Color-Kinematic Duality

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Prelude

$$\text{General Relativity} + \text{Quantum Mechanics} = \text{Inconsistency}$$

why?

- We don't understand the singularities of the classical solutions.
- We don't have proper microscopic understanding of the horizon of black holes (Hawking's information loss).
- If the universe is a wave function, what is the observer?

Prelude

$$\text{General Relativity} + \text{Quantum Mechanics} = \text{Inconsistency}$$

why?

- We are physicists, we should first ask what is the physical observable?

$$\langle \phi(x_1)\,,\phi(x_1),\cdots\,\phi(x_1)\rangle$$

Prelude

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why?

- We are physicists, we should first ask what is the physical observable?

$$\langle \phi(x_1), \phi(x_1), \cdots \phi(x_1) \rangle +$$

$\leftarrow$ Not gauge invariant!!

Information loss requires approximate locality

$|A\rangle \otimes |B\rangle$

Pure Entangled State

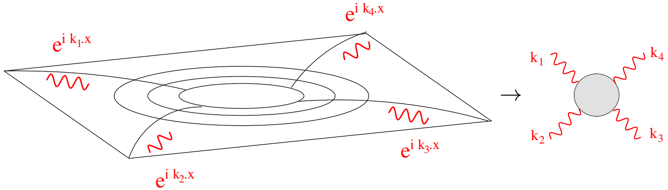
Approximate Locality

Horizon

Prelude

$$\text{General Relativity} + \text{Quantum Mechanics} = \text{Inconsistency}$$

Instead, we can consider the scattering of “quantized ripple of space-time” $g^{\mu\nu}$



Only the existence of asymptotic flat space-time is required.

Prelude

Ok, then what is wrong with gravity amplitudes?

- Short distance sickness (I) $\rightarrow$ Violation of unitarity bounds at high energy:

$$\text{Froissart bound} \;\; \sigma \sim \log^{\mathrm{D}-2} E,$$

but

$$\sigma_{\text{Grav}} \sim E^{2(\mathrm{D}-2)/(\mathrm{D}-4)}$$

- Short distance sickness (II) $\rightarrow$ UV-divergences $\rightarrow \infty$ counter terms $\rightarrow$ lost of predictability

$$\mathcal{L} = \int d^D x \sqrt{g}(R + \alpha_1 R^3 + \alpha_2 R^4 + \cdots)$$

Valuable constraint on understanding UV physics!
Exp:
Tree-unitarity: massive vector bosons must come from symmetry breaking J. Cornwall, D. Levin, G. Tiktopoulos

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Prelude

Is the problem truly tied to space-time structure?

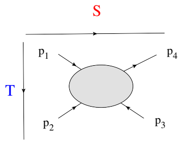
- Short distance sickness (I) $\to$ Violation of unitarity bounds at high energy:
- Short distance sickness (II) $\to$ UV-divergences $\to$ ∞ counter terms $\to$ lost of predictability

The exact same problem for both gauge and gravity in $D > 4$
That is why string theory contains both!

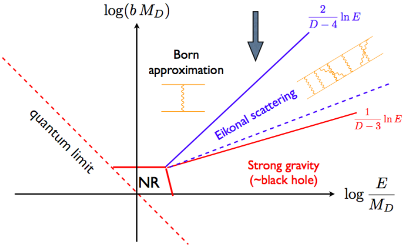
Prelude

WHY gravity amplitudes?

A laboratory :



$\rightarrow$



Giddings(EriceLectures)

Prelude

We want to challenge:

- Short distance sickness (I) $\rightarrow$ UV-divergences $\rightarrow$ ∞ counter terms $\rightarrow$ lost of predictability
-
$$\mathcal{L} = \frac{1}{\kappa^2}\int dx^D R$$
Implies $[\kappa^2] = M^{2-D}$
$$\mathcal{L} = \frac{1}{g^2}\int dx^D F^2$$
Implies $[g^2] = M^{4-D}$
- One can construct infinite number of invariant operators
$$R^3, R^4, R^5, D^2R^2, D^2R^3, D^2R^4, \cdots$$
$$F^3, F^4, F^5, D^2F^2, D^2F^3, D^2F^4, \cdots$$
- Loops amplitude are dressed $(\kappa^2)^{L-1}$ (gravity) $(g^2)^{L-1}$ (YM)
$$R^4 \rightarrow D = 4,\ L = 3 \text{ or } D = 5,\ L = 2$$

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Prelude

Gell-Mann's Totalitarian Principle states: "Everything not forbidden is compulsory"

$\rightarrow$ the symmetry allows for infinite divergences!

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"The unknown unknown " $\rightarrow$ Do we know all the symmetries?

Prelude

- UV finiteness of $\mathcal{N} = 8$ supergravity at $\mathrm{L} = 3, 4$ loops $(\mathrm{R}^4,\, \mathrm{D}^2\mathrm{R}^4)$. Z. Bern, J.J Carasco,L. Dixon, H. Johansson, R. Roiban,
- UV finiteness of $\mathcal{N} = 5$ supergravity at $\mathrm{L} = 4$ loops $(\mathrm{R}^4,\, \mathrm{D}^2\mathrm{R}^4)$. Z. Bern, T. Dennen, S. Davies
- UV finiteness of half-maximal supergravity in D=4 $\mathrm{L} = 3$ loops (R^4). Z. Bern, T. Dennen, S. Davies, Y-t H.
- UV finiteness of half-maximal supergravity in D=5 $\mathrm{L} = 2$ loops (R^4). Z. Bern, T. Dennen, S. Davies, Y-t H.

Outline

A duality between color and Kinematics

We often hear that Gravity=(YM)^2,

- True at 3-points

$$A_3(g_1^-, g_2^-, g_3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}, \qquad M_3(h_1^-, h_2^-, h_3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}$$

- Beyond 3-points become complicated: KLT relations Kawai, Lewellen, Tye

$$M_4^{\text{Closed}}(\{1,2,3,4\}) = \pi^{-1}\sin(\alpha'\pi s_{12})A_4^{\text{Open}}(1,2,3,4)A_4^{\text{Open}}(2,1,3,4)\,.$$

Not surprising: M_4^{Closed} has much more poles than $A_4^{\text{Open}}(1,2,3,4)$

$\rightarrow$ consider fully color-dressed amplitudes

Consider the fully dressed three-point amplitude:

$$A_3 = f^{123}\left[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot(p_1-p_2))+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot(p_2-p_3))+(\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot(p_3-p_1))\right]$$

$$M_3 = \left[(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot(p_1-p_2))+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot(p_2-p_3))+(\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot(p_3-p_1))\right]^2$$

- The kinematic factor has the same symmetry as the color factor
- The gravity amplitude is simply replacing the color in YM with the kinematic factor

BCJ-Duality

Bern-Carrasco-Johansson(BCJ):

Duality between color and kinematics for (super)Yang-Mills:

$$\mathcal{A}_5^{\text{tree}} = g^3\Big(\frac{n_1c_1}{s_{12}s_{45}} + \frac{n_2c_2}{s_{23}s_{51}} + \frac{n_3c_3}{s_{34}s_{12}} + \frac{n_4c_4}{s_{45}s_{23}} + \frac{n_5c_5}{s_{51}s_{34}} + \frac{n_6c_6}{s_{14}s_{25}} + \frac{n_7c_7}{s_{32}s_{14}} + \frac{n_8c_8}{s_{25}s_{43}} + \frac{n_9c_9}{s_{13}s_{25}} + \frac{n_{10}c_{10}}{s_{42}s_{13}} + \frac{n_{11}c_{11}}{s_{51}s_{42}} + \frac{n_{12}c_{12}}{s_{12}s_{35}} + \frac{n_{13}c_{13}}{s_{35}s_{24}} + \frac{n_{14}c_{14}}{s_{14}s_{35}} + \frac{n_{15}c_{15}}{s_{13}s_{45}}\Big),$$

$$\mathrm{n_i} = (\mathrm{k}_4 \cdot \mathrm{k}_5)(\mathrm{k}_3 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5) + \cdots$$

$$\mathrm{c}_1 = \mathrm{f}^{\,45a}\mathrm{f}^{\,a3b}\mathrm{f}^{\,b12}, \quad \mathrm{c}_2 = \mathrm{f}^{\,23a}\mathrm{f}^{\,a4b}\mathrm{f}^{\,b15}, \quad \mathrm{c}_3 = \mathrm{f}^{\,34a}\mathrm{f}^{\,a5b}\mathrm{f}^{\,b12}, \cdots$$

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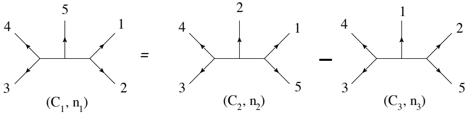
$$A_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

$$[T_a,[T_b,T_c]] + [T_b,[T_c,T_a]] + [T_c,[T_a,T_b]] = 0$$



$$c_1 = f^{34a} f^{a5b} f^{b12}, \quad c_2 = f^{34a} f^{a2b} f^{b15}, \quad c_3 = f^{34a} f^{a1b} f^{b25}$$

$$\boxed{c_1 = c_2 - c_3 \leftrightarrow n_1 = n_2 - n_3}$$

One can always find a representation where n_i satisfies the same Jacobi relations as c_i!

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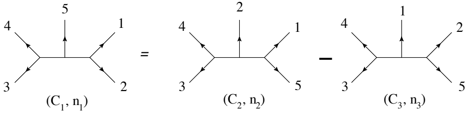

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$$\begin{aligned}
&n_3 - n_5 + n_8 = 0\,,\\
&n_3 - n_1 + n_{12} = 0\,,\\
&n_4 - n_1 + n_{15} = 0\,,\\
&n_4 - n_2 + n_7 = 0\,,\\
&n_5 - n_2 + n_{11} = 0\,,\\
&n_7 - n_6 + n_{14} = 0\,,\\
&n_8 - n_6 + n_9 = 0\,,\\
&n_{10} - n_9 + n_{15} = 0\,,\\
&n_{10} - n_{11} + n_{13} = 0\,,\\
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BCJ-Duality

Supporting data:

- $\mathrm{n_i}$ for 5,6-points was given in (BCJ)
- Consequence of BCJ-duality

$$\mathrm{C_1 = C_2 - C_3 \leftrightarrow n_1 = n_2 - n_3} \rightarrow$$

$$s_{24}A(1,2,4,3,5) = (s_{14} + s_{45})A(1,2,3,4,5) + s_{14}A(1,2,3,5,4) \quad s_{i,j} \equiv p_i \cdot p_j$$

  Proven via string theory Tye, Zhang, or BCFW recursion Chen, Du, Feng, Huang, Jia

- Explicit solutions for $\mathrm{n_i}$:

  Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove,C. R. Mafra, O. Schlotterer and S. Stieberger

Conjecture: Proven

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BCJ-Duality (Loops)

For loops one can always find a representation where n_i satisfies the same Jacobi relations as c_i

$$\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^{L} \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod_{\alpha_j} p_{\alpha_j}^2}$$

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In[323]:= ExploreGraph[INTEGRALBASIS[[1]]]
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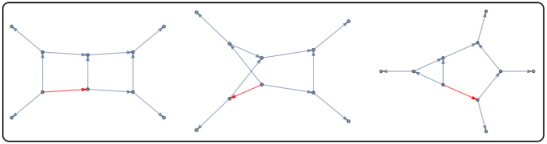
Out[323]=



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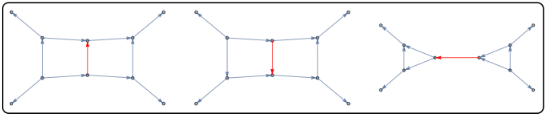
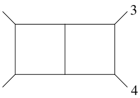
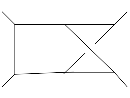


BCJ-Duality (Loops)

$$A_4^{2\text{–loop}} = stA_4^{\text{tree}} \left(s \right.$$



$$+ \; s$$

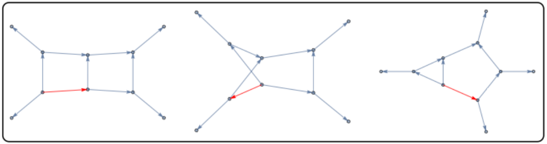


$$\left. + \; \cdots \right)$$

$$stA_4^{\text{tree}} = \frac{[34][41]}{\langle 12\rangle\langle 23\rangle} = \frac{[34][42]}{\langle 21\rangle\langle 13\rangle} \quad \text{permutation invariant}$$

In[323]:= `ExploreGraph[INTEGRALBASIS[[1]]]`

Out[323]=



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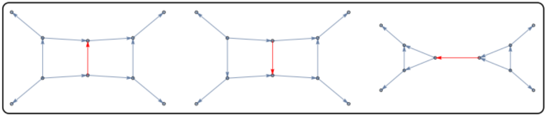
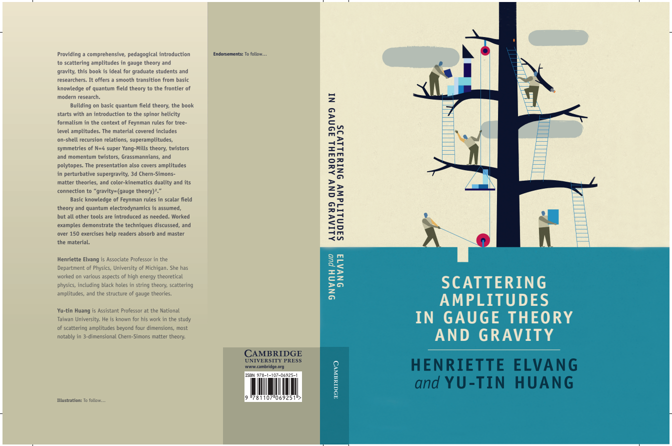


BCJ-Duality (Loops)

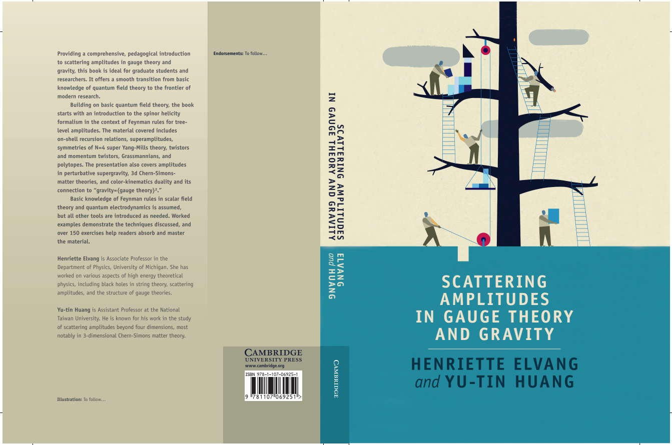
Non-trivial ividence:

- Up to 4-loops for 4-point in $\mathcal{N} = 4$ SYM [262, 232].
- Up to 2-loops for 5-point in $\mathcal{N} = 4$ SYM [263].

- At 1-loop up to 7-points in $\mathcal{N} = 4$ SYM [264].
- Up to 2-loops for 4-point for the all-plus pure Yang-Mills amplitude [262].
- 1-loop 4-point for pure Yang-Mills theory in arbitrary dimensions [240].
- 1-loop n-point all-plus or single-minus helicity amplitudes in pure Yang-Mills theory [265].
- 1-loop 4-point amplitudes in theories with less than maximally supersymmetry [266].
- 1-loop 4-point for an abelian orbifold of $\mathcal{N} = 4$ SYM [267].
- 1-loop 4-point Yang-Mills theory with matter [268].



arXiv:1308.1697



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Is there a Lagrangian derivation for the numerators ?

They are generated by deformations of YM Lagrangian Bern, Dennen, Kiermiaer, Y-T

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \ldots .$$

Each term is non-local, non-gauge invariance, but zero!

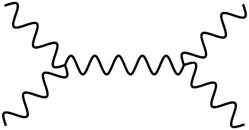
$$\mathcal{L}'_5 \sim \mathrm{Tr}\,[A^\nu, A^\rho]\frac{1}{\Box}\Big([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho]\Big)$$

Lagrangians are insensible to (very important) zero !

BCJ-Duality

Ok, but so what ?

Scattering of gravitons = (Scattering of gluons)^2



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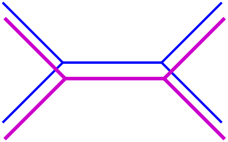

Consequences

$$\boxed{c_1 = c_2 - c_3 \leftrightarrow n_1 = n_2 - n_3}$$

Once a BCJ n_i is found, one obtains gravity:

$$\mathrm{YM}: \quad A_n^{\mathrm{tree}} = g^{n-2} \sum_i \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

$$\mathrm{Gravity}: \quad M_n^{\mathrm{tree}} = i \left(\frac{\kappa}{2}\right)^{n-2} \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

Most importantly, the same for loops:

$$\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^{L} \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod_{\alpha_j} p_{\alpha_j}^2}$$

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} M_n^L = \sum_j \int \prod_{\ell=1}^{L} \frac{d^D p_\ell}{(2\pi)^D} \frac{\tilde{n}_j n_j}{S_j \prod_{\alpha_j} p_{\alpha_j}^2}$$

Proven to all order in perturbation theory: Bern, Dennen, Kiermiaer, Y-T

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Consequences

Many supergravity theories can be obtained:

- $$\begin{aligned} \mathcal{N} = 8 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 4 \text{ sYM}), \\ \mathcal{N} = 6 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 2 \text{ sYM}), \\ \mathcal{N} = 5 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 1 \text{ sYM}), \\ \mathcal{N} = 4 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 0 \text{ sYM}), \end{aligned}$$
- $\mathcal{N} \leq 4$ supergravity with matter: ($\mathcal{N} \leq 2$ SYM with matter)^2 Johansson, Ochirov
- Gauged supergravity: ($\mathcal{N} \leq 2$ SYM with matter) $\times$ (D$>$4 pure YM) Chiodaroli, Gunaydin, Johansson, Roiban

Consequences

In fact we only need one copy to satisfy Color-Kinematic duality

$$c_i + c_j + c_k = 0\,,$$

Generalized gauge invariance

$$n_i \to n_i + s_i\Delta, \quad n_j \to n_j + s_j\Delta, \quad n_k \to n_k + s_k\Delta\,.$$

Define the difference between two valid representations as:

$$\Delta_i \equiv n_i - \tilde{n}_i$$

Since both are valid:

$$\sum_{i\in\text{cubic}} \frac{c_i\Delta_i}{\prod_{\alpha_i} p^2_{\alpha_i}} = 0\,.$$

The fact that one numerator satisfies the duality:

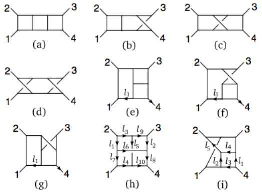
$$\sum_{i\in\text{cubic}} \frac{n_i\Delta_i}{\prod_{\alpha_i} p^2_{\alpha_i}} = 0$$

Consequences

$\mathcal{N} = 8$ Super Gravity: At L=3,4 Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, R. Roiban

$$D_c(L) = \frac{6}{L} + 4 \quad \text{for} \quad L > 1$$

The critical dimension of $\mathcal{N} = 8$ Super Gravity is the same as $\mathcal{N} = 4$ SYM



Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills	$\mathcal{N} = 8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1 + k_4)^2$	$[s\,(l_1 + k_4)^2]^2$
(h)	$sl_{1,2}^2 + tl_{3,4}^2 - sl_5^2 - tl_6^2 - st$	$(sl_{1,2}^2 + tl_{3,4}^2 - st)^2 - s^2(2(l_{1,2}^2 - t) + l_5^2)l_5^2 - t^2(2(l_{3,4}^2 - s) + l_6^2)l_6^2$ $- s^2(2l_1^2l_8^2 + 2l_2^2l_7^2 + l_1^2l_7^2 + l_2^2l_8^2) - t^2(2l_3^2l_{10}^2 + 2l_4^2l_9^2 + l_3^2l_9^2 + l_4^2l_{10}^2) + 2stl_5^2l_6^2$
(i)	$sl_{1,2}^2 - tl_{3,4}^2 - \frac{1}{3}(s - t)l_5^2$	$(sl_{1,2}^2 - tl_{3,4}^2)^2 - (s^2l_{1,2}^2 + t^2l_{3,4}^2 + \frac{1}{3}stu)l_5^2$

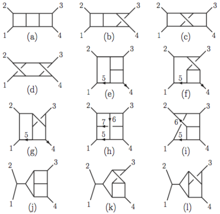
Miraculous cancellations required!

Consequences

$\mathcal{N}=8$ Super Gravity: At L=3,4 Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, R. Roiban

$$D_c(L)=\frac{6}{L}+4 \quad \text{for} \quad L>1$$

The critical dimension of $\mathcal{N}=8$ Super Gravity is the same as $\mathcal{N}=4$ SYM



Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
(i)	$\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right)\right.$ $\left.+t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{i,j}=k_i\cdot\ell_j$$

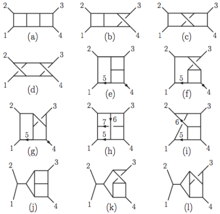
Color-kinematic numerators manifests the equivalent UV behavior!

Consequences

$\mathcal{N} = 4$ Super Gravity: Super-symmetric SU(2)/U(1) duality invariant operator exists G. Bossard, P.S. Howe, K.S. Stelle

$$R^4$$

Replace one numerator by a Yang-Mills numerator (Feynman diagrams) Bern, Davies, Dennen, Y-T



Graph	(divergence)$/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3} + \frac{205}{27648}\frac{1}{\epsilon^2} + \left(-\frac{5551}{768}\zeta_3 + \frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3} - \frac{1}{4}\frac{1}{\epsilon^2} + \left(\frac{593}{288}\zeta_3 - \frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3} + \frac{2057}{6912}\frac{1}{\epsilon^2} + \left(\frac{10769}{2304}\zeta_3 - \frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3} - \frac{41}{1536}\frac{1}{\epsilon^2} + \left(\frac{3227}{2304}\zeta_3 - \frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3} - \frac{29}{1024}\frac{1}{\epsilon^2} + \left(-\frac{2087}{2304}\zeta_3 - \frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3} + \frac{9}{64}\frac{1}{\epsilon^2} + \left(\frac{101}{12}\zeta_3 - \frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3} + \frac{89}{1152}\frac{1}{\epsilon^2} + \left(-\frac{377}{144}\zeta_3 + \frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3} - \frac{251}{1152}\frac{1}{\epsilon^2} + \left(-\frac{835}{144}\zeta_3 + \frac{7385}{3456}\right)\frac{1}{\epsilon}$

There is no UV divergence even though counter terms exists !

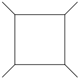
Consequences

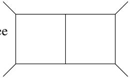
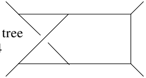
Let's consider a "Note-Pad" example G. Bossard, P.S. Howe, K.S. Stelle

Half-maximal supergravity in $D = 5$ at $L = 2 \rightarrow R^4$ operator is again a valid counter term

Consequences

The four-point N=4 SYM BCJ representation does not contain loop momenta in n_i:

1–Loop: $st\, A_4^{tree}$ 

2–Loop: $s^2t\, A_4^{tree}$  $+\ s^2t\, A_4^{tree}$  $+$ perm

$$\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^{L} \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod_{\alpha_j} p^2_{\alpha_j}}$$

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} M_n^L = \sum_j \int \prod_{\ell=1}^{L} \frac{d^D p_\ell}{(2\pi)^D} \frac{\tilde{n}_j n_j}{S_j \prod_{\alpha_j} p^2_{\alpha_j}}$$

Consequences

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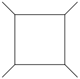
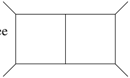
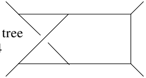
1–Loop: $st\, A_4^{tree}$ 

2–Loop: $s^2t\, A_4^{tree}$  $+\ s^2t\, A_4^{tree}$  $+$ perm

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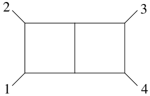
Consequences

Two-loop D=5, R^4 is a valid counter term for $\mathcal{N}=4$ super gravity
Color-kinematics duality says it is zero! Bern, Davies, Dennen, Y-T

$$\mathcal{A}_4 = \Big[c^P_{1234}A^P(1,2,3,4)+c^P_{3421}A^P(3,4,2,1)+c^{NP}_{1234}A^{NP}(1,2,3,4)+c^{NP}_{3421}A^{NP}(3,4,2,1) +cyclic(2,3,4)\Big]$$



$C^P(1,2,3,4) = f^{b1a}f^{a2c}f^{cde}f^{e3g}f^{g4h}f^{hdb}$

$$\begin{aligned}\mathcal{M}_4 &= \mathcal{A}_4|_{c^P_{1234},c^{NP}_{1234}\to s^2tA^{\text{tree}}}\\ &= stA^{\text{tree}}\Big[s(A^P(1,2,3,4)+A^P(3,4,2,1)+A^{NP}(1,2,3,4)+A^{NP}(3,4,2,1))\\ &\quad +cyclic(2,3,4)\Big]\end{aligned}$$

Gravity UV-div $\sim$ YM UV-div (F^3)

$$A^P(1,2,3,4)+A^P(3,4,2,1)+A^{NP}(1,2,3,4)+A^{NP}(3,4,2,1)|_{\frac{1}{\epsilon}}=0$$

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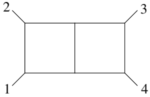

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$$= st A^{\text{tree}} \Big[s(A^P(1,2,3,4) + A^P(3,4,2,1) + A^{NP}(1,2,3,4) + A^{NP}(3,4,2,1)) + cyclic(2,3,4) \Big]$$

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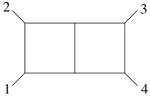

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$$A^P(1,2,3,4)+A^P(3,4,2,1)+A^{NP}(1,2,3,4)+A^{NP}(3,4,2,1)|_{\frac{1}{\epsilon}} = 0$$

Is this the only Gravity=Gauge^2 example ?

Double double copy in three-dimensions

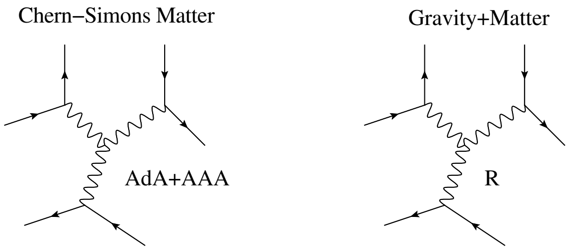
Double double copy in 3D

Three-dimensional supergravity amplitudes: $4D \to 3D$

- Supersymmetric matter coupled to topological gravity (local supersymmetric sigma model)
- $\mathcal{N} = 8$ Sugra $\to \mathcal{N} = 16$ Sugra: **128** $\phi_v \in \frac{E_8}{SO(16)}$
- $\mathcal{M}_n = 0$ for $n = odd$ compared to YM $\mathcal{A}_n \neq 0$ for $n = odd$

Remarkable cancellation for Gravity= (Yang-Mills)^2

Can this cancellation be manifest ?



Perturbation in a topological theory

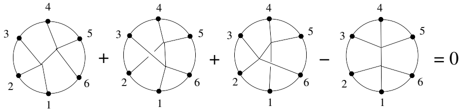
Double double copy in 3D

Chern-Simons matter theory can be associated with a Lie-3 algebra:

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

- The algebraic property of $f^{abc}{}_d$, and hence gauge group, dependent on $\mathcal{N}$ of SUSY
- Maximal $\mathcal{N} = 8$ has a unique gauge group SO(4)
- The “structure constants” satisfies:

$$f^{a\bar{c}b}{}_i f^{i\bar{e}d\bar{f}} + f^{a\bar{f}b}{}_i f^{i\bar{c}d\bar{e}} + f^{a\bar{e}d}{}_i f^{i\bar{f}b\bar{c}} + f^{d\bar{e}b}{}_i f^{i\bar{f}a\bar{c}} = 0 \,.$$



$$c_1 + c_2 + c_3 - c_4 = 0 \leftrightarrow n_1 + n_2 + n_3 - n_4 = 0$$

S. He, T. Mclaughlin, T. Bargeer

Double double copy in 3D

Duality between Color-Kinematic Dualities: $(\text{Lie 2 Algebra})^2 = (\text{Lie 3 Algebra})^2$

Summary of verified double copy constructions: Huang, H.J.

SG theory	$\text{CSm}_\text{L}\times\text{CSm}_\text{R}$ = supergravity	$\text{sYm}_\text{L}\times\text{sYM}_\text{R}$ = supergravity	coset
$\mathcal{N}=16$	$16^2=256$	$16^2=256$	$E_{8(8)}/SO(16)$
$\mathcal{N}=12$	$8^2+\bar{8}^2 = 16\times(4+\bar{4}) = 128$	$16\times 8=128$	$E_{7(-5)}/SO(12)\otimes SO(3)$
$\mathcal{N}=10$	$8\times 4+\bar{8}\times\bar{4} = 16\times(2+\bar{2}) = 64$	$16\times 4=64$	$E_{6(-14)}/SO(10)\otimes SO(2)$
$\mathcal{N}=8,\ n=2$	$4^2+\bar{4}^2=8\times 2+\bar{8}\times\bar{2}=32$	$16\times 2=32$	$SO(8,2)/SO(8)\otimes SO(2)$
$\mathcal{N}=8,\ n=1$	$16\times 1=16$	$16\times 1=16$	$SO(8,1)/SO(8)$

Examples 4pts:

$$\mathcal{M}_4^{\mathcal{N}=12}(\bar{1},2,\bar{3},4) = (A_4^{\mathcal{N}=6})^2 = \left(\frac{\delta^{(6)}(\sum_i \lambda^\alpha \eta_i^I)}{\langle 1\,2\rangle\langle 2\,3\rangle}\right)^2$$

$$\mathcal{M}_{4,n=2}^{\mathcal{N}=8}(\bar{1},2,\bar{3},4) = (A_4^{\mathcal{N}=4})^2 = \left(\frac{\delta^{(4)}(\sum_i \lambda^\alpha \eta_i^I)\langle 1\,3\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle}\right)^2$$

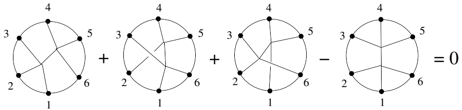
$$\mathcal{M}_{4,n=1}^{\mathcal{N}=8} = \frac{1}{2}\frac{\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^I)(s^2+t^2+u^2)}{\langle 1\,2\rangle^2\langle 2\,3\rangle^2\langle 1\,3\rangle^2}$$

checked double copy up to 6pts!

Trouble beyond six-point

Beyond six-points: This new color-kinematic duality fails for $\mathcal{N} < 8$! H. Johansson, S. Lee, Y-t Huang

Why not ABJM? General gauge invariance



$$c_1 + c_2 + c_3 - c_4$$

$$n_i \to n_i + \Delta_i, \quad \Delta_i = \Delta p^2_{\alpha_i}, \quad \sum_i \frac{\Delta_i c_i}{\prod_{\alpha_i} p^2_{\alpha_i}} = 0$$

ABJM partial amplitudes are not invariant under the BLG general gauge invariance

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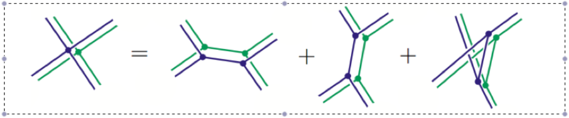
Beyond six-points: This new color-kinematic duality fails for $\mathcal{N} < 8$! H. Johansson, S. Lee, Y-t Huang

Unlike "Gravity=(YM)^2", "Gravity=(Chern-Simons matter)^2" is unique

- ($\mathcal{N} = 16$ sugra) the only supergravity that allows a double-double copy



Might there be something miraculous in the UV?

Assuming counter terms (CT) obtained from dimensional reduction
Prediction from SUSY: first valid CT "dim reduction-R^4"$\rightarrow$Diverges at 6-loops

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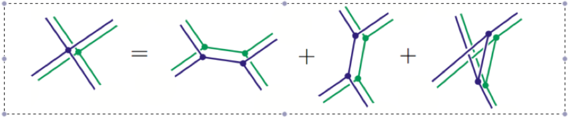

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Conclusions

- Gravity scattering amplitudes are a perfect Lab for issues of Quantum Gravity
- The existence of hidden dualities: color factors $\leftrightarrow$ kinematics.
- Partially responsible for UV finiteness in SUGRA
- Double dose of duality in $D = 3$ $\mathcal{N} = 16$ sugra
- Preliminary analysis shows M_4 is finite up to L=16! (Beyond in progress)