Color-Kinematic Duality

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Prelude

General Relativity $+$ Quantum Mechanics $=$ Inconsistency

why?

- We don't understand the singularities of the classical solutions.
- We don’t have proper microscopic understanding of the horizon of black holes (Hawking’s information loss).
- If the universe is a wave function, what is the observer?
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General Relativity + Quantum Mechanics = Inconsistency

why?

We are physicists, we should first ask what is the physical observable?

$$\langle \phi(x_1), \phi(x_1), \cdots \phi(x_1) \rangle$$
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General Relativity + Quantum Mechanics = Inconsistency

why?

- We are physicists, we should first ask what is the physical observable?

\[ \langle \phi(x_1), \phi(x_1), \cdots \phi(x_1) \rangle + \]

\[ \leftarrow \text{Not gauge invariant!!} \]

Information loss requires approximate locality

\[ |A \rangle \otimes |B \rangle \]

Pure Entangled State

Approximate Locality

Horizon

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General Relativity + Quantum Mechanics = Inconsistency

Instead, we can consider the scattering of “quantized ripple of space-time” $g^{\mu\nu}$

Only the existence of asymptotic flat space-time is required.
Ok, then what is wrong with gravity amplitudes?

- Short distance sickness (I) $\rightarrow$ Violation of unitarity bounds at high energy:

  Froissart bound $\sigma \sim \log^{D-2} E,$

  but

  $\sigma_{\text{Grav}} \sim E^{2(D-2)/(D-4)}$

- Short distance sickness (II) $\rightarrow$ UV-divergences $\rightarrow \infty$ counter terms $\rightarrow$ lost of predictability

$$\mathcal{L} = \int d^D x \sqrt{g} (R + \alpha_1 R^3 + \alpha_2 R^4 + \cdots)$$

Valuable constraint on understanding UV physics!

Exp:
Tree-unitarity: massive vector bosons must come from symmetry breaking J. Cornwall, D. Levin, G. Tiktopoulos
Ok, then what is wrong with gravity amplitudes?

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  but
  
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\]

Valuable constraint on understanding UV physics!

**Exp:**

Tree-unitarity: massive vector bosons must come from symmetry breaking J. Cornwall, D. Levin, G. Tiktopoulos
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Is the problem truly tied to space-time structure?

- Short distance sickness (I) $\rightarrow$ Violation of unitarity bounds at high energy:
- Short distance sickness (II) $\rightarrow$ UV-divergences $\rightarrow \infty$ counter terms
  $\rightarrow$ lost of predictability

The exact same problem for both gauge and gravity in $D > 4$
That is why string theory contains both!
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WHY gravity amplitudes?

A laboratory:

\[ S \]

\[ T \]

\[ p_1 \quad p_4 \]

\[ p_3 \quad p_2 \]

\[ \log(bM_D) \]

\[ \log \left( \frac{E}{M_D} \right) \]

\[ \frac{2}{D-4} \ln E \]

\[ \frac{1}{D-3} \ln E \]

Born approximation

Eikonal scattering

Strong gravity (~black hole)

Giddings (Erice Lectures)
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We want to challenge:

- Short distance sickness (I) $\rightarrow$ UV-divergences $\rightarrow \infty$ counter terms $\rightarrow$ lost of predictability

\[
\mathcal{L} = \frac{1}{\kappa^2} \int d^D R
\]

Implies $[\kappa^2] = M^{2-D}$

\[
\mathcal{L} = \frac{1}{g^2} \int d^D F^2
\]

Implies $[g^2] = M^{4-D}$

- One can construct infinite number of invariant operators

\[
R^3, R^4, R^5, D^2 R^2, D^2 R^3, D^2 R^4, \ldots
\]

\[
F^3, F^4, F^5, D^2 F^2, D^2 F^3, D^2 F^4, \ldots
\]

- Loops amplitude are dressed $(\kappa^2)^{L-1}$ (gravity) $(g^2)^{L-1}$ (YM)

$R^4 \rightarrow D = 4, \ L = 3$ or $D = 5, \ L = 2$
Prelude

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\[ R^3, R^4, R^5, D^2 R^2, D^2 R^3, D^2 R^4, \cdots \]

\[ F^3, F^4, F^5, D^2 F^2, D^2 F^3, D^2 F^4, \cdots \]

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Implies $[\kappa^2] = M^{2-D}$

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  $R^3, R^4, R^5, D^2R^2, D^2R^3, D^2R^4, \ldots$
  
  $F^3, F^4, F^5, D^2F^2, D^2F^3, D^2F^4, \ldots$

- Loops amplitude are dressed $(\kappa^2)^{L-1}$ (gravity) $(g^2)^{L-1}$ (YM)
  
  $R^4 \rightarrow D = 4, \ L = 3$ or $D = 5, \ L = 2$
Prelude

Gell-Mann’s Totalitarian Principle states: "Everything not forbidden is compulsory"

→ the symmetry allows for infinite divergences!
Prelude

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→ the symmetry allows for infinite divergences!

“The unknown unknown” → Do we know all the symmetries?
Prelude

- UV finiteness of $\mathcal{N} = 8$ supergravity at $L = 3, 4$ loops ($R^4, D^2 R^4$). Z. Bern, J.J Carasco, L. Dixon, H. Johansson, R. Roiban,

- UV finiteness of $\mathcal{N} = 5$ supergravity at $L = 4$ loops ($R^4, D^2 R^4$). Z. Bern, T. Dennen, S. Davies

- UV finiteness of half-maximal supergravity in $D=4$ $L = 3$ loops ($R^4$). Z. Bern, T. Dennen, S. Davies, Y-t H.

- UV finiteness of half-maximal supergravity in $D=5$ $L = 2$ loops ($R^4$). Z. Bern, T. Dennen, S. Davies, Y-t H.
Outline

1. Color-Kinematic Duality
2. D=3 Gravity=(?)^2
A duality between color and Kinematics
We often hear that Gravity=(YM)^2,

- True at 3-points

\[ A_3(g_1^-, g_2^-, g_3^+) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \quad M_3(h_1^-, h_2^-, h_3^+) = \frac{\langle 12 \rangle^6}{\langle 23 \rangle^2 \langle 31 \rangle^2} \]

- Beyond 3-points become complicated: KLT relations Kawai, Lewellen, Tye

\[ M_4^{\text{Closed}}(\{1, 2, 3, 4\}) = \pi^{-1} \sin(\alpha' \pi s_{12}) A_4^{\text{Open}}(1, 2, 3, 4) A_4^{\text{Open}}(2, 1, 3, 4). \]

Not surprising: \( M_4^{\text{Closed}} \) has much more poles than \( A_4^{\text{Open}}(1, 2, 3, 4) \)

→ consider fully color-dressed amplitudes
Consider the fully dressed three-point amplitude:

\[ A_3 = f^{123} [(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot (p_1 - p_2)) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot (p_2 - p_3)) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot (p_3 - p_1))] \]

\[ M_3 = [(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot (p_1 - p_2)) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot (p_2 - p_3)) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot (p_3 - p_1))]^2 \]

- The kinematic factor has the same symmetry as the color factor
- The gravity amplitude is simply replacing the color in YM with the kinematic factor
BCJ-Duality

Bern-Carrasco-Johansson (BCJ):

Duality between color and kinematics for (super)Yang-Mills:

\[ \mathcal{A}_5^{\text{tree}} = g^3 \left( \frac{n_1 c_1}{s_{12} s_{45}} + \frac{n_2 c_2}{s_{23} s_{51}} + \frac{n_3 c_3}{s_{34} s_{12}} + \frac{n_4 c_4}{s_{45} s_{23}} + \frac{n_5 c_5}{s_{51} s_{34}} + \frac{n_6 c_6}{s_{14} s_{25}} ight. \\
+ \frac{n_7 c_7}{s_{32} s_{14}} + \frac{n_8 c_8}{s_{25} s_{43}} + \frac{n_9 c_9}{s_{13} s_{25}} + \frac{n_{10} c_{10}}{s_{42} s_{13}} + \frac{n_{11} c_{11}}{s_{51} s_{42}} + \frac{n_{12} c_{12}}{s_{12} s_{35}} \\
\left. \right) \\
+ \frac{n_{13} c_{13}}{s_{35} s_{24}} + \frac{n_{14} c_{14}}{s_{14} s_{35}} + \frac{n_{15} c_{15}}{s_{13} s_{45}} \right), \]

\[ n_i = (k_4 \cdot k_5)(k_3 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5) + \cdots \]

\[ c_1 = f^{45a} f^{a3b} f^{b12}, \quad c_2 = f^{23a} f^{a4b} f^{b15}, \quad c_3 = f^{34a} f^{a5b} f^{b12}, \cdots \]
One can always find a representation where $n_i$ satisfies the same Jacobi relations as $c_i$. 

$$A_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0$$

$$c_1 = f^{34a} f^{5b} f^{12}, \quad c_2 = f^{34a} f^{2b} f^{15}, \quad c_3 = f^{34a} f^{1b} f^{25}$$

$$c_1 = c_2 - c_3 \iff n_1 = n_2 - n_3$$
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A_{\text{tree}}^5 = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}
\]

\[
[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0
\]

\[
\begin{align*}
c_1 &= f^{34a_f} a^{5b_f} b^{12}, \\
c_2 &= f^{34a_f} a^{2b_f} b^{15}, \\
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\[ A_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \]

\[ n_3 - n_5 + n_8 = 0, \]
\[ n_3 - n_1 + n_{12} = 0, \]
\[ n_4 - n_1 + n_{15} = 0, \]
\[ n_4 - n_2 + n_7 = 0, \]
\[ n_5 - n_2 + n_{11} = 0, \]
\[ n_7 - n_6 + n_{14} = 0, \]
\[ n_8 - n_6 + n_9 = 0, \]
\[ n_{10} - n_9 + n_{15} = 0, \]
\[ n_{10} - n_{11} + n_{13} = 0, \]
\[ (n_{13} - n_{12} + n_{14} = 0), \]

One can always find a representation where \( n_i \) satisfies the same Jacobi relations as \( c_i \)!
Supporting data:
- \( n_i \) for 5,6-points was given in \((BCJ)\)
- Consequence of BCJ-duality

\[
C_1 = C_2 - C_3 \leftrightarrow n_1 = n_2 - n_3 \rightarrow
\]

\[
s_{24} A(1, 2, 4, 3, 5) = (s_{14} + s_{45}) A(1, 2, 3, 4, 5) + s_{14} A(1, 2, 3, 5, 4)
\]

Proven via string theory \(\text{Tye, Zhang, or BCFW recursion}\) \(\text{Chen, Du, Feng, Huang, Jia}\)

- Explicit solutions for \(n_i\):
  - Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; C. R. Mafra, O. Schlotterer and S. Stieberger

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**Conjecture:** Proven
For loops one can always find a representation where $n_i$ satisfies the same Jacobi relations as $c_i$

\[
\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^{L} \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod_{\alpha_j} p_{\alpha_j}^2} \]

\text{Explo} \text{rGraph} \text{[INTEGRALBASIS[[1]]]}

\text{Out[320]=}

\text{Explo} \text{rGraph} \text{[INTEGRALBASIS[[1]]]}

\text{Out[318]=}
BCJ-Duality (Loops)

\[
A_{4}^{2\text{-loop}} = \text{st}A_{4}\text{tree} = \frac{[34][41]}{\langle 12\rangle\langle 23\rangle} = \frac{[34][42]}{\langle 21\rangle\langle 13\rangle} \quad \text{permutation invariant}
\]
BCJ-Duality (Loops)

Non-trivial evidence:

- Up to 4-loops for 4-point in $\mathcal{N} = 4$ SYM [262, 232].
- Up to 2-loops for 5-point in $\mathcal{N} = 4$ SYM [263].

- At 1-loop up to 7-points in $\mathcal{N} = 4$ SYM [264].
- Up to 2-loops for 4-point for the all-plus pure Yang-Mills amplitude [262].
- 1-loop 4-point for pure Yang-Mills theory in arbitrary dimensions [240].
- 1-loop $n$-point all-plus or single-minus helicity amplitudes in pure Yang-Mills theory [265].
- 1-loop 4-point amplitudes in theories with less than maximally supersymmetry [266].
- 1-loop 4-point for an abelian orbifold of $\mathcal{N} = 4$ SYM [267].
- 1-loop 4-point Yang-Mills theory with matter [268].
Providing a comprehensive, pedagogical introduction to scattering amplitudes in gauge theory and gravity, this book is ideal for graduate students and researchers. It offers a smooth transition from basic knowledge of quantum field theory to the frontier of modern research.

Building on basic quantum field theory, the book starts with an introduction to the spinor helicity formalism in the context of Feynman rules for tree-level amplitudes. The material covered includes on-shell recursion relations, superamplitudes, symmetries of N=4 super Yang-Mills theory, twistors and momentum twistors, Grassmannians, and polytopes. The presentation also covers amplitudes in perturbative supergravity, 3d Chern-Simons-matter theories, and color-kinematic duality and its connection to “gravity=(gauge theory)/2”.

Basic knowledge of Feynman rules in scalar field theory and quantum electrodynamics is assumed, but all other tools are introduced as needed. Worked examples demonstrate the techniques discussed, and over 150 exercises help readers absorb and master the material.

Henriette Elvang is Associate Professor in the Department of Physics, University of Michigan. She has worked on various aspects of high energy theoretical physics, including black holes, string theory, scattering amplitudes, and the structure of gauge theories.

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Endorsements:

To follow…

Illustration:

To follow…

arXiv:1308.1697

Yu-tin Huang

NTU
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arXiv:1308.1697
Is there a Lagrangian derivation for the numerators?

They are generated by deformations of YM Lagrangian Bern, Dennen, Kiermiaer, Y-T

\[ \mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}_5' + \mathcal{L}_6' + \ldots. \]

Each term is non-local, non-gauge invariance, but zero!

\[ \mathcal{L}_5' \sim \text{Tr} [A^\nu, A^\rho] \frac{1}{\square} \left( [\partial_\mu A_\nu, A_\rho], A^\mu \right) + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho] \]

Lagrangians are insensible to (very important) zero!
BCJ-Duality

Ok, but so what?

Scattering of gravitons = (Scattering of gluons)^2
Consequences

\[ c_1 = c_2 - c_3 \leftrightarrow n_1 = n_2 - n_3 \]

Once a BCJ \( n_i \) is found, one obtains gravity:

\[
YM: \quad A_n^{\text{tree}} = g^{n-2} \sum_i \frac{n_i c_i}{\prod \alpha_i p_{\alpha_i}^2}
\]

\[
\text{Gravity:} \quad M_n^{\text{tree}} = i \left( \frac{\kappa}{2} \right)^{n-2} \sum_i \frac{n_i \tilde{n}_i}{\prod \alpha_i p_{\alpha_i}^2}
\]

Most importantly, the same for loops:

\[
\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^L \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod \alpha_j p_{\alpha_j}^2}
\]

\[
\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} M_n^L = \sum_j \int \prod_{\ell=1}^L \frac{d^D p_\ell}{(2\pi)^D} \frac{\tilde{n}_j n_j}{S_j \prod \alpha_j p_{\alpha_j}^2}
\]

Proven to all order in perturbation theory: Bern, Dennen, Kiermaier, Y-T
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Consequences

Many supergravity theories can be obtained:

- $N = 8$ supergravity: $(N = 4 \text{ sYM}) \times (N = 4 \text{ sYM})$, 
- $N = 6$ supergravity: $(N = 4 \text{ sYM}) \times (N = 2 \text{ sYM})$, 
- $N = 5$ supergravity: $(N = 4 \text{ sYM}) \times (N = 1 \text{ sYM})$, 
- $N = 4$ supergravity: $(N = 4 \text{ sYM}) \times (N = 0 \text{ sYM})$,

- $N \leq 4$ supergravity with matter: $(N \leq 2 \text{ SYM with matter })^2$ Johansson, Ochirov

- Gauged supergravity: $(N \leq 2 \text{ SYM with matter }) \times (D > 4 \text{ pure YM})$ Chiodaroli, Gunaydin, Johansson, Roiban
Consequences

In fact we only need one copy to satisfy Color-Kinematic duality

\[ c_i + c_j + c_k = 0, \]

Generalized gauge invariance

\[ n_i \rightarrow n_i + s_i \Delta, \quad n_j \rightarrow n_j + s_j \Delta, \quad n_k \rightarrow n_k + s_k \Delta. \]

Define the difference between two valid representations as:

\[ \Delta_i \equiv n_i - \tilde{n}_i \]

Since both are valid:

\[ \sum_{i \in \text{cubic}} \frac{c_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0. \]

The fact that one numerator satisfies the duality:

\[ \sum_{i \in \text{cubic}} \frac{n_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0 \]
\[ \mathcal{N} = 8 \text{ Super Gravity: At } L=3,4 \] 

\[ D_c(L) = \frac{6}{L} + 4 \quad \text{for} \quad L > 1 \]

The critical dimension of \( \mathcal{N} = 8 \) Super Gravity is the same as \( \mathcal{N} = 4 \) SYM

Miraculous cancellations required!

$$D_c(L) = \frac{6}{L} + 4 \quad \text{for} \quad L > 1$$

The critical dimension of $\mathcal{N} = 8$ Super Gravity is the same as $\mathcal{N} = 4$ SYM

$\tau_{i,j} = k_i \cdot \ell_j$

Color-kinematic numerators manifests the equivalent UV behavior!
$\mathcal{N} = 4$ Super Gravity: Super-symmetric SU(2)/U(1) duality invariant operator exists \( g \).

Bossard, P.S. Howe, K.S. Stelle

\[ R^4 \]

Replace one numerator by a Yang-Mills numerator (Feynman diagrams) Bern, Davies, Dennen, Y-T

There is no UV divergence even though counter terms exists!
Consequences

Let’s consider a “Note-Pad” example G. Bossard, P.S. Howe, K.S. Stelle

Half-maximal supergravity in $D = 5$ at $L = 2 \rightarrow R^4$ operator is again a valid counter term
The four-point N=4 SYM BCJ representation does not contain loop momenta in $n_i$:

1-Loop: $\text{st} A_4^{\text{tree}}$

2-Loop: $s^2 t A_4^{\text{tree}} + s^2 t A_4^{\text{tree}} + \text{perm}$

$$\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod \alpha_j p_{\alpha_j}^2}$$

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} M_n^L = \sum_j \int \frac{d^D p_\ell}{(2\pi)^D} \frac{\tilde{n}_j n_j}{S_j \prod \alpha_j p_{\alpha_j}^2}$$
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1-Loop: $s^t A_4^{\text{tree}}$

2-Loop: $s^2t A_4^{\text{tree}} + s^t A_4^{\text{tree}} + \text{perm}$

\[
\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^L \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod_{\alpha} p_{\alpha_j}^2}
\]

\[
\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} M_n^L = \sum_j \int \prod_{\ell=1}^L \frac{d^D p_\ell}{(2\pi)^D} \frac{\tilde{n}_j n_j}{S_j \prod_{\alpha} p_{\alpha_j}^2}
\]
Consequences

Two-loop $D=5$, $R^4$ is a valid counter term for $\mathcal{N} = 4$ super gravity

Color-kinematics duality says it is zero! Bern, Davies, Dennen, Y-T

\[ A_4 = \left[ c_{1234}^P A^P (1, 2, 3, 4) + c_{3421}^P A^P (3, 4, 2, 1) + c_{1234}^{NP} A^{NP} (1, 2, 3, 4) + c_{3421}^{NP} A^{NP} (3, 4, 2, 1) \right. \]
\[ + \text{cyclic}(2, 3, 4) \]

\[ C^P(1,2,3,4) = f^{h1a} f^{a2c} f^{cde} f^{e3g} f^{g4h} f^{hdb} \]

\[ M_4 = A_4 \mid_{c_{1234}^P, c_{1234}^{NP} \to s^2} s_t A^\text{tree} \]
\[ = s t A^\text{tree} \left[ s (A^P (1, 2, 3, 4) + A^P (3, 4, 2, 1) + A^{NP} (1, 2, 3, 4) + A^{NP} (3, 4, 2, 1)) \right. \]
\[ + \text{cyclic}(2, 3, 4) \]

Gravity UV-div $\sim$ YM UV-div ($F^3$)

\[ A^P (1, 2, 3, 4) + A^P (3, 4, 2, 1) + A^{NP} (1, 2, 3, 4) + A^{NP} (3, 4, 2, 1) \mid_{\frac{1}{\epsilon}} = 0 \]
Consequences

Two-loop $D=5$, $R^4$ is a valid counter term for $\mathcal{N} = 4$ super gravity
Color-kinematics duality says it is zero! Bern, Davies, Dennen, Y-T

$$\mathcal{A}_4 = \left[ c_{1234}^P A^P (1, 2, 3, 4) + c_{3421}^P A^P (3, 4, 2, 1) + c_{1234}^{NP} A^{NP} (1, 2, 3, 4) + c_{3421}^{NP} A^{NP} (3, 4, 2, 1) + \text{cyclic}(2, 3, 4) \right]$$

$$\mathcal{M}_4 = \mathcal{A}_4 \big|_{c_{1234}^P, c_{1234}^{NP} \rightarrow s^2 t A^{tree}}$$

$$= st A^{tree} \left[ s(A^P (1, 2, 3, 4) + A^P (3, 4, 2, 1) + A^{NP} (1, 2, 3, 4) + A^{NP} (3, 4, 2, 1)) + \text{cyclic}(2, 3, 4) \right]$$

Gravity UV-div $\sim$ YM UV-div $(F^3)$

$$A^P (1, 2, 3, 4) + A^P (3, 4, 2, 1) + A^{NP} (1, 2, 3, 4) + A^{NP} (3, 4, 2, 1) \big|_{\epsilon} = 0$$
Consequences

Two-loop $D=5$, $R^4$ is a valid counter term for $\mathcal{N} = 4$ super gravity. Color-kinematics duality says it is zero! Bern, Davies, Dennen, Y-T

\[
\mathcal{A}_4 = \left[ c_{1234}^{P} A_P(1, 2, 3, 4) + c_{3421}^{P} A_P(3, 4, 2, 1) + c_{1234}^{NP} A_{NP}(1, 2, 3, 4) + c_{3421}^{NP} A_{NP}(3, 4, 2, 1) \\
+ cyclic(2, 3, 4) \right]
\]

\[
\mathcal{M}_4 = \mathcal{A}_4 |_{c_{1234}^{P}, c_{1234}^{NP} \rightarrow s^2 t A_{tree}}
\]

\[
= st A_{tree} \left[ s(A^P(1, 2, 3, 4) + A^P(3, 4, 2, 1) + A^{NP}(1, 2, 3, 4) + A^{NP}(3, 4, 2, 1)) \right. \\
\left. + cyclic(2, 3, 4) \right]
\]

Gravity UV-div $\sim$ YM UV-div $(F^3)$

\[
A^P(1, 2, 3, 4) + A^P(3, 4, 2, 1) + A^{NP}(1, 2, 3, 4) + A^{NP}(3, 4, 2, 1) |_{\epsilon} = 0
\]
Is this the only Gravity=Gauge^2 example?

Double double copy in three-dimensions
Double double copy in 3D

Three-dimensional supergravity amplitudes: $4D \rightarrow 3D$

- Supersymmetric matter coupled to topological gravity (local supersymmetric sigma model)
- $\mathcal{N} = 8$ Sugra $\rightarrow \mathcal{N} = 16$ Sugra: $128 \phi_v \in \frac{E_8}{SO(16)}$
- $\mathcal{M}_n = 0$ for $n = \text{odd}$ compared to YM $\mathcal{A}_n \neq 0$ for $n = \text{odd}$

Remarkable cancellation for Gravity$= (\text{Yang-Mills})^2$
Can this cancellation be manifest?

Chern–Simons Matter

Gravity+Matter

Perturbation in a topological theory
Double double copy in 3D

Chern-Simons matter theory can be associated with a Lie-3 algebra:

\[ [T^a, T^b, T^c] = f^{abc} d T^d \]

- The algebraic property of \( f^{abc} d \), and hence gauge group, dependent on \( \mathcal{N} \) of SUSY
- Maximal \( \mathcal{N} = 8 \) has a unique gauge group SO(4)
- The “structure constants” satisfies:

\[
\begin{aligned}
&f^{a\bar{c}b} i_{\bar{f}i\bar{e}d\bar{i}} + f^{a\bar{f}b} i_{\bar{f}i\bar{c}d\bar{e}} + f^{a\bar{e}d} i_{\bar{f}i\bar{f}b\bar{c}} + f^{d\bar{e}b} i_{\bar{f}i\bar{f}a\bar{c}} = 0.
\end{aligned}
\]

\[
\begin{aligned}
\text{Diagram: } &\sum_{\text{all orientations}} c_1 + c_2 + c_3 - c_4 = 0 \iff n_1 + n_2 + n_3 - n_4 = 0
\end{aligned}
\]

S. He, T. Mclaughlin, T. Bargeer

Yu-tin Huang
Double double copy in 3D

Duality between Color-Kinematic Dualities: $(\text{Lie 2 Algebra})^2 = (\text{Lie 3 Algebra})^2$

Summary of verified double copy constructions: Huang, H.J.

<table>
<thead>
<tr>
<th>SG theory</th>
<th>$C\text{Sm}_L \times C\text{Sm}_R = \text{supergravity}$</th>
<th>$sY\text{m}_L \times sY\text{m}_R = \text{supergravity}$</th>
<th>coset</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N} = 16$</td>
<td>$16^2 = 256$</td>
<td>$16^2 = 256$</td>
<td>$E_{8}(8)/SO(16)$</td>
</tr>
<tr>
<td>$\mathcal{N} = 12$</td>
<td>$8^2 + 8^2 = 16 \times (4 + 4) = 128$</td>
<td>$16 \times 8 = 128$</td>
<td>$E_{7(-5)}/SO(12) \otimes SO(3)$</td>
</tr>
<tr>
<td>$\mathcal{N} = 10$</td>
<td>$8 \times 4 + 8 \times 4 = 16 \times (2 + 2) = 64$</td>
<td>$16 \times 4 = 64$</td>
<td>$E_{6(-14)}/SO(10) \otimes SO(2)$</td>
</tr>
<tr>
<td>$\mathcal{N} = 8, n = 2$</td>
<td>$4^2 + 4^2 = 8 \times 2 + 8 \times 2 = 32$</td>
<td>$16 \times 2 = 32$</td>
<td>$SO(8,2)/SO(8) \otimes SO(2)$</td>
</tr>
<tr>
<td>$\mathcal{N} = 8, n = 1$</td>
<td>$16 \times 1 = 16$</td>
<td>$16 \times 1 = 16$</td>
<td>$SO(8,1)/SO(8)$</td>
</tr>
</tbody>
</table>

Examples 4pts:

$\mathcal{M}_{4,4}^{\mathcal{N}=12}(1, 2, 3, 4) = (A_{4}^{\mathcal{N}=6})^2 = \left(\frac{\delta^{(6)}}{12\ 23}\sum_{i} \lambda^{\alpha} \eta_{i}^{I}\right)^2$

$\mathcal{M}_{4,n=2}^{\mathcal{N}=8}(1, 2, 3, 4) = (A_{4}^{\mathcal{N}=4})^2 = \left(\frac{\delta^{(4)}}{12\ 23}\sum_{i} \lambda^{\alpha} \eta_{i}^{I}\right)^2$

$\mathcal{M}_{4,n=1}^{\mathcal{N}=8} = \frac{1}{2}\frac{\delta^{(8)}}{12\ 23\ 23\ 23}\sum_{i} \lambda^{\alpha} \eta_{i}^{I}(s^2 + t^2 + u^2)

checked double copy up to 6pts!
Beyond six-points: This new color-kinematic duality fails for $\mathcal{N} < 8$! H. Johansson, S. Lee, Y-t Huang

Why not ABJM? General gauge invariance

$$\sum_i \frac{\Delta_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0$$

ABJM partial amplitudes are not invariant under the BLG general gauge invariance
Trouble beyond six-point

Beyond six-points: This new color-kinematic duality fails for $\mathcal{N} < 8!$ H. Johansson, S. Lee, Y-t Huang

Unlike “Gravity=$(YM)^2$”, “Gravity=$(\text{Chern-Simons matter})^2$” is unique

- $(\mathcal{N} = 16 \text{ sugra})$ the only supergravity that allows a double-double copy

Might there be something miraculous in the UV?

Assuming counter terms (CT) obtained from dimensional reduction
Prediction from SUSY: first valid CT “dim reduction-$R^4$” $\rightarrow$ Diverges at 6-loops
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Conclusions

- Gravity scattering amplitudes are a perfect Lab for issues of Quantum Gravity
- The existence of hidden dualities: color factors ↔ kinematics.
- Partially responsible for UV finiteness in SUGRA
- Double dose of duality in $D = 3 \mathcal{N} = 16$ sugra
- Preliminary analysis shows $M_4$ is finite up to $L=16! \ (Beyond \ in \ progress)$