Geometry of nilpotent orbits

Baohua FU (AMSS)
9 August 2014, Daejeon
Question: Why nilpotent orbits?
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They are important in representation theory...
They form a very nice class of examples of *symplectic varieties*
and they have many interesting geometrical properties...
Birational geometry: well-understood by works of Namikawa and Myself, connected by stratified Mukai flops.
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What problems on the geometry?

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**Today:**

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Symplectic varieties

## Definition (Beauville)

(i) A **symplectic variety** is a variety $W$, smooth in codimension 1, with a symplectic form $\omega$ on $W_{\text{reg}}$ such that for any resolution $\pi : Z \to W$, the pull-back $\pi^*\omega$ extends to a regular 2-form $\Omega$ on $Z$.

(ii) If $\Omega$ is everywhere non-degenerate, then $\pi : Z \to W$ is called a **symplectic resolution** of $W$.

## Remark

- For a symplectic variety $W$, a resolution $\pi : Z \to W$ is symplectic iff $\pi$ is crepant, i.e. $K_Z = \mathcal{O}_Z$.
- The normalization of a symplectic variety is again a symplectic variety and it has only rational Gorenstein singularities.
Theorem (Namikawa)

Normal symplectic variety = rational Gorenstein + symplectic form.

and

Symplectic resolution = crepant resolution
(i) ADE-singularities: $\mathbb{C}^2/H$, where $H \subset SL(2)$ is a finite subgroup. Symplectic resolution = Minimal resolution
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(ii) [Fu-Namikawa] \( \text{Sym}^n(\mathbb{C}^2) = \mathbb{C}^{2n} / S_n \) has a unique symplectic resolution given by \( \text{Hilb}^n(\mathbb{C}^2) \to \text{Sym}^n(\mathbb{C}^2) \).

(iii) \( \text{Sym}^n(\mathbb{C}^2 / H) \) has many symplectic resolutions obtained by Mukai flops from \( \text{Hilb}^n(\hat{\mathbb{C}}^2 / H) \to \text{Sym}^n(\mathbb{C}^2 / H) \), where \( \hat{\mathbb{C}}^2 / H \to \mathbb{C}^2 / H \) is the minimal resolution.

(iv) Except two more examples in dim. 4, no other (irreducible) \( \mathbb{C}^{2n} / H \) is expected to admit a symplectic resolution. For example, \( \mathbb{C}^{2n} / \pm 1 \) has no symplectic resolution if \( n \geq 2 \).
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Finite symplectic quotient $\mathbb{C}^{2n}/H$

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[Note: The text is a compilation of points about symplectic resolutions and minimal resolutions, emphasizing specific examples and cases where such resolutions exist or do not exist.]
Another important classes of symplectic varieties are given by nilpotent orbit closures in semi-simple Lie algebras.
### Definition

- A $A \in M_{n \times n}(\mathbb{C})$ is called **nilpotent** if $A^n = 0$.
- The **nilpotent orbit** of $A$ is the set of all matrices conjugate to $A$.
- The **nilpotent cone** $\mathcal{N}$ is the set of all nilpotent matrices.
Nilpotent matrices

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Every nilpotent matrix has its Jordan form $\text{Diag}(J_{d_1}, \cdots, J_{d_k})$, where $J_{d_i}$ is the Jordan matrix of size $d_i$ with zeros on diagonal. Hence **nilpotent orbits in** $M_{n \times n}(\mathbb{C})$ **are parametrized by partitions of** $n$. 

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Note that $\mathcal{N}$ is defined by polynomials in $M_{n \times n}(\mathbb{C}) \simeq \mathbb{C}^{n^2}$, hence it is an affine algebraic variety.
Let $A \in sl_2(\mathbb{C})$ be as

$$
\begin{pmatrix}
  a & b \\
  c & -a
\end{pmatrix}
$$

$A$ is nilpotent iff $A^2 = 0$ iff $a^2 + bc = 0$. The nilpotent cone consists of two orbits: $[2], [1, 1]$. 
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- The nilpotent cone is the union of nilpotent orbits
- Nilpotent orbits are not closed
- Nilpotent orbits are stable under dilations
- Nilpotent orbits are of even dimension
Nilpotent orbit closures

g a complex simple Lie algebra and \( G \) the adjoint Lie group of \( g \).
\( 0 \neq x \in g \) a nilpotent element, \( O_x := G \cdot x \) the nilpotent orbit of \( x \).

This homogeneous manifold enjoys the following properties:

- \( O_x \) admits a holomorphic symplectic structure \( \omega \) (K-K-S form)
- \( O_x \) is stable under the dilation action of \( \mathbb{C}^* \) and we have \( \lambda^* \omega = \lambda \omega \).
- The closure \( \bar{O}_x \) is singular and it is a union of finitely many nilpotent orbits.
- Nilpotent orbits are classified by weighted Dynkin diagrams (and by partitions in the case of classical types).

**Theorem (Hinich, Payushev)**

*Nilpotent orbit closures in semisimple Lie algebras are symplectic varieties.*
Partial order between orbits: denote by $\mathcal{O}' < \mathcal{O}$ if $\mathcal{O}' \subsetneq \overline{\mathcal{O}}$.

- Unique maximal nilpotent orbit $\mathcal{O}_{\text{reg}}$ of dimension equal to the number of roots. Called the regular or principal nilpotent orbit.

- Unique orbit $\mathcal{O}_{\text{subreg}}$ containing all non-regular orbits in its closure. It is of codimension 2 in $\overline{\mathcal{O}}_{\text{reg}}$. Called the subregular nilpotent orbit.

- $\mathcal{O}_{\text{min}}$ is the minimal non-zero nilpotent orbit. Its closure $\overline{\mathcal{O}}_{\text{min}}$ is normal with an isolated singularity at 0. It is the orbit of a highest root vector.

$0 < \mathcal{O}_{\text{min}} < \cdots$ could be very complicated $\cdots < \mathcal{O}_{\text{subreg}} < \mathcal{O}_{\text{reg}}$
Hasse diagram of $A_5$

```
(6)   30

(5, 1)  28

(4, 2)  26

(4, 1^2)  (3^2)  24

(3, 2, 1)  22

(3, 1^3)  (2^3)  18

(2^2, 1^2)  16

(2, 1^4)  10

(1^6)  0
```
If $P \subset G$ is a parabolic subgroup, we have the moment map $T^*(G/P) \to g \simeq g^*$, whose image is a nilpotent orbit closure (by Richardson), which gives

$$\pi : T^*(G/P) \to \bar{O}.$$ 

$\pi$ is only generically finite. If it is birational, we get a Springer resolution which is also a crepant resolution.

**Theorem (Fu)**

*For nilpotent orbit closures, crepant resolutions $=$ Springer resolution.*
In general, $\tilde{O}$ (or equivalently $\bar{O}$) does not admit any crepant resolution. Examples:

- $\bar{O}_{\text{min}} \subset \mathfrak{sp}_{2n}$ is isomorphic to $\mathbb{C}^{2n}/\pm 1$, which has no symplectic resolution if $n \geq 2$.
- In general, $\bar{O}_{\text{min}}$ admits a sympl. resol. iff $\mathfrak{g}$ is of type $A$. In this case, we get $T^*\mathbb{P}^n \to \bar{O}_{\text{min}} = \text{rank} \leq 1$ matrices in $\mathfrak{sl}_{n+1}$

**Conjecture**

A normal isolated symplectic singularity of dim $\geq 4$ which admits a crepant resolution is analytically isomorphic to $\bar{O}_{\text{min}} \subset \mathfrak{sl}_n$.

Known in dimension 4 by the work of Wierzba-Wisniewski.
Problem

How to characterize nilpotent orbit closures among symplectic varieties?
Problem

*How to characterize nilpotent orbit closures among symplectic varieties?*

**Special features of nilpotent orbits:**
affine variety in $g$, stable under the dilation action of $\mathbb{C}^*$ and the symplectic form satisfies $\lambda^*\omega = \lambda\omega$. 
A conical symplectic variety is an affine symplectic variety \( \mathcal{W} \subset \mathbb{C}^N \), stable under the dilation action of \( \mathbb{C}^* \) such that \( \lambda^* \omega = \lambda \omega \).

i) Nilpotent orbit closures are conical symplectic varieties.

ii) Some birational projections of nilpotent orbit closures are conical symplectic varieties: \( \mathcal{O} \subset \mathfrak{g} \) and \( L \subset \mathfrak{g} \) a linear subspace. \( \pi : \mathfrak{g} \to \mathfrak{g}/L \) the projection, then \( \pi(\bar{\mathcal{O}}) \) is conical if \( \mathcal{O} \to \pi(\mathcal{O}) \) is birational and \( \pi(\bar{\mathcal{O}}) \) is smooth in codim. 1.

Problem (Namikawa)

Are conical symplectic varieties either nilpotent orbits in semisimple Lie algebras or their linear birational projections?
Recall that the nilpotent cone $\mathcal{N}$ is the union of all nilpotent elements in $\mathfrak{g}$, which is the closure of the maximal orbit $O_{\text{reg}}$. 

Theorem (Namikawa 2013)
If a conical symplectic variety $W \subset \mathbb{C} \mathcal{N}$ is a complete intersection, then it is the nilpotent cone $\mathcal{N} \subset g$.

Proposition (Brion-Fu 2014)
If a conical symplectic variety $W \subset \mathbb{C} \mathcal{N}$ has only isolated singularities, then it is the minimal orbit closure $\overline{O}_{\text{min}} \subset g$.

These give characterizations of nilpotent cones and minimal nilpotent orbit closures, the two extremities in Hasse diagram.
Recall that the nilpotent cone $\mathcal{N}$ is the union of all nilpotent elements in $\mathfrak{g}$, which is the closure of the maximal orbit $\mathcal{O}_{\text{reg}}$.

**Theorem (Namikawa 2013)**

*If a conical symplectic variety $W \subset \mathbb{C}^N$ is a complete intersection, then it is the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$.*
Recall that the nilpotent cone $\mathcal{N}$ is the union of all nilpotent elements in $\mathfrak{g}$, which is the closure of the maximal orbit $\mathcal{O}_{\text{reg}}$.

**Theorem (Namikawa 2013)**

*If a conical symplectic variety $W \subset \mathbb{C}^N$ is a complete intersection, then it is the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$.***

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Symplectic resolutions for conical symplectic varieties

Theorem (Brion-Fu 2014)

If a conical symplectic variety has a symplectic resolution $\pi : Z \rightarrow W$, then $W$ is a birational linear quotient of a nilpotent orbit closure $\bar{O}$ and $\pi$ factorizes through a Springer resolution of $\bar{O}$.

- This generalizes my previous classification of symplectic resolutions for nilpotent orbit closures.
- The proof uses contact geometry.
A contact structure on a smooth variety $Y$ is a corank $1$ sub-bundle $D \subset T_Y$ such that $\left[ , \right] : D \times D \to T_Y / D =: L$ is non-degenerate.

Equivalently, there exist $L \in \text{Pic}(Y)$ and $\theta \in H^0(\Omega_Y \otimes L)$ such that $\theta \wedge (d\theta)^n$ everywhere non-vanishing. Hence get $K_Y \simeq L^{-(n+1)}$.

Remark

Let $p : L^\times = L^* \setminus \text{zero \, section} \to Y$ be the principal $\mathbb{C}^*$-bundle. Then a symplectic form $\omega$ on $L^\times$ such that $\lambda^* \omega = \lambda \omega$ if and only if $\omega = d(p^* \theta)$ for some contact structure $\theta$ on $Y$.

Examples of contact manifolds:
1) $\mathbb{P}(T^* M)$ with $L^\times = T^* M \setminus \text{zero \, section}$. 
2) $\mathbb{P}\mathcal{O}$ is a contact manifold with $L^\times = \mathcal{O}$. 
Theorem (Kebekus-Peternell-Sommese-Wisniewski, Demailly)

Let \((Y, L)\) be a projective contact manifold, then
\((Y, L) \simeq (\mathbb{P} T^* M, \mathcal{O}_{\mathbb{P} T^* M}(1))\) unless \(Y\) is Fano contact with \(b_2 = 1\).

Conjecture (LeBrun-Salamon)

A Fano contact manifold with \(b_2 = 1\) is isomorphic to \(\mathbb{P} \mathcal{O}_{\text{min}}\), for \(\mathcal{O}_{\text{min}}\) in a simple Lie algebra.
Sketch of the proof

Let $\pi : Z \to W$ be a symplectic resolution with $W$ being conical.

(i) [Namikawa] $\mathbb{C}^*$-action on $W$ lifts to $Z$ and

$$\tilde{\pi} : \mathbb{P}Z := (Z \setminus \pi^{-1}(0))/\mathbb{C}^* \to \mathbb{P}W$$

is a resolution and $Z$ is a contact manifold with contact bundle $L \cong \tilde{\pi}^*\mathcal{O}_{\mathbb{P}W}(1)$. 

(ii) if $\mathbb{P}W$ is smooth, then $W = \mathcal{O}_{\min}$. This is the characterization of $\mathcal{O}_{\min}$.

(iii) By (ii) and [KPSW+D], get

$$\left(\mathbb{P}Z, \bar{L}\right) \cong \left(\mathbb{P}T^*M, \mathcal{O}_{\mathbb{P}T^*M}(1)\right).$$

$L$ is pull-back of an ample Line bundle, hence globally generated and so is $TM$. Thus $M$ is homogeneous, i.e. $M \cong G/P \times A$.

Then get $\mathbb{P}Z \cong \mathbb{P}(G/P)$. 

(iv) Recover $Z$ from $\mathbb{P}Z$ and $L$ and $\pi$ from the Springer map of $T^*(G/P) \to \mathcal{O}_{\mathbb{P}W}$. 
Let \( \pi : Z \to W \) be a symplectic resolution with \( W \) being conical.

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(ii) if \( \mathbb{P}W \) is smooth, then \( W = \bar{O}_{\text{min}} \). This is the characterization of \( O_{\text{min}} \).

(iii) By (ii) and [KPSW+D], get \((\mathbb{P}Z, L) \simeq (\mathbb{P}T^*M, \mathcal{O}_{\mathbb{P}T^*M}(1))\). \( L \) is pull-back of an ample Line bundle, hence globally generated and so is \( TM \). Thus \( M \) is homogeneous, i.e. \( M \simeq G/P \times A \).

Then get \( \mathbb{P}Z \simeq \mathbb{P}T^*(G/P) \).
Sketch of the proof

Let $\pi : Z \to W$ be a symplectic resolution with $W$ being conical.

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(iv) Recover $Z$ from $\mathbb{P}Z$ and $L$ and $\pi$ from the Springer map of $T^*(G/P) \to \bar{O}_P$. 

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Recall our second problem:

Problem

Determine generic singularities of nilpotent orbit closures.
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**Problem**

_Determine generic singularities of nilpotent orbit closures._

**Question:** Why generic singularities?

Motivated by representation theory (modular Springer correspondence...)

This gives isolated symplectic singularities...
Let \((Y, L)\) be a projective contact manifold with \(b_2 = 1\), then \(L^*\) can be contracted along the zero section, giving \(f : L^* \to W\) with an isolated symplectic singularity. As \(L^\times\) has a symplectic form, hence \(f\) becomes an isolated symplectic singularity.
Let \((Y, L)\) be a projective contact manifold with \(b_2 = 1\), then \(L^*\) can be contracted along the zero section, giving \(f: L^* \to W\) with an isolated symplectic singularity. As \(L^\times\) has a symplectic form, hence \(f\) becomes an isolated symplectic singularity.

Motivated by LeBrun-Salamon conjecture,

### Speculation

*A normal isolated symplectic singularity is analytically locally isomorphic to a finite quotient of either \(\mathbb{C}^{2n}\) or of \(\overline{O}_{\text{min}}\).*

**Evidences:**

(i) [Beauville] OK if the projective tangent cone is smooth.

(ii) A conjecture of Kollár implies \(\pi_1(W_{\text{reg}})\) is finite.
Theorem (Jacobson-Morozov)

Let $e \in \mathcal{N}$ be a nilpotent element, then there exists an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{g}$, i.e. find $h$ and $f$ such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

We denote by $\mathfrak{g}^f$ the centralizer of $f$ in $\mathfrak{g}$. One can show that the affine subspace

$$S_e := e + \mathfrak{g}^f$$

is transverse to the orbit $O_e$ of $e$ in $\mathfrak{g}$.

The variety $S_e \cap \mathcal{N}$ carries all essential information of singularities of $\mathcal{N}$ to $O_e$. 

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Geometry

Nilp

Orbits
Definition

\( \mathcal{O}' \subset \mathcal{O} \) is a **minimal degeneration** if there is no other orbits lies between \( \mathcal{O} \) and \( \mathcal{O}' \).

Take an element \( e \in \mathcal{O}' \), the singularity of \( S_e \cap \mathcal{O} \) at \( e \) is called **generic singularity** of \( \mathcal{O} \) along \( \mathcal{O}' \), denoted by \( \text{Sing}(\mathcal{O}, \mathcal{O}') \).

Each irreducible component of \( \text{Sing}(\mathcal{O}, \mathcal{O}') \) is an isolated symplectic singularity of dimension \( = \text{codim}_{\mathcal{O}}(\mathcal{O}') \).

Problem

**Determine/recognize** generic singularities of nilpotent orbit closures.
The regular nilpotent orbit $O_{\text{reg}}$ is dense in $\mathcal{N}$, and there is a unique subregular nilpotent orbit $O_{\text{subreg}}$ open in $\mathcal{N} - O_{\text{reg}}$. It is of codimension 2.

**Theorem (Brieskorn, 1970)**

Suppose that $g$ is of $ADE$ type $\Gamma$. Then

$$\text{Sing}(\overline{O}_{\text{reg}}, O_{\text{subreg}}) = \Gamma$$

is a simple surface singularity of the same type, and Grothendieck’s simultaneous resolution restricted to the slice gives a versal deformation.
Non simply-laced cases

Slodowy explained what happens for non simply-laced types:

\[ B_n = A_n + 2^{n-1} : A_{2n-1} \text{ with } S_2 \text{-action} \]

\[ C_n = D_n + n + 1 : D_{n+1} \text{ with } S_2 \text{-action} \]

\[ F_4 = E_6 + 6 : E_6 \text{ with } S_2 \text{-action} \]

\[ G_2 = D_{++} + 4 : D_4 \text{ with } S_3 \text{-action} \]
Non simply-laced cases

Slodowy explained what happens for non simply-laced types:

- \( B_n = A^{+}_{2n-1} := A_{2n-1} \) with \( \mathfrak{g}_2 \)-action
- \( C_n = D^{+}_n := D_{n+1} \) with \( \mathfrak{g}_2 \)-action
- \( F_4 = E^{+}_6 := E_6 \) with \( \mathfrak{g}_2 \)-action
- \( G_2 = D^{++}_4 := D_4 \) with \( \mathfrak{g}_3 \)-action
Motivated by the normality problem for nilpotent orbit closures, Kraft and Procesi described minimal degenerations in nilpotent cones in classical type.

**Theorem (Kraft-Procesi, case of $\mathfrak{sl}_n$)**

If the first lines and the first columns of $\lambda$ and $\mu$ are identical, and if $\hat{\lambda}$ and $\hat{\mu}$ are the partitions obtained by removing those common lines and columns, then $\text{Sing}(O_\lambda, O_\mu) = \text{Sing}(O_{\hat{\lambda}}, O_{\hat{\mu}})$.

It follows that all minimal degenerations are either $A_k$ (codimension 2) or $a_k$ (codimension $>2$)!
Motivated by the normality problem for nilpotent orbit closures, Kraft and Procesi described minimal degenerations in nilpotent cones in classical type.

**Theorem (Kraft-Procesi, case of $\mathfrak{sl}_n$)**

If the $r$ first lines and the $s$ first columns of $\lambda$ et $\mu$ are identical, and if $\hat{\lambda}$ and $\hat{\mu}$ are the partitions obtained by removing those common lines and columns, then

$$\text{Sing}(\mathcal{O}_\lambda, \mathcal{O}_\mu) = \text{Sing}(\mathcal{O}_{\hat{\lambda}}, \mathcal{O}_{\hat{\mu}})$$
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$$\text{Sing}(\overline{O}_\lambda, O_\mu) = \text{Sing}(\overline{O}_{\hat{\lambda}}, \overline{O}_{\hat{\mu}})$$

It follows that all minimal degenerations are
- either $A_k$ (codimension 2)
- or $a_k$ (codimension $> 2$)!
Example: type $A_5$

\[
\begin{array}{ccc}
(6) & & 30 \\
| A_5 & & \\
(5, 1) & & 28 \\
| A_3 & & \\
(4, 2) & & 26 \\
\end{array}
\]

\[
\begin{array}{ccc}
(4, 1^2) & A_1 & (3^2) \\
& \searrow & \swarrow \\
A_2 & (3, 2, 1) & A_2 \\
& \searrow & \swarrow \\
(3, 1^3) & a_2 & (2^3) \\
& \searrow & \swarrow \\
& a_1 & (2^2, 1^2) \\
& \searrow & \swarrow \\
& & a_3 \\
& & (2, 1^4) \\
& & \searrow \\
& & a_5 \\
& & (1^6) \\
\end{array}
\]

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Theorem (Kraft-Procesi)

In other classical types, if $O' < O$ a minimal degeneration, then $\text{Sing}(\overline{O}, O')$ is equivalent to one of the following:

- $A_{2k-1}$
- $D_k$
- $A_{2k-1} \cup A_{2k-1}$
- $b_k$
- $c_k$
- $d_k$
Assume $g$ is of classical type and $\mathcal{O}' < \mathcal{O}$ a minimal degeneration.

- $\text{Sing}(\mathcal{O}, \mathcal{O}')$ is isom. to some $\bar{\mathcal{O}}_{\text{min}}$ if $\text{dim} \geq 4$.
- $\text{Sing}(\mathcal{O}, \mathcal{O}')$ is non-normal $\iff$ it is isom. to $2A_{2k-1}$.
- The non-normality of $\bar{\mathcal{O}}$ can be detected by its generic singularities of dim. 2.

**Upshot of this work**

Many funny phenomena appear when $g$ is of exceptional type.
The component group $A(O')$ acts on $\text{Sing}(O, O')$. We keep track of this action.

- $A_n, D_n, E_n$ (no action);
- $A_{2n-1}$ with $S_2$-action, denoted $B_n$;
- $D_{n+1}$ with $S_2$-action, denoted $C_n$;
- $E_6$ with $S_2$-action, denoted $F_4$;
- $D_4$ with $S_3$-action, denoted $G_2$;
- $A_2$ and $A_4$ with $S_2$-action, denoted $A_2^+$ and $A_4^+$;
- $\text{Sing}(\tilde{A}_1, A_1)$ in $G_2$ is denoted by $m$. 
the minimal orbit closure $a_n, b_n, c_n, d_n, g_2, e_n, f_4$ with trivial action of $A(O');$
the minimal orbit closure in $d_n$ with $S_2$-action, denoted $d_n^+$;
the minimal orbit closure in $d_4$, with $S_3$-action, denoted $d_4^{++};$
the minimal orbit closure in $a_2, a_3, a_4, or a_5$ with $S_2$-action, denoted $a_n^+.$
We determined all general singularities of nilpotent orbit closures in exceptional Lie algebras (up to normalization for a few surface cases in $E_7, E_8$).
Main theorem in more details

**Theorem (FJLS)**

Let $\mathfrak{g}$ be a simple exceptional Lie algebra and $\mathcal{O}' < \mathcal{O}$ a minimal degeneration. Let $S = \text{Sing}(\mathcal{O}, \mathcal{O}')$. Then

(a) If $\dim S = 2$, then $S$ is a union of surfaces of one of the following types

- $A_k(k = 1, 2, 5), A^+_k(k = 2, 4),$
- $B_3, C_k(k = 2, 3, 4, 5, 6), D_6,$
- $E_k(k = 6, 7, 8), F_4, G_2,$
- $m$

(b) If $\dim S \geq 4$, then it is equivalent to one of the following

- $a_k(k = 1, 2, 5), a^+_k(k = 2, 3, 4),$
- $b_k(k = 2, 3, 4, 5, 6), c_k(k = 3, 4), d_6, d_4^{++},$
- $e_k(k = 6, 7, 8), e_6^+,$
- one of four exceptional singularities of codimension 4, each occurring only once.
Main Theorem itself: $G_2$ (Kraft)

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All orbit closures are normal except for $\tilde{A}_1$, for which the normalization is a homeomorphism.
Main Theorem itself: $F_4$
Main Theorem itself: $E_6$

\[
\begin{align*}
E_6 & \quad 72 \\
E_6(a_1) & \quad 70 \\
D_5 & \quad 68 \\
E_6(a_3) & \quad 66 \\
A_5 & \quad 64 \\
A_4 + A_1 & \quad 62 \\
A_4 & \quad 60 \\
D_4(a_1) & \quad 58 \\
\end{align*}
\]
Main Theorem itself: $E_7, E_8$
The margin is too narrow to contain…
The singularity $m$

**Proposition**

The singularity $m$ can be described as the image of the morphism:

$$\mathbb{C}^2 \rightarrow \mathbb{C}^7 = \mathbb{C}^3 \oplus \mathbb{C}^4$$

$$(u, v) \mapsto (u^2, uv, v^2; u^3, u^2v, uv^2, v^3)$$
The exceptional codimension 4 singularities

\[ \text{Sing}(2A_2 + A_1, A_2 + 2A_1)_{E_6} = \mathbb{C}^4/\mu_3 \]

Here \( \mu_3 \) acts on \( \mathbb{C}^4 \) by \( \xi \cdot (z_1, z_2, z_3, z_4) = (\xi z_1, \xi z_2, \xi^{-1} z_3, \xi^{-1} z_4) \)
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\[ \text{Sing}(A_4 + A_1, A_3 + A_2 + A_1)_{E_7} = a_2 / S_2 \]
The exceptional codimension 4 singularities

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Here \( \mu_3 \) acts on \( \mathbb{C}^4 \) by \( \xi \cdot (z_1, z_2, z_3, z_4) = (\xi z_1, \xi z_2, \xi^{-1} z_3, \xi^{-1} z_4) \)

\[ \text{Sing}(A_4 + A_1, A_3 + A_2 + A_1)_{E_7} = a_2/S_2 \]

\[ \text{Sing}(A_3 + 2A_1, 2A_2 + 2A_1)_{E_8} = \text{Spec}(\mathbb{C} \oplus \mathbb{C}[s, t, u, v]_{\geq 2}) \]

It is a pinched \( \mathbb{C}^4 \), in the same way as \( m \) is a pinched \( \mathbb{C}^2 \).
The exceptional codimension 4 singularities

Sing(2A₂ + A₁, A₂ + 2A₁)_{E_6} = \mathbb{C}^4/\mu_3

Here \( \mu_3 \) acts on \( \mathbb{C}^4 \) by \( \xi \cdot (z_1, z_2, z_3, z_4) = (\xi z_1, \xi z_2, \xi^{-1} z_3, \xi^{-1} z_4) \)

Sing(A₄ + A₁, A₃ + A₂ + A₁)_{E_7} = a_2/S_2

Sing(A₃ + 2A₁, 2A₂ + 2A₁)_{E_8} = \text{Spec}(\mathbb{C} \oplus \mathbb{C}[s, t, u, v]_{\geq 2})

It is a pinched \( \mathbb{C}^4 \), in the same way as \( m \) is a pinched \( \mathbb{C}^2 \).

Sing(A₄ + A₃, A₄ + A₂ + A₁)_{E_8}

Consider the dihedral group \( \Gamma \) of order 10 acting on its reflection representation \( V \). Our singularity is the blow-up of \( (V \oplus V^*)/\Gamma \) at the singular locus.
Immediate Corollaries

**Corollary**

The isolated symplectic singularities coming from generic singularities of nilpotent orbit closures are finite quotient of either $\bar{O}_{\text{min}}$ or $\mathbb{C}^{2n}$, except possibly the case $\tau'$ in $E_8$.

**Conjecture (Andreatta-Wisniewski, Arxiv.1101.4884)**

There is no symplectic resolution $\pi : Z \to X$ with a codimension 2 locus of $A_{2n}$ singularities of $X$ and a non-trivial numerical equivalence in $Z$ of curves in a general fiber of $\pi$ over this locus.

**Corollary**

The conjecture of Andreatta-Wisniewski is false. We have an example of $A_2^+$ and $A_4^+$ in $E_7$ and $E_8$. 
Recall: Observation for classical types

Assume $g$ is of classical type and $\mathcal{O}' < \mathcal{O}$ a minimal degeneration.

- $\text{Sing}(\mathcal{O}, \mathcal{O}')$ is isom. to some $\bar{\mathcal{O}}_{min}$ if $\text{dim} \geq 4$.

- $\text{Sing}(\mathcal{O}, \mathcal{O}')$ is non-normal $\iff$ it is isom. to $2A_{2k-1}$.

- The non-normality of $\bar{\mathcal{O}}$ can be detected by its generic singularites of dim. 2.
Observation for exceptional types

$\mathfrak{g}$ is of exceptional type and $S = \text{Sing}(\mathcal{O}, \mathcal{O}')$ is a generic singularity of a nilpotent orbit closure.

- A completely new non-normal surface singularity $m$ appears, whose normalization is $\mathbb{C}^2$. It appears in each type several times.
\( \mathfrak{g} \) is of exceptional type and \( S = \text{Sing}(\mathcal{O}, \mathcal{O}') \) is a generic singularity of a nilpotent orbit closure.

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- Several other non-normal surfaces singularities like \( 4G_2 \) etc appear.
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- A non-normal 4-dimensional isolated singularity appears in \( E_8 \).
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- A quotient singularity $\mathbb{C}^4/\mathbb{Z}_3$ appears.
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- A quotient of \( \bar{O}_{\text{min}} \in \mathfrak{sl}_3 \) by \( \mathbb{Z}_2 \) appears in \( E_7 \).
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- A quotient singularity \( \mathbb{C}^4/\mathbb{Z}_3 \) appears.
- A quotient of \( \bar{\mathcal{O}}_{\text{min}} \in \mathfrak{sl}_3 \) by \( \mathbb{Z}_2 \) appears in \( E_7 \).
- An isolated symplectic singularity appears which looks like not to be a finite quotient of \( \bar{\mathcal{O}}_{\text{min}} \) or \( \mathbb{C}^{2n} \).
[Kraft-Procesi] + [Sommers]: in the classical groups, the non-normality of $\bar{O}$ is detected by its generic singularities.
[Kraft-Procesi]+[Sommers]: in the classical groups, the non-normality of $\tilde{O}$ is detected by its generic singularities.

In the exceptional groups, normality can fail in more ways:

- it is branched at a minimal degeneration (e.g. $3.a_1$ or $2.g_2$).
- it is branched at a point farther down (detected by Green functions).
- the singularity $m$ arises (non-normal and unibranched).
- the one case in $E_8$ (non-normal and unibranched).
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- it is branched at a minimal degeneration (e.g. $3.a_1$ or $2.g_2$).
- it is branched at a point farther down (detected by Green functions).
- the singularity $m$ arises (non-normal and unibranched).
- the one case in $E_8$ (non-normal and unibranched).

Actually, these are not all: the orbits $B_2, \tilde{A}_2$ in $F_4$ are non-normal while not of the above cases.
Three methods are applied to accomplish the proof.

- **Use the slice of reductive centralizers**
  This determines most of singularities with $\dim \geq 4$ or those of type $A_1$.

- **Use geometrical method via minimal resolutions**
  This determines all surface singularities (up to normalization)

- **Use computer-aided ad hoc methods**
  This deals with the remaining hard cases, which also removes ”up to normalization” in several cases.
(Namikawa, Fu) Every nilpotent orbit closure has an explicit minimal model given by generalized Springer maps or by normalizations.

The preimage over the slice gives the minimal resolution of the surface singularity.

We can use Springer correspondence and a formula of Borho-MacPherson to compute the number of $\mathbb{P}^1$'s in the minimal resolution and also the monodromy action.
Move to the Geometry of Special Pieces
**Special nilpotent orbits**

_Springer correspondence:_

\[ W = \text{Weyl Group}. \]

\[ \{\text{irreducible} \ W - \text{modules}\} \leftrightarrow \{ (\mathcal{O}, \phi) | \mathcal{O} \text{ nilp. orbit}, \phi \text{ irredu. representation of } A(\mathcal{O}) \}. \]

A nilp. orbit \( \mathcal{O} \) is called **special** if the irredu. rep. \( \rho(\mathcal{O}, 1) \) is a special \( W \)-representation.

Special nilpotent orbits play a key role in several problems in representation theory:

- Classification of irreducible complex rep. of a reductive gp over a finite field
- Classification of primitive ideals in the enveloping alg. of a semi-simple Lie algebra
Special orbits in $F_4$

\[
\begin{array}{c}
F_4 \\
\mid F_4 \\
F_4(a_1) \\
\mid A_3 \\
F_4(a_2) \\
\mid A_1 \\
B_3 \\
\mid G_2 \\
F_4(a_3) \\
\mid A_1 \\
C_3 \\
\mid 4.D_4 \\
C_3(a_1) \\
\mid 2.A_1 \\
A_1\tilde{A}_2 \\
\mid m \\
B_2 \\
\mid A_1 \\
A_2\tilde{A}_1 \\
\mid a_2 \\
\tilde{A}_2 \\
\mid s_2 \\
A_1\tilde{A}_1 \\
\mid a_1 \\
\tilde{A}_1 \\
\mid a_1 \\
A_1\tilde{A}_1 \\
\mid a_3^+ \\
\tilde{A}_1 \\
\mid c_3 \\
A_1 \\
\mid f_4 \\
0 \\
\end{array}
\]

\[\text{Baohua FU} \hspace{1cm} \text{GeometryNilpOrbits}\]
Definition

For $O$ a special orbit, the special piece of $O$ is

\[ \mathcal{P}(O) = \overline{O} - \bigcup_{O' < O \text{ special}} \overline{O'} \]

- The special pieces form a partition of the nilpotent cone, $N$ (Spaltenstein).
- Lusztig conjectured in 1981 that every special piece is rationally smooth (i.e. for any $x \in \mathcal{P}(O)$, we have $IH^i_x(\mathcal{P}(O), \mathbb{Q}) = \mathbb{Q}$ if $i = 0$ and zero otherwise). This has been proved by Kraft-Procesi (for classical types), Beynon-Spaltenstein (for $E_n$), Shoji (for $F_4$) and Lusztig (for $G_2$).
In 1997, Lusztig formulated the following conj. to explain the rationally smoothness:

**Conjecture (Lusztig)**

Every special piece $\mathcal{P}(O)$ is a finite quotient of a smooth variety $P/H$, and the orbits in $\mathcal{P}(O)$ correspond to the images of points in $P$ whose $H$-stabilizer are in the same conjugacy classes of $H$.

- Lusztig has predicted the group $H$ and the correspondence between conj. classes in $H$ and nilpotent orbits contained in $\mathcal{P}(O)$.

**Conjecture (Achar-Sage, 2009)**

Every special piece is normal.
The special piece of $F_4(a_3)$

The right-hand diagram is nothing but the Hasse diagram of strata in $W := (C_3 \oplus (C_3)*)/S_4$. If $S$ is a transverse slice to the special piece, then $S$ and $W$ are of the same dimension, both symplectic with the same Hasse diagram.
Observation: The right-hand diagram is nothing but the Hasse diagram of strata in \( W := (\mathbb{C}^3 \oplus (\mathbb{C}^3)*)/\mathcal{G}_4 \). If \( S \) is a transverse slice to the special piece, then \( S \) and \( W \) are of the same dimension, both symplectic with the same Hasse diagram...
A special piece in $E_8$

\[
\begin{align*}
E_8(a_7) & \quad a_1 \\
E_7(a_5) & \quad 2a_1 \\
E_6(a_3) + A_1 & \quad m \\
A_5 + A_1 & \quad m \\
A_4 + A_3 & \quad m \\
A_2 & \quad D_6(a_2) \\
& \quad D_5(a_1) + A_2 \\
& \quad (1^5) \\
& \quad (21^3) \\
& \quad (31^2) \\
& \quad (32) \\
& \quad (5) \\
& \quad (2^21) \\
& \quad (2^1) \\
& \quad (41) \\
& \quad (4)
\end{align*}
\]
Brave guess

Summing up the observations so far:

- The group $H$ is predicted to be $\mathfrak{S}_k$, $k = 2, 3, 4, 5$.
- The slice $S$ of a special piece looks like very similar to $W := (\mathbb{C}^{k-1} \oplus (\mathbb{C}^{k-1})^*)/\mathfrak{S}_k$.

**Could it be that $S$ is isomorphic to $W$?**
Summing up the observations so far:

- The group $H$ is predicted to be $\mathfrak{S}_k$, $k = 2, 3, 4, 5$.
- The slice $S$ of a special piece looks like very similar to

$$W := (\mathbb{C}^{k-1} \oplus (\mathbb{C}^{k-1})^*) / \mathfrak{S}_k.$$ 

Could it be that $S$ is isomorphic to $W$?

Not always! There are examples of special pieces consisting of two orbits, with generic singularities being $c_n$. Note that $c_n = \mathbb{C}^{2n} / \pm 1$, which can be written as $(\mathbb{C} \oplus \mathbb{C})^{\oplus n} / \mathfrak{S}_2$. 
Consider a special piece in a simple Lie algebra. A Slodowy transverse slice to the minimal orbit in the piece is isomorphic to

\[(\mathfrak{h}_n \oplus \mathfrak{h}_n^*)^k / \mathfrak{S}_{n+1}\]

where \(k\) and \(n\) are (uniquely determined) integers and \(\mathfrak{h}_n\) is the \(n\)-dimensional reflection representation of the symmetric group \(\mathfrak{S}_{n+1}\).
Immediate consequences

Going back to Lusztig’s conjecture, we obtain

**Corollary**

For every special piece $\mathcal{P}(O)$, there exists a vector space $V$ endowed with an action of $\mathfrak{S}_{n+1}$ (for some $n$ uniquely determined by $O$) such that we have a surjective smooth morphism

$$G \times V/\mathfrak{S}_{n+1} \to \mathcal{P}(O).$$

Going back to the conjecture of Achar-Sage, we have

**Corollary**

Every special piece is rationally smooth and normal.
For classical types, this can be proved based on previous work of Kraft-Procesi.

If the special piece consists of two orbits, then the result follows from our previous result on generic singularities of nilpotent orbit closures. (all of type $c_k$)

For the special piece $[G_2(a_1), \tilde{A}_1, A_1]$ in $G_2$, a slice to it is isomorphic to $(\mathfrak{h}_2 \oplus \mathfrak{h}_2^*)/S_3$. This solves several others in $E_6, E_7, E_8$.

The remaining cases are: $[D_4(a_1) + A_1, 2A_2 + 2A_1]_{E_8}$ is equivalent to $(\mathfrak{h}_2 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_2)/S_3$; $[F_4(a_3), A_2 + \tilde{A}_1]_{F_4}$ is equivalent to $(\mathfrak{h}_3 \oplus \mathfrak{h}_3)/S_4$ and $[E_8(a_7), A_4 + A_3]_{E_8}$ is equivalent to $(\mathfrak{h}_4 \oplus \mathfrak{h}_4)/S_5$. 
A geometrical characterization of nilpotent orbits with symplectic resolutions.

We have determined all general singularities of nilpotent orbits in exceptional types (up to normalization for a handful surface cases in $E_7, E_8$).

A transverse slice to any special piece is isomorphic to a natural finite quotient.

Duality between general singularities in nilpotent orbit closures.
Type $F_4$

$$
\begin{array}{c}
F_4 \\
| F_4 \\
F_4(a_1) \\
| C_3 \\
F_4(a_2) \\
| A_1 \\
A_1 \\
B_3 \\
| G_2 \\
F_4(a_3) \\
| A_1 \\
C_3(a_1) \\
| A_1 \\
A_1 \tilde{A}_2 \\
| m \\
A_2 \tilde{A}_1 \\
| g_2 \\
\tilde{A}_2 \\
\tilde{A}_2 \\
A_1 \tilde{A}_1 \\
| a_3^+ \\
\tilde{A}_1 \\
| c_3 \\
A_1 \\
| f_4 \\
0
\end{array}
$$

$$
\begin{array}{c}
F_4 \\
| F_4 \\
F_4(a_1) \\
| C_3 \\
F_4(a_2) \\
| A_1 \\
C_3 \\
4. G_2 \\
F_4(a_3) \\
| A_1 \\
A_1 \tilde{A}_2 \\
| m \\
B_2 \\
| 2. A_1 \\
A_2 \tilde{A}_1 \\
| a_2^+ \\
\tilde{A}_2 \\
A_2 \\
| a_2^+ \\
A_1 \tilde{A}_1 \\
| a_3^+ \\
\tilde{A}_1 \\
| f_4^{sp} \\
0
\end{array}
$$
Thanks!