt{r^2 - L^2} \cosh \frac{t}{L}, \quad y_i = r \sinh u e^i \]

- The resulting metric is

\[ ds^2 = - \left( \frac{r^2}{L^2} - 1 \right) dt^2 + \frac{dr^2}{r^2 \left( \frac{r^2}{L^2} - 1 \right)} + r^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2) \]

- Cover half of the whole AdS\(_{d+1}\) space with Euclidean boundary \( S^1 \times \mathbb{H}^{d-1} \)

- Event horizon at \( r = L \) with \( \beta = 2\pi L \)
Hyperbolic coordinates

The coordinate transformations:

\[ y_{-1} = r \cosh u , \quad y_0 = \sqrt{r^2 - L^2} \sinh \frac{t}{L} \]

\[ y_d = \sqrt{r^2 - L^2} \cosh \frac{t}{L} , \quad y_i = r \sinh u e^i \]

The resulting metric is

\[ ds^2 = -\left( \frac{r^2}{L^2} - 1 \right) dt^2 + \frac{dr^2}{r^2 - L^2} + r^2 \left( du^2 + \sinh^2 u d\Omega_{d-2}^2 \right) \]

Cover half of the whole AdS\(_{d+1}\) space with Euclidean boundary \( \mathbb{S}^1 \times \mathbb{H}^{d-1} \)

Event horizon at \( r = L \) with \( \beta = 2\pi L \)
Hyperbolic coordinates

The coordinate transformations:

\[ y_{-1} = r \cosh u , \quad y_0 = \sqrt{r^2 - L^2} \sinh \frac{t}{L} \]
\[ y_d = \sqrt{r^2 - L^2} \cosh \frac{t}{L} , \quad y_i = r \sinh u e^i \]

The resulting metric is

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The AdS/CFT and GKP-W relation

- The AdS/CFT relates the partition functions

**GKP-W relation**

\[ e^{-I_{\text{bulk}}[\mathcal{B}=\text{AdS}_{d+1}]} = Z_{\text{CFT}}[\partial \mathcal{B}] \]

- Consider the Einstein-Hilbert action

\[ I_{\text{bulk}}[\mathcal{B}] = -\frac{1}{16\pi G_N} \int_{\mathcal{B}} d^{d+1}x \sqrt{g} \left( \mathcal{R} + \frac{d(d - 1)}{L^2} \right) \]
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Holographic entanglement entropy

Following the GKP-W

\[ S_A = \lim_{n \to 1} \partial_n (I_{\text{bulk}}[\mathcal{B}_n] - n I_{\text{bulk}}[\mathcal{B}]) \]

AdS space

\[ z = \epsilon \]
Holographic entanglement entropy

Following the GKP-W

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Holographic formula [Ryu-Takayanagi 06]

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \]
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Holographic formula [Ryu-Takayanagi 06]

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \]

Reproduce the area law divergence

\[ S_A = \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + \cdots \]

\( \epsilon \): UV cutoff at \( z = \epsilon \)
Holographic proof of strong subadditivity

SSA follows from the **minimality**
[Headrick-Takayanagi 07]
Holographic proof of strong subadditivity

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$$S_{A \cup B \cup C} + S_B \leq S_{A \cup B} + S_{B \cup C}$$
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$$S_A + S_C \leq S_{A \cup B} + S_{B \cup C}$$
Spherical entangling surface in CFT\(_d\)

- In the Poincaré patch, 
\[ \Sigma = \{ \rho = R, t = 0 \} \]

\[ ds^2 = L^2 \frac{dz^2 + dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2}{z^2} \]

- The area functional for \( z = z(\rho) \)
\[ \text{Area}(\gamma_A) = L^{d-1} \text{Vol}(S^{d-2}) \int_0^R d\rho \frac{\rho^{d-2}}{z^{d-1}(\rho)} \sqrt{1 + (\partial_\rho z)^2} \]

- The minimal surface
\[ z(\rho) = \sqrt{R^2 - \rho^2} \]
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Spherical entangling surface in $\text{CFT}_d$

**Holographic EE**

$$S_A = \frac{L^{d-1} \text{Vol}(\mathbb{S}^{d-2})}{4G_N} \int_{\epsilon/R}^1 dy \frac{(1 - y^2)^{\frac{d-3}{2}}}{y^{d-1}}$$

$$= \frac{L^{d-1} \text{Vol}(\mathbb{S}^{d-2})}{4G_N} \left[ \frac{1}{d-2} \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots \right]$$
Spherical entangling surface in $\text{CFT}_d$

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In odd dimensions

\[ F = \frac{L^{d-1}}{4G_N} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \]
Spherical entangling surface in CFT\(_d\)

- **Holographic EE**

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\]

- **In odd dimensions**

\[
F = \frac{L^{d-1}}{4G_N} \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}
\]

- **In even dimensions**

\[
c_0 = (-1)^{\frac{d}{2}+1} A = (-1)^{\frac{d}{2}+1} \frac{L^{d-2}}{2G_N} \frac{\pi^{\frac{d}{2}-1}}{\Gamma\left(\frac{d}{2}\right)}
\]
The entangling surface $\Sigma$ is at the spatial infinity in the hyperbolic coordinates.

- The minimal surface is anchored on $\Sigma$.
- It coincides with the BH horizon!

Holographic EE for spherical entangling surface

$$S_A(R) = S_{\text{BH}}(T) = S_{\text{therm}}(T)$$
Viewpoint from the hyperbolic coordinates

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Holographic EE for spherical entangling surface:

$$S_A(R) = S_{\text{BH}}(T) = S_{\text{therm}}(T)$$
Outline

1. Basics of entanglement entropy
2. Field theoretic methods
3. Conformal field theory
4. Holographic method
5. Renormalization group flow
6. Perturbation
7. Summary
RG flow and $c$-function

UV fixed point

RG flow

IR fixed point
RG flow and $c$-function

UV fixed point

IR fixed point

RG flow

$c$

$c_{UV}$

$c_{IR}$

$1/\text{Energy}$
$c$-function can be a measure of degrees of freedom!

[Zamolodchikov 86, Cardy 88, Komargodski-Shwimmer 11]
Entropic $c$-theorem

- **2d entropic $c$-function:**

$$c(r) \equiv 3r \frac{dS_A(r)}{dr}$$

- Interpolate two fixed points

$$c(r) \rightarrow c_{UV} \quad (r \rightarrow 0), \quad c(r) \rightarrow c_{IR} \quad (r \rightarrow \infty)$$

- SSA + Lorentz invariance $\Rightarrow$ monotonicity [Casini-Huerta 04]

$$c'(r) \leq 0$$
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Proof of entropic $c$-theorem
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\[ S_A + S_B \geq S_{A \cup B} + S_{A \cap B} \]
Proof of entropic $c$-theorem

\[ 2S(\sqrt{rR}) \geq S(R) + S(r) \quad \Rightarrow \quad c'(r) \leq 0 \]
Entanglement and $c$-theorem in 2d

- **Entropic $c$-function** (not stationary at a fixed point)

  $$c(t) = c \quad \text{for CFT} \ , \quad c'(t) \leq 0$$

- **Zamolodchikov’s $c$-function** (stationary at a fixed point)

  $$c'(t) = -\frac{3}{2} G_{ij} \beta^i \beta^j \leq 0 \ , \quad \frac{\partial c}{\partial g^i} = G_{ij} \beta^j$$

- **Thermal $c$-function**

  $$F_{\text{Therm}} \sim c T^2$$

- Every $c$-function coincides at a conformal fixed point
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  \[ C_T|_{\text{UV}} \geq C_T|_{\text{IR}} \, , \quad \langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle_{\text{CFT}} = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6} \]

- **$F$-theorem:** [Jafferis-Klebanov-Pufu-Safdi 11, Myers-Sinha 10]
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For CFT$_3$ [Casini-Huerta-Myers 11]

\[ S_A(R) = \log Z[\mathbb{S}^3] \]
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$$S_A(R) = \log Z[S^3] = \alpha \frac{R}{\epsilon} - F(S^3)$$
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Proof of the $F$-theorem by using entanglement entropy?
Renormalized entanglement entropy

- Interpolating function between $F_{UV}$ and $F_{IR}$
Renormalized entanglement entropy

- Interpolating function between $F_{\text{UV}}$ and $F_{\text{IR}}$
- Monotonically decreasing under RG flow
Renormalized entanglement entropy

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Renormalized entanglement entropy [Liu-Mezei 12]

$$\mathcal{F}(R) \equiv (R \partial_R - 1) S_A(R)$$
Renormalized entanglement entropy

\[ \mathcal{F}(R) \equiv (R \partial_R - 1) S_A(R) \]

- For CFT$_3$

\[ S_A(R) = \alpha \frac{2\pi R}{\epsilon} - F(S^3) \quad \Rightarrow \quad \mathcal{F}(R) = F(S^3) \]
Renormalized entanglement entropy

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- For CFT_3

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- Proof of monotonicity [Casini-Huerta 12]

\[ SSA + \text{Lorentz invariance} \quad \Rightarrow \quad \mathcal{F}'(R) = R S''(R) \leq 0 \]
EE in gapped phase

- Large $m$ expansion: [cf. Grover-Turner-Vishwanath 11]

\[ S_A(R) = \alpha \frac{\ell_{\Sigma}}{\epsilon} + \beta m \ell_{\Sigma} - \gamma + \sum_{l=0}^{\infty} \frac{c_{-1-2l}}{m^{2l+1}} \]

- $\ell_{\Sigma}$: length of $\Sigma = \partial A$
- $\gamma$: topological entanglement entropy [Kitaev-Preskill 05, Levin-Wen 05]

\[ c_{-1-2l} = \int_{\Sigma} f(\kappa, \partial_s \kappa, \partial_s^2 \kappa, \cdots) \]

$f$: even for $\kappa \rightarrow -\kappa$ \hspace{1cm} ($S_A = S_{\overline{A}}$)
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$1/m$
Dimensional reduction for free massless fields

- Dimensional reduction:
  \[ \mathbb{R}^{2,1} \times S^1 \rightarrow \mathbb{R}^{2,1} \]
  [Huerta 11, Klebanov-TN-Pufu-Safdi 12]

- Entangling surface: \( \Sigma \times S^1 \rightarrow \Sigma \)

\[
S^{(3+1)}_{\Sigma \times S^1} = \sum_{n \in \mathbb{Z}} S^{(2+1)}_{\Sigma} \left( m = \left| \frac{2\pi n}{L} \right| \right)
\]
Dimensional reduction for free massless fields

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4d EE from 3d EE

\[
S_{\Sigma \times S^1}^{(3+1)} = \sum_{n \in \mathbb{Z}} S_{\Sigma}^{(2+1)} \left( m = \left| \frac{2\pi n}{L} \right| \right)
\]
Log divergence in the large $L$ limit

$$S^{(3+1)}_{\Sigma_2 = \Sigma \times S^1} = \sum_{n \in \mathbb{Z}} S^{(2+1)}_{\Sigma} (m_n)$$

4d EE has a logarithmic divergence

$$S^{(3+1)}_{\Sigma_2} \bigg|_{\log} = \frac{c}{2\pi} \int_{\Sigma_2} \left( \mathcal{K}^a_{\mu \nu} \mathcal{K}^a_{\mu \nu} - \frac{1}{2} (\mathcal{K}^a_{\mu})^2 \right) \log \epsilon ,$$

From 6d anomaly,

$$c^\Sigma_{-3} = \# \int_{\Sigma} ds \kappa^4 + \# \int_{\Sigma} ds \left( \frac{d\kappa}{ds} \right)^2$$
Shape dependence of EE in 3d
[Klebanov-TN-Pufu-Safdi 12]

- Log divergence in the large $L$ limit

$$S_{\Sigma_2=\Sigma \times S^1}^{(3+1)} \xrightarrow{L \to \infty} \frac{L}{\pi} \int_0^{1/\epsilon} dp \ S_{\Sigma}^{(2+1)}(m=p)$$

- 4d EE has a logarithmic divergence

$$S_{\Sigma_2}^{(3+1)} \bigg|_{log} = \frac{c}{2\pi} \int_{\Sigma_2} \left( \mathcal{K}_\mu^a \mathcal{K}_\mu^{a \nu} - \frac{1}{2} (\mathcal{K}_\mu^a \mathcal{K}_\mu^a)^2 \right) \log \epsilon ,$$

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- Log divergence in the large $L$ limit

$$S^{(3+1)}_{\Sigma_2=\Sigma \times \mathbb{S}^1} \xrightarrow{L \to \infty} \frac{L}{\pi} \int_0^{1/\epsilon} dp \, S^{(2+1)}_{\Sigma} (m = p) \sim c_{-1} \log \epsilon$$

- 4d EE has a logarithmic divergence

$$S^{(3+1)}_{\Sigma_2} \bigg|_{\log} = \frac{c}{2\pi} \int_{\Sigma_2} \left( K^a_{\mu \nu} K^a_{\mu \nu} - \frac{1}{2} (K^a_{\mu})^2 \right) \log \epsilon,$$

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4d EE has a logarithmic divergence

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S_{\Sigma_2}^{(3+1)} \bigg|_{\text{log}} = \frac{c}{2\pi} \int_{\Sigma_2} \left( K_{\mu\nu}^a K^a_{\mu\nu} - \frac{1}{2} (K^a_{\mu})^2 \right) \log \epsilon,
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From 6d anomaly,

\[
c_{-3}^\Sigma = \# \oint_{\Sigma} ds \kappa^4 + \# \oint_{\Sigma} ds \left( \frac{d\kappa}{ds} \right)^2
\]

Shape dependence of EE in 3d
[Klebanov-TN-Pufu-Safdi 12]
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- Log divergence in the large $L$ limit

\[
S^{(3+1)}_{\Sigma_2=\Sigma \times S^1} \xrightarrow{L \to \infty} \frac{L}{\pi} \int_0^{1/\epsilon} dp \, S^{(2+1)}_\Sigma (m = p) \sim c_{-1} \log \epsilon
\]

- 4d EE has a logarithmic divergence

\[
S^{(3+1)}_{\Sigma_2} \bigg|_{\log} = \frac{c}{2\pi} \int_{\Sigma_2} \left( \mathcal{K}_\mu^\alpha \mathcal{K}^\alpha_{\mu \nu} - \frac{1}{2} (\mathcal{K}_\mu^\alpha)^2 \right) \log \epsilon
\]

For free $n_0$ scalar fields and $n_{1/2}$ Dirac fermions

\[
c_{-1} = \frac{1}{480} (n_0 + 3n_{1/2}) \int_{\Sigma} ds \, \kappa^2
\]

- From 6d anomaly,

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REE: $\Sigma = \text{a circle of radius } R \implies \kappa = \frac{1}{R}$

REE is monotonically decreasing to zero in IR!

What happens in small mass region?
REE of free massive scalar in IR region

- REE: $\Sigma = \text{a circle of radius } R \quad \Rightarrow \quad \kappa = \frac{1}{R}$

$$\mathcal{F}(R) = \frac{\pi}{120mR} + O \left( \frac{1}{(mR)^3} \right)$$

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Numerical study of REE for free massive scalar

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- Numerical method [Huerta 11]: \( \mathcal{F}(0) \simeq 0.0638 = F_{UV}(S^3) \)

- \( \mathcal{F} \) is not stationary at UV fixed point!
  \[
  (\partial_{(mR)^2}\mathcal{F}|_{(mR)^2=0} \sim \langle \phi^2 \rangle \neq 0) 
  
  [\text{Klebanov-TN-Pufu-Safdi 12, TN 14}]
  \]
  \[
  \mathcal{F} = F_{UV} - 0.13(mR)^2
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- IR divergence?
Asymptotically AdS space
\[ ds^2 = \frac{L^2}{z^2} \left[ \frac{dz^2}{f(z)} - dt^2 + dx_{d-1}^2 \right] \]
if \( f(z) \to 1 \) as \( z \to 0 \)

Consider the Einstein gravity coupled to matters
\[ I = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} [\mathcal{R} + \mathcal{L}_{\text{matter}}] \]
Holographic RG flow

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Null energy condition

\[ T^\text{matter}_{\mu\nu} \xi^\mu \xi^\nu \geq 0 \quad \text{for any null vector} \quad (\xi_\mu \xi^\mu = 0) \]
Holographic RG flow

The null vector

\[ \xi^z = \sqrt{f(z)} , \quad \xi^t = 1 , \quad \xi^i = 0 \quad (i \neq t, z) \]

The Einstein equation

\[ T_{\mu\nu}^{\text{matter}} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \]

Applying the NEC

\[ f'(z) \geq 0 \]
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Applying the NEC

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A counterpart of SSA?
A solution interpolating two AdS spaces of radius $L$ and $L_{IR}$

\[ f(z) = 1 + \eta g(z) , \quad \eta = \frac{L^2}{L_{IR}^2} - 1 \ll 1 \]

where $g'(z) \geq 0$, $g(0) = 0$, $g(\infty) = 1$

Perturbatively calculate the entropy across $S^{d-2}$ with

\[ \rho_0(z) = \sqrt{R^2 - z^2} \]

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which gives the expansion of the area functional

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The variation of the $\mathcal{F}$-function [Liu-Mezei 12]

$$\Delta \mathcal{F}(R) = -\eta \frac{\pi L^2}{2G_N} \int_0^1 dz \, g(zR)$$

The difference between UV and IR

$$\Delta \mathcal{F}(R \to \infty) = -\eta \frac{\pi L^2}{2G_N}$$
Domain wall RG flow

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Monotonically decreases as $R$ increases!

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The difference between UV and IR

$$\Delta \mathcal{F}(R \to \infty) = -\eta \frac{\pi L^2}{2G_N}$$

$$= F_{\text{IR}} - F_{\text{UV}} + O(\eta^2)$$
Cap off in IR of AdS space \iff\ confining gauge theory

Two minimal surfaces:
- disk type
- cylinder type

A phase transition happens from the disk type to the cylinder type as the radius is increased (\sim a confinement/deconfinement transition)
Cap off in IR of AdS space \iff\text{confining gauge theory}

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Cap off in IR of AdS space ⇔ confining gauge theory

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HEE in confining geometry

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Holographic example: CGLP solution

Dual to a gapped \( (2 + 1) \)-dim QFT \(^{[\text{Cvetic-Gibbons-Lu-Pope 00}]}\)

- Relevant deformation at UV: \( S = S_{\text{UV}} + g \int d^3x \mathcal{O}(x) \)
  \[ \Delta[\mathcal{O}] = \frac{7}{3}, \quad \Delta[g] = \frac{2}{3} \]
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When is REE stationary for a relevant perturbation?
Outline

1. Basics of entanglement entropy
2. Field theoretic methods
3. Conformal field theory
4. Holographic method
5. Renormalization group flow
6. Perturbation
7. Summary
Relevant perturbation in AdS/CFT

- **Perturbation of CFT$_d$:** $S = S_{CFT} + g \int d^d x \mathcal{O}$

- Holographically described by a free massive scalar $\Phi$ of mass $M$ in AdS$_{d+1}$

\[
\Delta_\pm = \frac{d}{2} \pm \sqrt{(ML_{AdS})^2 + \frac{d^2}{4}}
\]

- Two boundary conditions near $z \to 0$ [Klebanov-Witten 99]

\[
\Phi(z, \vec{x}) \to z^{\Delta_+} [A(\vec{x}) + \cdots] + z^{\Delta_-} [B(\vec{x}) + \cdots]
\]

- $\Delta = \Delta_+: A = \langle \mathcal{O} \rangle$, $B = g$ (standard quantization)
- $\Delta = \Delta_-: A = g$, $B = \langle \mathcal{O} \rangle$ (alternative quantization)
Perturbation of CFT \(d\): \( S = S_{\text{CFT}} + g \int d^d x \mathcal{O} \)

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Two boundary conditions near \(z \to 0\) [Klebanov-Witten 99]

\[
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Relevant perturbation in AdS/CFT

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- $\Delta = \Delta_-: A = g, \quad B = \langle \mathcal{O} \rangle$ (alternative quantization)
The backreacted geometry by the scalar field

\[ f(z) = 1 + \begin{cases} 
\#z^{2\Delta} + \cdots, & \Delta \neq d/2 \\
\#z^d(\log z)^2 + \cdots, & \Delta = d/2 
\end{cases} \]

- HEE of a disk in the backreacted geometry \((t \equiv gR^{d-\Delta})\)

\[ \frac{dS}{dt} = \begin{cases} 
-\#t^{2\Delta}/(d-\Delta)-1 + \cdots, & \Delta \neq d/2 \\
-\#t\log^2 t + \cdots, & \Delta = d/2 
\end{cases} \]

- Near the UV fixed point \((t = 0)\)
The backreacted geometry by the scalar field

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HEE of a disk in the backreacted geometry \( (t \equiv gR^{d-\Delta}) \)

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Near the UV fixed point \((t = 0)\)

Classification of EE for a relevant perturbation

1. \(d/2 < \Delta < d\) : stationary \((dS/dt|_{t=0} = 0)\)
2. \(d/3 < \Delta \leq d/2\) : stationary, but the perturbation fails
3. \(d/2 - 1 < \Delta \leq d/3\) : neither stationary nor perturbative
Comparison to field theory results

- Free massive scalar in 3d: \( \Delta = 1 \quad \Rightarrow \ (3) \)

\[
S(mR) = S(0) - #(mR)^2 + \cdots
\]

Consistent with the numerical computation of the \( \mathcal{F} \)-function!  
[Klebanov-TN-Pufu-Safdi 12, TN 14]

- Free massive fermion in 2d (an interval): \( \Delta = 1 \quad \Rightarrow \ (2) \)

\[
S(mR) = S(0) - #(mR)^2 \log^2(mR) + \cdots
\]

Agree with the known results!  [Casini-Fosco-Huerta 05, Herzog-TN 13]
Perturbation of EE

- Under the variation $\rho_A \rightarrow \rho_A + \delta \rho_A$

1st law of entanglement

$$\delta S_A = -\text{tr}_A (\delta \rho_A \log \rho_A)$$

where $\delta \langle \mathcal{O} \rangle \equiv \text{tr}_A (\delta \rho_A \mathcal{O})$

- The modular Hamiltonian $H_A \equiv -\log \rho_A$ is given by the stress-energy tensor

- For a spherical entangling surface in CFT [Casini-Huerta-Myers 11]

$$H_A = 2\pi \int_{r \leq R} d^{d-1}x \frac{R^2 - r^2}{2R} T_{00}(x)$$
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$$\delta S_A = -\text{tr}_A(\delta\rho_A \log \rho_A) = \delta\langle H_A \rangle$$

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Perturbation of $\rho_A$ for spherical entangling surface

- For a spherical entangling surface in CFT

The variation of the density matrix $\rho_A \rightarrow \rho_A + \delta \rho_A$

$$\delta S_A = 2\pi \int_{r \leq R} d^{d-1}x \frac{R^2 - r^2}{2R} \delta \langle T_{00}(x) \rangle$$

- In the gravity theory with the holographic stress tensor

$\delta \langle T_{00}(x) \rangle \propto h_{\mu\nu}(x, z \rightarrow 0)$

The variation of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$

$$\frac{\delta \text{Area}}{4G_N} = 2\pi \int_{r \leq R} d^{d-1}x \frac{R^2 - r^2}{2R} \delta \langle T_{00}(x) \rangle$$
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It yields the linearized Einstein equation $\delta E_{\mu\nu}[h] = 0!$

1. Basics of entanglement entropy
2. Field theoretic methods
3. Conformal field theory
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Summary

- EE a useful measure of degrees of freedom defined in arbitrary dimensions
  - For even $d$, central charge dependence in the logarithmic term
  - For odd $d$, the finite part as an analogue of central charge

- The entropic $c$-theorem in two dimensions and the $F$-theorem in three dimensions

- REE not a $c$-function in the Zamolodchikov’s sense
  (non-stationarity $\sim$ IR divergence)

- 1st law of entanglement = the linearized Einstein equation through the holographic formula
Open problems

- Proof of the $a$- and $F$-theorem with SSA in higher dimensions? (The holographic $c$-theorem [Myers-Sinha 10, Freedman-Gubser-Pilch-Warner 99, · · · ])

- Perturbative computation of EE in QFT? IR divergence? [Rosenhaus-Smolkin 14]

- Holographic Rényi entropy formula? (For a spherical entangling surface, [Hung-Myers-Smolkin-Yale 11])
Open problems

- Is SSA equivalent to the Null energy condition?

- Einstein gravity from entanglement at non-linear level? (with MERA [Swingle 09, Raamsdonk 09, Nozaki-Ryu-Takayanagi 12, · · · ])

- · · · and more!

See the slides of the workshop “Quantum Information in Quantum Gravity"
http://www.maths.dur.ac.uk/~dma0mr/qiqg-ubc/
Open problems


- Einstein gravity from entanglement at non-linear level? (with MERA [Swingle 09, Raamsdonk 09, Nozaki-Ryu-Takayanagi 12, · · · ])

- · · · and more!

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