Into the Grassmannian

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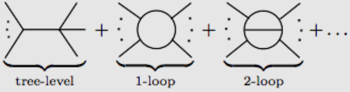
Prelude

Locality and Unitarity largely determines the S-matrix

$$A_n = \sum (\text{Feynman diagrams}) =$$



- Is there other formulations where the form of a Lagrangian is not needed?
- How does locality and unitarity emerge in such formulation?
- How universal are such constructions?

Prelude

The best shot: $\mathcal{N}=4$ Super-Yang-Mills

$$\mathcal{A}_n(\lambda_i,\tilde{\lambda}_i,\eta_i^I)$$

The on-shell variables form a representation for SUSY

$$P=\sum_i \lambda_i^\alpha\tilde{\lambda}_i^{\dot{\alpha}},\quad Q_I^\alpha=\sum_i \lambda_i^\alpha\frac{\partial}{\partial\eta_i^I},\quad \tilde{Q}^{\dot{\alpha}I}=\sum_i \tilde{\lambda}_i^{\dot{\alpha}}\eta_i^I$$

$$\mathcal{A}_n(\lambda_i,\tilde{\lambda}_i,\eta_i^I)=\langle\quad G_1(\Lambda_1)G_2(\Lambda_2)\cdots G_n(\Lambda_n)\quad\rangle$$

$$G(\eta)=A^+ +\eta^I\psi_I+\eta^I\eta^J\phi_{IJ}+\cdots+(\eta^4)A^-$$

SU(4) invariance $\rightarrow$ degree $4k$ polynomial in η (k negative helicity gluons)

Prelude

The best shot: $\mathcal{N} = 4$ Super-Yang-Mills

$$\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i^I)$$

Can we construct super conformal invariant building blocks ?

$$P = \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \quad Q_I^\alpha = \sum_i \lambda_i^\alpha \frac{\partial}{\partial \eta_i^I}, \quad \tilde{Q}^{\dot{\alpha} I} = \sum_i \tilde{\lambda}_i^{\dot{\alpha}} \eta_i^I$$

$$D = \sum_i \left(\lambda_i \frac{\partial}{\partial \lambda_i} + \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i} - 1 \right)$$

$$K = \sum_i \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \tilde{\lambda}_i}, \quad S_\alpha^I = \sum_i \eta_i^I \frac{\partial}{\partial \lambda_i^\alpha}, \quad \tilde{S}_{\dot{\alpha} I} = \sum_i \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \frac{\partial}{\partial \eta_i^I}$$

The generators are non-linear

Prelude

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The generators are non-linear

Prelude

Half-Fourier transform:

$$\tilde{\lambda}_i \to \frac{\partial}{\partial \mu}, \quad \frac{\partial}{\partial \tilde{\lambda}_i} \to \mu$$

The amplitude is now a function of twistor variables:

$$\mathcal{A}_n(\lambda_i, \mu_i, \eta_i^I)$$

$$P = \sum_i \lambda_i \frac{\partial}{\partial \mu_i}, \quad Q^{\alpha I} = \sum_i \lambda_i^\alpha \frac{\partial}{\partial \eta_i}, \quad \tilde{Q}_I^{\dot{\alpha}} = \sum_i \frac{\partial}{\partial \mu} \eta_i$$

$$D = \sum_i \left(\lambda_i \frac{\partial}{\partial \lambda_i} - \mu_i \frac{\partial}{\partial \mu_i} \right)$$

$$K = \sum_i \frac{\partial}{\partial \lambda_i} \mu_i, \quad S_\alpha^I = \sum_i \eta_i^I \frac{\partial}{\partial \lambda_i^\alpha}, \quad \tilde{S}_{\dot{\alpha} I} = \sum_i \mu_i \frac{\partial}{\partial \eta_i^I}$$

The generators are linear:

$$G^{\mathcal{A}}{}_{\mathcal{B}} = \mathcal{Z}^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}^{\mathcal{B}}} \quad \mathcal{Z}^{\mathcal{B}} = (\lambda_i, \mu_i, \eta_i^I)$$

Prelude

$$G^{\mathcal{A}}{}_{\mathcal{B}} = \mathcal{Z}^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}^{\mathcal{B}}}$$

We want super conformal invariants:

$$G^{\mathcal{A}}{}_{\mathcal{B}} \quad \mathcal{A}_n(\ \{\mathcal{Z}_i\} \ ) = 0$$

A simple solution

$$\mathcal{A}_n(\quad \{\mathcal{Z}_i\} \quad) \sim \delta^{4k|4k}(\sum C_{\alpha i}\mathcal{Z}_i)$$

$$C_{\alpha i} = \begin{pmatrix} c_{11} & c_{12} & \cdots & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{1k} & c_{2k} & \cdots & \cdots & c_{nk} \end{pmatrix}$$

Invariant under GL(k) rotations $\rightarrow$The space of k-planes in n-dimensional space.

Prelude

Represent the amplitude as an integral over a Grassmannian ! Arkani-Hamed, Cachazo, Cheung, Kaplan

$$\mathcal{A}_n = \int_{\mathcal{C}} d^{k\times n}C \frac{1}{\prod_i M_i} \delta^{4k|4k}(\sum C_{\alpha i}\mathcal{Z}_i)$$

$$M_2 = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \end{pmatrix}$$

The contours are localized by the minors.

Represent the amplitude as an integral over a Grassmannian ! Arkani-Hamed, Cachazo, Cheung, Kaplan

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$$C = \begin{pmatrix} 1 & 0 & 0 & c_{14} & c_{15} & c_{16} & c_{17} \\ 0 & 1 & 0 & c_{24} & c_{25} & c_{26} & c_{27} \\ 0 & 0 & 1 & c_{34} & c_{35} & c_{36} & c_{37} \end{pmatrix}$$

$$\int \frac{d^{(n-k)\times k}c}{\prod_{j=1}^{n} M_j} \prod_{\mathrm{a}=1}^{k} \delta^2\left([\mathrm{a}| + \sum_{l=k+1}^{n} c_{\mathrm{a}l}[l|\right) \delta^2\left(|\tilde{\mu}_{\mathrm{a}}\rangle + \sum_{l=k+1}^{n} |\tilde{\mu}_l\rangle c_{\mathrm{a}l}\right) \delta^{(4)}\left(\eta_{\mathrm{a}} + \sum_{l=k+1}^{n} c_{\mathrm{a}l}\,\eta_l\right)$$

$$\int \frac{d^{(n-k)\times k}c}{\prod_{j=1}^{n} M_j} \left[\prod_{\mathrm{a}=1}^{k} \delta^2\left([\mathrm{a}| + \sum_{l=k+1}^{n} c_{\mathrm{a}l}[l|\right) \delta^{(4)}\left(\eta_{\mathrm{a}} + \sum_{l=k+1}^{n} c_{\mathrm{a}l}\,\eta_l\right)\right] \times \left[\prod_{i=k+1}^{n} \delta^2\left(|i\rangle - \sum_{\mathrm{a}=1}^{k} |\mathrm{a}\rangle c_{\mathrm{a}i}\right)\right].$$

An $(n-k)k-(2k+2(k-n))-4 = (k-2)(n-k-2)$-dimensional integral

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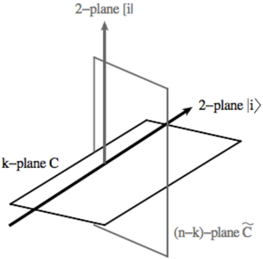

Represent the amplitude as an integral over a Grassmannian ! Arkani-Hamed, Cachazo, Cheung, Kaplan

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$(k-2)(n-k-2)$-dimensional integral

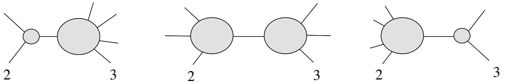
$$M_2 = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \end{pmatrix}$$

The contours are localized by the minors.

$$\{M_6\} = \frac{\langle 62\rangle^4[35]^4}{\langle 6|1+2|3]\langle 2|6+1|5][43][45]\langle 16\rangle\langle 12\rangle P^2_{612}}$$

$$\{M_2\} = \frac{\langle 24\rangle^4[51]^4}{\langle 2|3+4|5]\langle 4|2+3|1][65][61]\langle 32\rangle\langle 34\rangle P^2_{234}} \qquad \{M_4\} = \frac{\langle 46\rangle^4[13]^4}{\langle 4|5+6|1]\langle 6|4+5|3][21][23]\langle 54\rangle\langle 56\rangle P^2_{456}}$$

These are precisely the BCFW terms:



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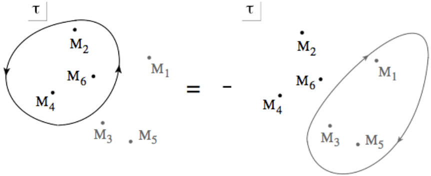
The contours are localized by the minors $\rightarrow$ linear dependency of columns in $C_{\alpha i}$.

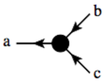


Building blocks of amplitudes corresponds to cells of $C_{\alpha i}$! (No linear dependency is a Top Cell)

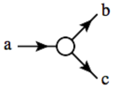
Is there a microscopic explanation ? ($k = 1, 2$)

$$\int \frac{d^{(n-k)\times k} c}{\prod_{j=1}^{n} M_j} \left[\prod_{a=1}^{k} \delta^2\left([a| + \sum_{l=k+1}^{n} c_{al} [l| \right) \delta^{(4)}\left(\eta_a + \sum_{l=k+1}^{n} c_{al}\, \eta_l \right) \right] \times \left[\prod_{i=k+1}^{n} \delta^2\left(|i\rangle - \sum_{a=1}^{k} |a\rangle c_{ai} \right) \right] .$$

$$\begin{array}{ccc} b & c & a \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & \alpha_b \\ 0 & 1 & \alpha_c \end{pmatrix}$$



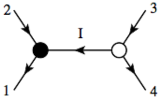
$$\begin{array}{ccc} a & b & c \\ \end{array}$$
$$\begin{pmatrix} 1 & \beta_b & \beta_c \end{pmatrix}$$



MHV: $\delta^2\left([b| + \alpha_b [a|\right)\, \delta^2\left([c| + \alpha_c [a|\right)$, anti-MHV: $\delta^2\left([a| + \beta_b [b| + \beta_c [c|\right)$

MHV $|1] \sim |2] \sim |3]$, $\overline{\text{MHV}}$ $|1\rangle \sim |2\rangle \sim |3\rangle$

The BCFW Bridge:

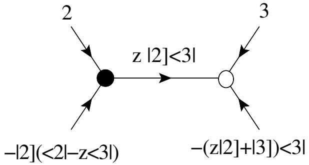


$$\int \frac{d\alpha_2}{\alpha_2}\frac{d\alpha_I}{\alpha_I}\frac{d\beta_4}{\beta_4}\frac{d\beta_I}{\beta_I}\frac{1}{U(1)}\;\delta^{2\times 2}(C_i[i|)\;\delta^{(4\times 2)}(C_i\eta_i)\;\delta^{2\times 2}(\tilde{C}_i\langle i|)$$

The bosonic constraints are simply

$$\delta^2\left([2|+\alpha_2[1|\right),\quad \delta^2\left([I|+\alpha_I[1|\right),\quad \delta^2\left([3|+\beta_4[4|+\beta_I[I|\right)$$

$$|\mathrm{I}] \sim |2],\quad |\mathrm{I}\rangle \sim |3\rangle \rightarrow |\mathrm{I}]\langle \mathrm{I}| = z|2]\langle 3|$$



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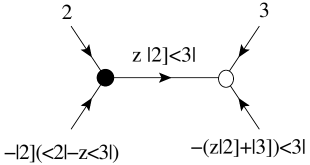
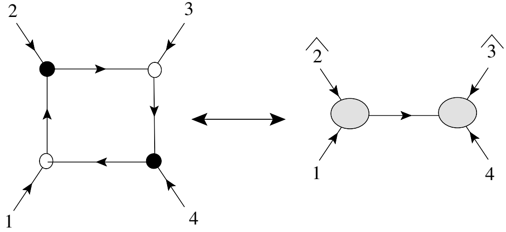
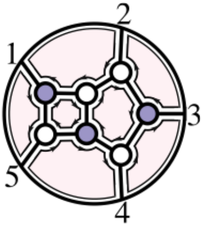




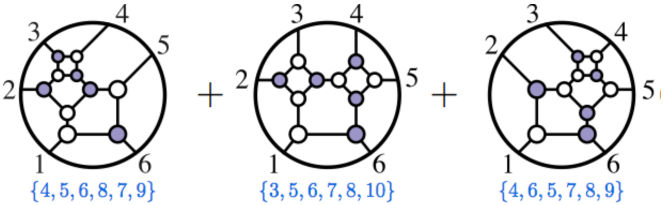
Each BCFW term is a cell in the Grassmannian

5-pt amplitude

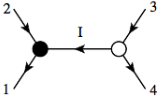


6-pt amplitude

$\mathcal{A}_6^{(3)} =$



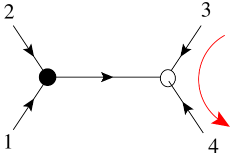
Which cell?



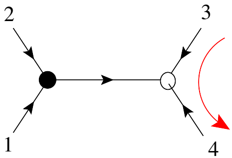
$$C_{ai} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$

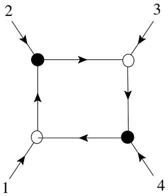
$$C_{ai} = \left(\begin{array}{cccc} \alpha_2 & 1 & 0 & 0 \\ -\beta_I\alpha_I & 0 & 1 & \beta_4 \end{array}\right)$$

Meet black (white) turn right (left):



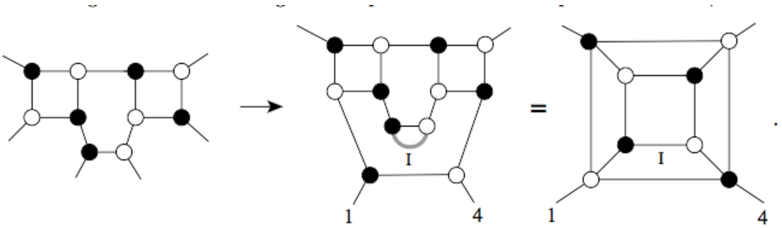
Which cell? Meet black (white) turn right (left):





The four-point amplitude is a top-cell

This works at loop-level as well



- The independent degrees of freedom in each cell diagram is $n_f - 1$.
- The cell diagrams constructed are always positive: $M_i > 0$ (the positive Grassmannian)
- The removal of each edge corresponds to singularities of each diagram.

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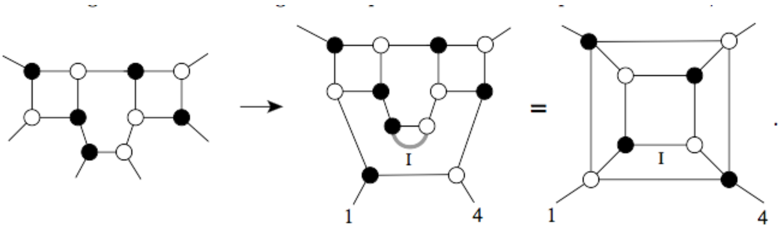

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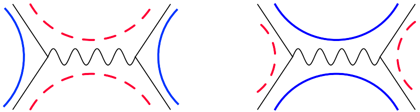
- The cell diagrams constructed are always positive: $M_i > 0$ (the positive Grassmannian)
- The removal of each edge corresponds to singularities of each diagram.

Is this the only example for such construction?

The suspect

Chern-Simons Matter:
1. $\mathcal{N}=6$ (ABJM): $\mathrm{U(N)}_k\times\mathrm{U(N)}_{-k}$ gauge fields$(\mathrm{A}^\mu,\bar{\mathrm{A}}^\mu)$,
SU(4) bi-fundamental matter $(\phi^{\mathrm{I}},\psi^{\mathrm{I}},\bar\phi_{\mathrm{I}},\bar\psi_{\mathrm{I}})$, $\mathrm{I}=1,2,3,4$

$$\mathcal{L}=\mathcal{L}_{C\ S}+\mathcal{L}_{\phi,Kin}+\mathcal{L}_{\psi,Kin}+\mathcal{L}_{4\phi^2\psi^2}+\mathcal{L}_{6\phi^6}$$



$$\begin{aligned}\Phi(\eta)&=\phi^4+\eta^{\mathrm{I}}\psi_{\mathrm{I}}+\frac{1}{2}\epsilon_{\mathrm{IJK}}\eta^{\mathrm{I}}\eta^{\mathrm{J}}\phi^{\mathrm{K}}+\frac{1}{3!}\epsilon_{\mathrm{IJK}}\eta^{\mathrm{I}}\eta^{\mathrm{J}}\eta^{\mathrm{K}}\psi_4,\\ \bar\Psi(\eta)&=\bar\psi^4+\eta^{\mathrm{I}}\bar\phi_{\mathrm{I}}+\frac{1}{2}\epsilon_{\mathrm{IJK}}\eta^{\mathrm{I}}\eta^{\mathrm{J}}\bar\psi^{\mathrm{K}}+\frac{1}{3!}\epsilon_{\mathrm{IJK}}\eta^{\mathrm{I}}\eta^{\mathrm{J}}\eta^{\mathrm{K}}\bar\phi_4,\end{aligned}$$

$$\mathcal{A}_n(\bar{1}2\bar{3}\cdots n)(\lambda,\eta)$$
$$\mathcal{A}_n(1\bar{2}3\cdots\bar{n})(\lambda,\eta)$$

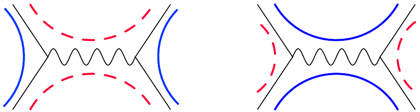
The suspect

Chern-Simons Matter:

1. $\mathcal{N}=6$ (ABJM): $\mathrm{U(N)}_k \times \mathrm{U(N)}_{-k}$ gauge fields$(\mathrm{A}^\mu, \bar{\mathrm{A}}^\mu)$,

SU(4) bi-fundamental matter $(\phi^{\mathrm{I}}, \psi^{\mathrm{I}}, \bar{\phi}_{\mathrm{I}}, \bar{\psi}_{\mathrm{I}})$, $\mathrm{I}=1,2,3,4$

$$\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_{\phi,Kin} + \mathcal{L}_{\psi,Kin} + \mathcal{L}_{4\phi^2\psi^2} + \mathcal{L}_{6\phi^6}$$



2. $\mathcal{N}=8$ (BLG): $\mathrm{SU(2)}_k \times \mathrm{SU(2)}_{-k}$ gauge fields$(\mathrm{A}^\mu, \bar{\mathrm{A}}^\mu)$,

SO(8) adjoint matter $(\phi^{\mathrm{I_v}}, \psi^{\mathrm{I_c}})$

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

The known for $\mathcal{N} = 4$ SYM:

- The planar theory enjoys SU(2,2|4) DSCI
- The string sigma model enjoys fermionic self T-duality
- The (super)amplitude is dual to a (super)Wilson-loop
- The IR-divergence structure captured by BDS
- The leading singularities is given by residues of $Gr(k, n)$
- The amplitude has uniform trancsendentality
- The amplitudehedron

Prelude

As comparison to ABJM:

- The planar theory enjoys SU(2,2|4) DSCI $\rightarrow$ OSp(6|4)
- The IR-divergence structure captured by BDS $\rightarrow$ Remarkably yes
- The leading singularities is given by residues of $Gr(k, n)$ $\rightarrow$ OG(k,2k)S. Lee
- The amplitude has uniform trancsendentality (*) $\rightarrow$ True so far

Known unknowns:

1. Why is the IR-divergence (Dual conformal anomaly equation) the same? Y-t, W. Chen, S. Caron-Huot

$$\mathcal{A}_4^{2-\text{loop}} = \left(\frac{N}{k}\right)^2 \frac{\mathcal{A}_4^{\text{tree}}}{2} \text{BDS}_4$$

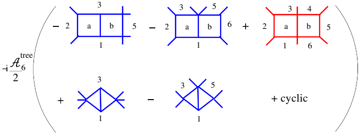
$$\mathcal{A}_6^{2-\text{loop}} = \left(\frac{N}{k}\right)^2 \left\{ \frac{\mathcal{A}_6^{\text{tree}}}{2} \left[\text{BDS}_6 + R_6 \right] + \frac{\mathcal{A}_{6,\text{shifted}}^{\text{tree}}}{4i} \left[\ln \frac{u_2}{u_3} \ln \chi_1 + \text{cyclic} \times 2 \right] \right\}$$

At four-point to all orders in ϵ M. Bianchi, M. Leoni, S Penati, exponentiation verified at three-loops M. Bianchi, M. Leoni

Known unknowns: 2. Why is the amplitude non-analytic?

$$\boxed{\begin{aligned}\mathcal{A}_6^{\text{1-loop}} = \frac{\mathcal{A}_6^{\text{tree}}}{\sqrt{2}}\Big[&I_{box}(3,4,5,1) + I_{box}(1,2,3,4) - I_{box}(4,5,6,1) - I_{box}(6,1,2,4)\Big]\\ &+ \frac{\mathcal{C}_1 + \mathcal{C}_1^*}{2} I_{tri}(1,3,5) + \frac{\mathcal{C}_2 + \mathcal{C}_2^*}{2} I_{tri}(2,4,6)\,.\end{aligned}}$$

$$\rightarrow \boxed{\mathcal{A}_6^{\text{1-loop}} = \left(\frac{N}{k}\right)\frac{-\pi}{2}\mathcal{A}_{6,\text{shifted}}^{\text{tree}}\left(\text{sgn}_c\langle 12\rangle\text{sgn}_c\langle 34\rangle\text{sgn}_c\langle 56\rangle + \text{sgn}_c\langle 23\rangle\text{sgn}_c\langle 45\rangle\text{sgn}_c\langle 61\rangle\right).}$$



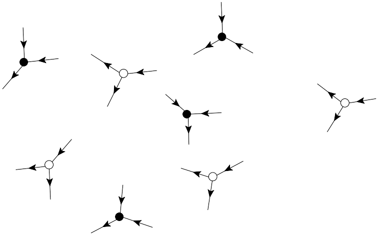
$$\rightarrow \boxed{\begin{aligned}\mathcal{A}_6^{\text{2-loop}} = \left(\frac{N}{k}\right)^2\Bigg\{&\frac{\mathcal{A}_6^{\text{tree}}}{2}\Big[BDS_6 + R_6\Big] + \frac{\mathcal{A}_{6,\text{shifted}}^{\text{tree}}}{2}\times\\ &\left[\text{sgn}_c(\langle 12\rangle)\text{sgn}_c(\langle 45\rangle)\frac{(\langle 34\rangle\langle 46\rangle + \langle 35\rangle\langle 56\rangle)}{\sqrt{(\langle 34\rangle\langle 46\rangle + \langle 35\rangle\langle 56\rangle)^2}}\log\frac{u_2}{u_3}\arccos(\sqrt{u_1}) + \text{cyclic}\times 2\right]\Bigg\}\end{aligned}}$$

Unknown knowns

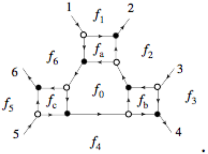
- The string sigma model enjoys fermionic self T-duality $\rightarrow$ Unsucessful
- The (super)amplitude is dual to a (super)Wilson-loop $\rightarrow$ Unsucessful

Prelude

Planar $\mathcal{N}=4$ SYM $\in \mathrm{Gr}(\mathrm{k},\mathrm{n})_+$ Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, J. Trnka



$\rightarrow$



$\rightarrow$ $\mathrm{C}_{\alpha \mathrm{i}}(\mathrm{f})$

$$C_{\alpha i}(f)=\begin{pmatrix} 1 & \frac{1}{f_1}+\frac{1}{f_1 f_a(1+f_0)} & 0 & \frac{f_4 f_5 f_6 f_c}{1+1/f_0} & 0 & \frac{f_6}{1+1/f_0} \\ 0 & \frac{f_2}{1+1/f_0} & 1 & \frac{1}{f_3}+\frac{1}{f_3 f_b(1+f_0)}, & 0 & \frac{f_1 f_2 f_6 f_a}{1+1/f_0} \\ 0 & \frac{f_3 f_4 f_2 f_b}{1+1/f_0} & 0 & \frac{f_4}{1+1/f_0} & 1 & \frac{1}{f_5}+\frac{1}{f_5 f_c(1+f_0)} \end{pmatrix}$$

$$\mathcal{A}_n=\sum_{\mathrm{dia}}\int\prod_i \frac{\mathrm{df_i}}{\mathrm{f_i}}\delta^{4\mathrm{k}|4\mathrm{k}}\left(\mathrm{C}\cdot\mathcal{W}\right)$$

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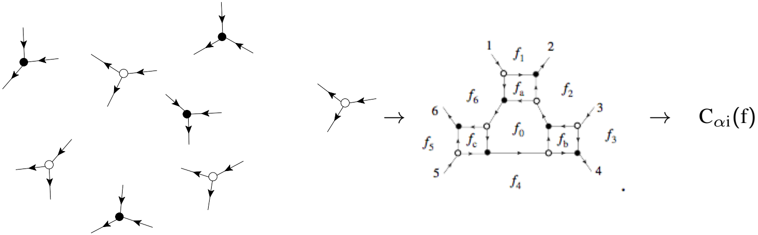

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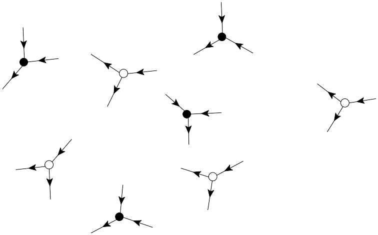
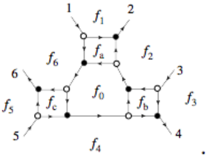


$\rightarrow$



$\rightarrow \quad \mathrm{C}_{\alpha\mathrm{i}}(\mathrm{f})$

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Conclusion

- The scattering amplitude of ABJM is given by integrals over cells in the positive orthogonal grassmannian $\mathrm{OG}_{\mathrm{k}+}$
- Each cell in the positive orthogonal grassmannian $\mathrm{OG}_{\mathrm{k}+} \to$ cell $\mathrm{Gr}(\mathrm{k}, 2\mathrm{k})_+$.
- The canonical form has logarithmic singularity at $\partial\mathrm{OG}_{\mathrm{k}+}$

Orthogonal Grasmmannian

Consider k-planes in n-dimensional space equipped with a symmetric bi-linear Q^{ij}

The orthogonal grassmannian $\equiv \mathrm{Q^{ij}C_{\alpha i}C_{\beta j}} = 0$

Consider $\mathrm{n} = 2\mathrm{k}$ and $\mathrm{Q^{ij}} = \eta^{\mathrm{ij}}$ signature $(+,+,+,\cdots,+)$

$$\begin{aligned} k &= 1, \qquad \mathrm{C_{\alpha i}} = (1, \pm \mathrm{i}) \\ k &= 2, \qquad \mathrm{C_{\alpha i}} = \begin{pmatrix} 1 & \pm i\cos z & 0 & -i\sin z \\ 0 & \pm i\sin z & 1 & i\cos z \end{pmatrix} \end{aligned}$$

$$\mathrm{A_n^{tree}} = \sum_{\mathrm{res}} \int \frac{\mathrm{dC}}{(1\cdots \mathrm{k})\cdots(\mathrm{k}\cdots \mathrm{n}-1)} \delta(\mathrm{Q^{ij}C_{\alpha i}C_{\beta j}})\delta^{2\mathrm{k}}(\mathrm{C}\cdot\lambda)\delta^{3\mathrm{k}}(\mathrm{C}\cdot\eta)$$

S. Lee, D. Gang, E. Koh, E. Koh, A. Lipstein, Y-t

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S. Lee, D. Gang, E. Koh, E. Koh, A. Lipstein, Y-t

Positive Orthogonal Grasmmannian

Positivity: $(\mathrm{i}, \mathrm{i}+1, \cdots, \mathrm{i}+\mathrm{k}) > 0$

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Positive for $0 \leq z \leq \pi/2$

Volume form w. logarithmic singularity at the boundary: $z = \pi/2$, $z = 0$

$$\frac{dz}{\cos z \sin z} = d \log \tan z$$

$$\int d \log \tan \quad \delta^4(C \cdot \lambda)\delta^6(C \cdot \eta)$$

This is not the amplitude $\mathcal{A}_4$!

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Branches of Positive Orthogonal Grasmmannian

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The four-point amplitude is given by the sum of two branches in OG_{2+}

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Why Two Branches of Positive Orthogonal Grasmmannian

$$k=2, C_{\alpha i} = \begin{pmatrix} 1 & \cos z & 0 & -\sin z \\ 0 & \sin z & 1 & \cos z \end{pmatrix}$$

$$\delta^4(C\cdot\lambda) \to \begin{array}{c} \lambda_1 + \cos z\lambda_2 - \sin z\lambda_4 = 0 \\ \lambda_3 + \sin z\lambda_2 + \cos z\lambda_4 = 0 \end{array} \to \langle 34\rangle = \langle 12\rangle$$

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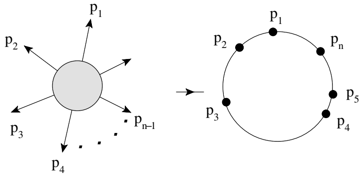
There are two branches in the kinematics as well:

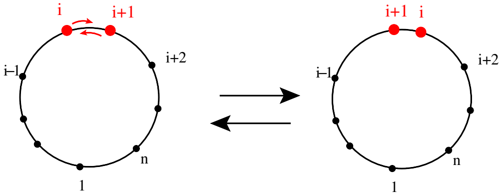
$$\langle 34\rangle^2 = s_{34} = s_{12} = \langle 12\rangle^2$$

Why Two Branches of Positive Orthogonal Grasmmannian

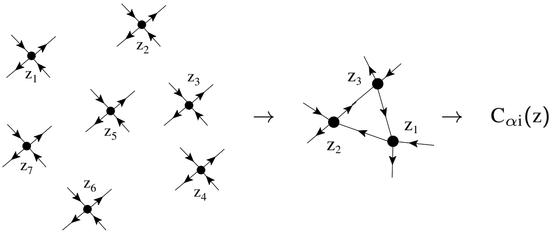
3D- kinematics is topologically a circle

$p_i = (1, \cos\theta_i, \sin\theta_i)$





On-shell diagrams in Orthogonal Grasmmannian



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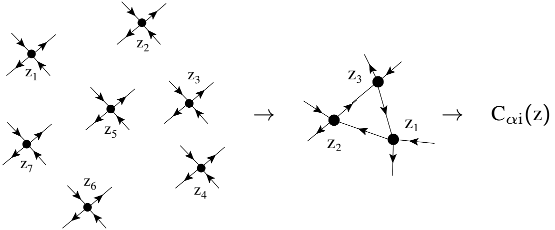
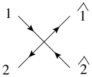
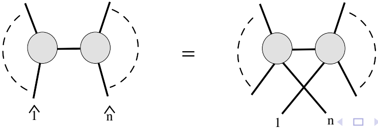


1. Are these diagrams related to $\mathcal{A}_n$? Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, J. Trnka



$$\delta^4(C\cdot\lambda) \rightarrow \begin{array}{l} \lambda_{\hat{1}} + \sec z\lambda_1 + \tan z\lambda_2 \\ \lambda_{\hat{2}} - \tan z\lambda_1 - \sec z\lambda_2 \end{array}$$



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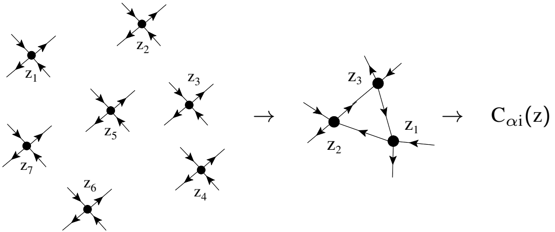
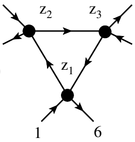


1. Are these diagrams related to $\mathcal{A}_n$?

$$\mathcal{A}_6 = \sum_{\text{branch}} \int \mathrm{d}\log\tan_1 \mathrm{d}\log\tan_2 \mathrm{d}\log\tan_3 \, \delta^{2k}(C\cdot\lambda)\delta^{3k}(C\cdot\eta)$$



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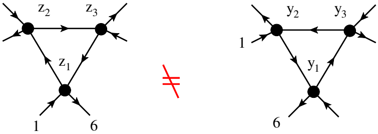
No



On-shell diagrams in Orthogonal Grasmmannian

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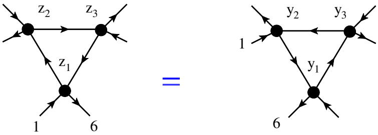
$$\mathcal{A}_6 = \sum_{\text{branch}} \int \mathrm{d}\log\tan_1 \mathrm{d}\log\tan_2 \mathrm{d}\log\tan_3 \,(1+\sin_1\sin_2\sin_3)\,\delta^{2k}(C\cdot\lambda)\,\delta^{3k}(C\cdot\eta)$$

Yes



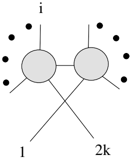
$$\mathcal{A}_6 = \sum_{\text{branch}} \int \mathrm{d}\log\tan_1 \mathrm{d}\log\tan_2 \mathrm{d}\log\tan_3 \,(1+\cos_1\cos_2\cos_3)\,\delta^{2k}(C\cdot\lambda)\,\delta^{3k}(C\cdot\eta)$$

No new singularities $0 \leq z \leq \pi/2$.

On-shell diagrams in Orthogonal Grasmmannian

In general

$$\sum_{\rm i}$$

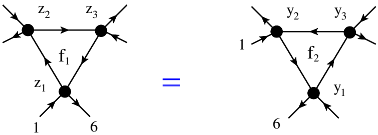


$$\mathcal{A}_n = \sum_{\rm branch} \sum_{\rm dia} \int \prod_{\rm i=1}^{\rm k} {\rm d}\log \tan_{\rm i}\, \mathcal{J} \delta^{\rm 2k}({\rm C}\cdot\lambda)\delta^{\rm 3k}({\rm C}\cdot\eta)$$

How to get $\mathcal{J}$?

On-shell diagrams in Orthogonal Grasmmannian

$$\mathcal{A}_6 = \sum_{\text{branch}} \int \mathrm{d}\log\tan_1 \,\mathrm{d}\log\tan_2\,\mathrm{d}\log\tan_3\,(1+\sin_1\sin_2\sin_3)\delta^{2k}(C\cdot\lambda)\delta^{3k}(C\cdot\eta)$$



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$\mathcal{J}$ is naturally associated with faces!

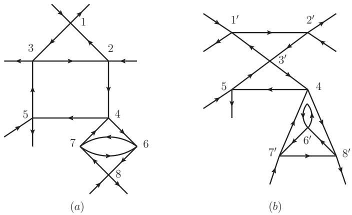
On-shell diagrams in Orthogonal Grasmmannian

$$\mathcal{J} = 1 + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_{13} + \mathcal{J}_{23}$$

- $\mathcal{J}_1$:

$$\mathcal{J}_1 = \sum_{\text{single}} J_i + \sum_{\text{disjoint pairs}} J_i J_j + \sum_{\text{disjoint triples}} J_i J_j J_k + \ldots$$

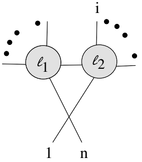
- $\mathcal{J}_2$: Two closed loops sharing a single vertex
- $\mathcal{J}_3$: Two closed loops sharing two vertices without sharing an edge.
- $\mathcal{J}_{13}$ and $\mathcal{J}_{23}$: The effect of the bigger loop from $\mathcal{J}_3$.



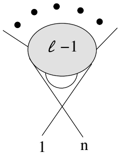
Loop-amplitude and on-shell diagrams in Orthogonal Grasmmannian

The loop-level recurssion Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, J. Trnka

$$\mathcal{A}_n^l = \sum_{l_1+l_2=l}\sum_{i=4}^{n-2}$$



$$+$$



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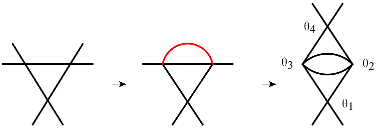

$$= \mathcal{A}_4^{\ell=1}$$

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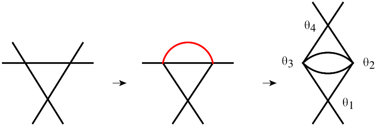

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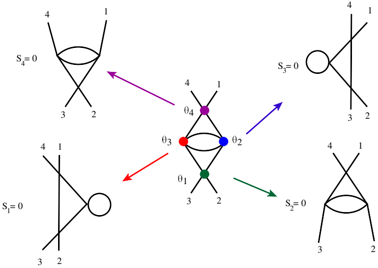
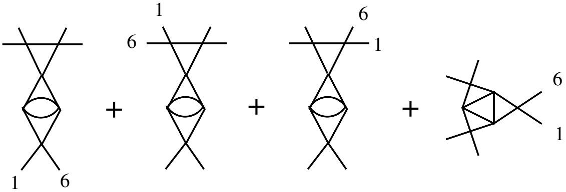
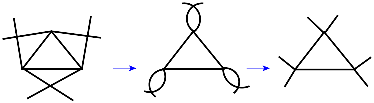


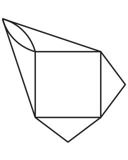
Loop-amplitude and on-shell diagrams in Orthogonal Grasmmannian

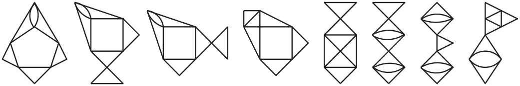
$$A_6^{1-\text{loop}} =$$





Loop-amplitude and on-shell diagrams in Orthogonal Grasmmannian

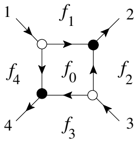
$A_4^{2-\text{loop}} =$  $+ \ (i \to i+2)$

$A_6^{2-\text{loop}} =$  $+ \ (i \to i+2)$

Loop-amplitude and on-shell diagrams in Orthogonal Grasmmannian

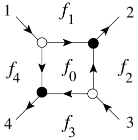
- The solution to BCFW is manifestly cylic $i \to i+2$
- For each cell, a single chart covers all singularities
- All loop: 4 and 6-point amplitudes is a product of independent $d\log$
- Proved all physical sing present, spurious cancels

Embedding $\mathrm{OG}(\mathrm{k}, 2\mathrm{k})$ into $\mathrm{G}(\mathrm{k}, 2\mathrm{k})$



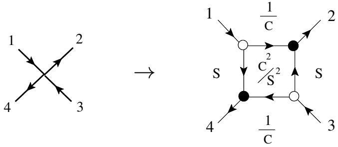
$$\mathrm{C} = \begin{pmatrix} 1 & 1/f_1 & 0 & -f_4 \\ 0 & f_2 & 1 & 1/f_3 \end{pmatrix} .$$

Embedding $\mathrm{OG}(k, 2k)$ into $\mathrm{G}(k, 2k)$



$$C = \begin{pmatrix} 1 & 1/f_1 & 0 & -f_4 \\ 0 & f_2 & 1 & 1/f_3 \end{pmatrix} .$$

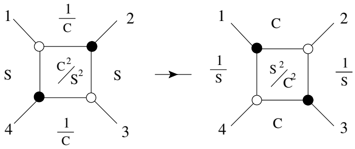
$$f_1 = \frac{1}{c},\ f_4 = s,\ f_2 = s,\ f_3 = \frac{1}{c}$$



OG_{2+} has an image in $Gr(2,4)_+$

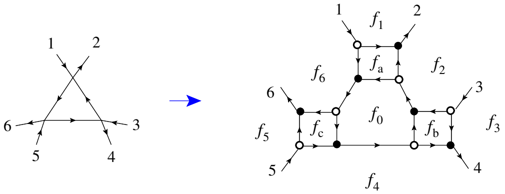
$$C = \begin{pmatrix} 1 & 1/f_1 & 0 & -f_4 \\ 0 & f_2 & 1 & 1/f_3 \end{pmatrix}.$$

Cluster transformation:



$c, s \rightarrow \frac{1}{c}, \frac{1}{s}$

Embedding $\mathrm{OG}(k,2k)$ into $\mathrm{G}(k,2k)$



$$(\mathrm{f_a},\mathrm{f_b},\mathrm{f_c}) = (\mathrm{c}_1^2/\mathrm{s}_1^2, \mathrm{c}_2^2/\mathrm{s}_2^2, \mathrm{c}_3^2/\mathrm{s}_3^2),\ \mathrm{f}_0 = \frac{1}{\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3}$$

$$\mathrm{f}_1 = \frac{1}{\mathrm{c}_1},\ \mathrm{f}_2 = \mathrm{s}_1\mathrm{s}_2,\ \mathrm{f}_3 = \frac{1}{\mathrm{c}_3},\ \mathrm{f}_4 = \mathrm{s}_2\mathrm{s}_3,\ \mathrm{f}_5 = \frac{1}{\mathrm{c}_3},\ \mathrm{f}_6 = \mathrm{s}_1\mathrm{s}_3$$

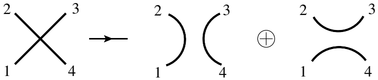
- The variable for the k new faces is simply $f = c^2/s^2$.
- Take a clockwise orientation on each face. The contribution from each vertex is $1/c$ if one first encounters the black vertex, otherwise the contribution is s.

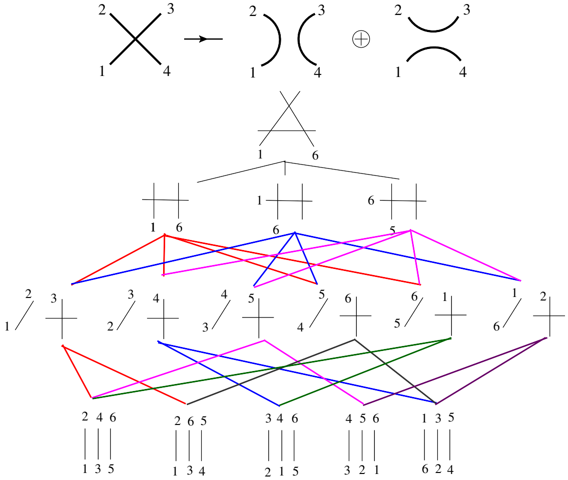
The combinatorics of the cells in Orthogonal Grasmmannian

$$k = 2, \qquad C_{\alpha i} = \begin{pmatrix} 1 & \cos z & 0 & -\sin z \\ 0 & \sin z & 1 & \cos z \end{pmatrix}$$

Volume form w. logarithmic singularity at the boundary: $z = \pi/2$, $z = 0$

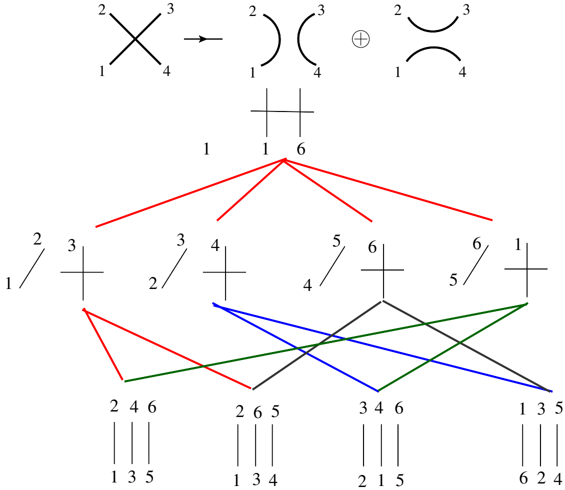
$$\frac{dz}{\cos z \sin z} = d \log \tan z$$





$$-1+3-6+5=1$$

The combinatorics of the cells in Orthogonal Grasmmannian



$$1 - 4 + 4 = 1$$

The combinatorics of the cells in Orthogonal Grasmmannian

A generating function for the number of cells J. Kim, S. Lee

$$T_k(q) = \sum_{l=0}^{k(k-1)/2} T_{k,l} q^l = \frac{1}{(1-q)^k} \sum_{j=-k}^{k} (-1)^j \left(\begin{array}{c} 2k \\ k+j \end{array} \right) q^{j(j-1)/2} \quad (1)$$

$l =$ number of vertices. For top-cells the Euler number is always 1

$$T_k(-1) = \sum_{l=0}^{k(k-1)/2} T_{k,l}(-1)^l = 1$$

Poset is Eulerian Thomas Lam

Conclusions

- Alternative formulations of scattering amplitudes where Locality and Unitarity are secondary
- Such formulations exposes the close relation between a 4-D CFT and a 3-D CFT.
- Extension to non-CFT and fewer susy feasible.
- Progress in non-planar sector
- Gravity?