Geometry of nilpotent orbits

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Question: Why nilpotent orbits?

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They are important in representation theory...
They form a very nice class of examples of **symplectic varieties** and they have many interesting geometrical properties...

What problems on the geometry?

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Today:

Problem

Geometrical characterizations of nilpotent orbit closures among symplectic varieties.

Problem

Determine generic singularities of nilpotent orbit closures.

Problem

Study the geometry of special pieces.

Symplectic varieties

Definition (Beauville)

(i) A **symplectic variety** is a variety W, smooth in codimension 1, with a symplectic form ω on W_{reg} such that for any resolution $\pi : Z \to W$, the pull-back $\pi^*\omega$ extends to a regular 2-form Ω on Z.

(ii) If Ω is everywhere non-degenerate, then $\pi : Z \to W$ is called a **symplectic resolution** of W.

Remark

- *For a symplectic variety W, a resolution $\pi : Z \to W$ is symplectic iff π is crepant, i.e. $K_Z = \mathcal{O}_Z$.*
- *The normalization of a symplectic variety is again a symplectic variety and it has only rational Gorenstein singularities.*

Memo: symplectic varieties

Theorem (Namikawa)

Normal symplectic variety = rational Gorenstein + symplectic form.

and

$\boxed{\text{Symplectic resolution} = \text{crepant resolution}}$

Finite symplectic quotient $\mathbb{C}^{2n}/H$

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(iii) $\mathrm{Sym}^n(\mathbb{C}^2/H)$ has $\boxed{\text{many}}$ symplectic resolutions obtained by Mukai flops from $\mathrm{Hilb}^n(\widehat{\mathbb{C}^2/H}) \to \mathrm{Sym}^n(\mathbb{C}^2/H)$, where $\widehat{\mathbb{C}^2/H} \to \mathbb{C}^2/H$ is the minimal resolution.
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(iv) Except two more examples in dim. 4, no other (irreducible) $\mathbb{C}^{2n}/H$ is expected to admit a symplectic resolution. For example, $\mathbb{C}^{2n}/\pm 1$ has no symplectic resolution if $n \geq 2$.

Another important classes of symplectic varieties are given by nilpotent orbit closures in semi-simple Lie algebras.

Nilpotent matrices

Definition

- *$A \in M_{n\times n}(\mathbb{C})$ is called* **nilpotent** *if $A^n = 0$.*
- *The* **nilpotent orbit** *of A is the set of all matrices conjugate to A.*
- *The* **nilpotent cone** *$\mathcal{N}$ is the set of all nilpotent matrices.*

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Every nilpotent matrice has its Jordan form $\mathrm{Diag}(J_{d_1}, \cdots, J_{d_k})$, where J_{d_i} is the Jordan matrix of size d_i with zeros on diagonal. Hence **nilpotent orbits in** $M_{n\times n}(\mathbb{C})$ **are parametrized by partitions of** n.

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Note that $\mathcal{N}$ is defined by polynomials in $M_{n\times n}(\mathbb{C}) \simeq \mathbb{C}^{n^2}$, hence it is an affine algebraic variety.

$sl_2(\mathbb{C})$

Let $A \in sl_2(\mathbb{C})$ be as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

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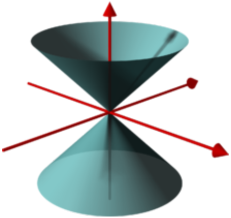

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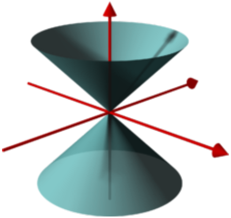


- The nilpotent cone is the union of nilpotent orbits
- Nilpotent orbits are not closed
- Nilpotent orbits are stable under dilations
- Nilpotent orbits are of even dimension

Nilpotent orbit closures

$\mathfrak{g}$ a complex simple Lie algebra and G the adjoint Lie group of $\mathfrak{g}$. $0 \neq x \in \mathfrak{g}$ a nilpotent element, $\mathcal{O}_x := G \cdot x$ the nilpotent orbit of x. This homogeneous manifold enjoys the following properties:

- $\mathcal{O}_x$ admits a holomorphic symplectic structure ω (K-K-S form)
- $\mathcal{O}_x$ is stable under the dilation action of $\mathbb{C}^*$ and we have $\lambda^* \omega = \lambda \omega$.
- The closure $\bar{\mathcal{O}}_x$ is singular and it is a union of finitely many nilpotent orbits.
- Nilpotent orbits are classified by weighted Dynkin diagrams (and by partitions in the case of classical types).

Theorem (Hinich, Payushev)

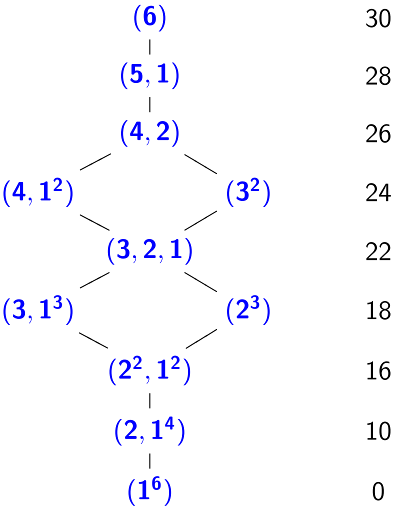
Nilpotent orbit closures in semisimple Lie algebras are symplectic varieties.

Hasse diagram

Partial order between orbits: denote by $\mathcal{O}' < \mathcal{O}$ if $\mathcal{O}' \subsetneq \overline{\mathcal{O}}$.

- Unique maximal nilpotent orbit $\mathcal{O}_{\text{reg}}$ of dimension equal to the number of roots. Called the regular or principal nilpotent orbit.
- Unique orbit $\mathcal{O}_{\text{subreg}}$ containing all non-regular orbits in its closure. It is of codimension 2 in $\overline{\mathcal{O}}_{reg}$. Called the subregular nilpotent orbit.
- $\mathcal{O}_{min}$ is the minimal non-zero nilpotent orbit. Its closure $\bar{\mathcal{O}}_{min}$ is normal with an isolated singularity at 0. It is the orbit of a highest root vector.

$$0 < \mathcal{O}_{min} < \cdots could\ be\ very\ complicated \cdots < \mathcal{O}_{subreg} < \mathcal{O}_{reg}$$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Symplectic resolutions of nilpotent orbit closures

If $P \subset G$ is a parabolic subgroup, we have the moment map $T^*(G/P) \to \mathfrak{g} \simeq \mathfrak{g}^*$, whose image is a nilpotent orbit closure (by Richardson), which gives

$$\pi : T^*(G/P) \to \bar{\mathcal{O}}.$$

π is only generically finite. If it is birational, we get a **Springer resolution** which is also a crepant resolution.

Theorem (Fu)

For nilpotent orbit closures, crepant resolutions = Springer resolution.

Orbits with no crepant resolutions

In general, $\tilde{\mathcal{O}}$ (or equivalently $\bar{\mathcal{O}}$) does not admit any crepant resolution. Examples:

- $\bar{\mathcal{O}}_{min} \subset \mathfrak{sp}_{2n}$ is isomorphic to $\mathbb{C}^{2n}/\pm 1$, which has no symplectic resolution if $n \geq 2$.
- In general, $\bar{\mathcal{O}}_{min}$ admits a sympl. resol. iff $\mathfrak{g}$ is of type A. In this case, we get $T^*\mathbb{P}^n \to \bar{\mathcal{O}}_{min}$ =rank ≤ 1 matrices in $\mathfrak{sl}_{n+1}$

Conjecture

A normal isolated symplectic singularity of dim ≥ 4 which admits a crepant resolution is analytically isomorphic to $\bar{\mathcal{O}}_{min} \subset \mathfrak{sl}_n$.

Known in dimension 4 by the work of Wierzba-Wisniewski.

Problem

How to characterize nilpotent orbit closures among symplectic varieties?

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Special features of nilpotent orbits:

affine variety in $\mathfrak{g}$, stable under the dilation action of $\mathbb{C}^*$ and the symplectic form satisfies $\lambda^*\omega = \lambda\omega$.

Conical symplectic varieties

Definition

A conical symplectic variety is an affine symplectic variety $W \subset \mathbb{C}^N$, stable under the dilation action of $\mathbb{C}^*$ such that $\lambda^* \omega = \lambda \omega$.

i) Nilpotent orbit closures are conical symplectic varieties.
ii) Some birational projections of nilpotent orbit closures are conical symplectic varieties: $\mathcal{O} \subset \mathfrak{g}$ and $L \subset \mathfrak{g}$ a linear subspace. $\pi : \mathfrak{g} \to \mathfrak{g}/L$ the projection, then $\pi(\bar{\mathcal{O}})$ is conical if $\mathcal{O} \to \pi(\mathcal{O})$ is birational and $\pi(\bar{\mathcal{O}})$ is smooth in codim. 1.

Problem (Namikawa)

Are conical symplectic varieties either nilpotent orbits in semisimple Lie algebras or their linear birational projections?

Characterization of maximal and minimal nilpotent orbits

Recall that the nilpotent cone $\mathcal{N}$ is the union of all nilpotent elements in $\mathfrak{g}$, which is the closure of the maximal orbit $\mathcal{O}_{reg}$.

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Proposition (Brion-Fu 2014)

If a conical symplectic variety $W \subset \mathbb{C}^N$ has only isolated singularities, then it is the minimal orbit closure $\bar{\mathcal{O}}_{min} \subset \mathfrak{g}$.

These gives characterisations of nilpotent cones and minimal nilpotent orbit closures, the two extremeties in Hasse diagram.

Symplectic resolutions for conical symplectic varieties

Theorem (Brion-Fu 2014)

If a conical symplectic variety has a symplectic resolution $\pi : Z \to W$, then W is a birational linear quotient of a nilpotent orbit closure $\bar{\mathcal{O}}$ and π factorizes through a Springer resolution of $\bar{\mathcal{O}}$.

- This generalizes my previous classification of symplectic resolutions for nilpotent orbit closures.
- The proof uses contact geometry.

Contact structure

Definition

A **contact structure** *on a smooth variety Y is a corank 1 sub-bundle $D \subset T_Y$ such that $[,] : D \times D \to T_Y/D =: L$ is non-degenerate.*

Equivalently, there exist $L \in \mathrm{Pic}(Y)$ and $\theta \in H^0(\Omega_Y \otimes L)$ such that $\theta \wedge (d\theta)^n$ everywhere non-vanishing. Hence get $K_Y \simeq L^{-(n+1)}$.

Remark

Let $p : L^\times = L^* \setminus zero - section \to Y$ be the principal $\mathbb{C}^*$-bundle. Then a symplectic form ω on $L^\times$ such that $\lambda^*\omega = \lambda\omega$ if and only if $\omega = d(p^*\theta)$ for some contact structre θ on Y.

Examples of contact manifolds:

1) $\mathbb{P}(T^*M)$ with $L^\times = T^*M \setminus zero - section$.

2) $\mathbb{P}\mathcal{O}$ is a contact manifold with $L^\times = \mathcal{O}$.

Projective contact manifolds

Theorem (Kebekus-Peternell-Sommese-Wisniewski, Demailly)

Let (Y, L) be a projective contact manifold, then $(Y, L) \simeq (\mathbb{P}T^*M, \mathcal{O}_{\mathbb{P}T^*M}(1))$ unless Y is Fano contact with $b_2 = 1$.

Conjecture (LeBrun-Salamon)

A Fano contact manifold with $b_2 = 1$ is isomorphic to $\mathbb{P}\mathcal{O}_{min}$, for $\mathcal{O}_{min}$ in a simple Lie algebra.

Sketch of the proof

Let $\pi : Z \to W$ be a symplectic resolution with W being conical.

(i) [Namikawa] $\mathbb{C}^*$-action on W lifts to Z and

$$\bar{\pi} : \mathbb{P}Z := (Z \setminus \pi^{-1}(0))//\mathbb{C}^* \to \mathbb{P}W$$

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(iii) By (ii) and [KPSW+D], get $(\mathbb{P}Z, L) \simeq (\mathbb{P}T^*M, \mathcal{O}_{\mathbb{P}T^*M}(1))$. L is pull-back of an ample Line bundle, hence globally generated and so is TM. Thus M is homogeneous, i.e. $M \simeq G/P \times A$. Then get $\mathbb{P}Z \simeq \mathbb{P}T^*(G/P)$.

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(iv) Recover Z from $\mathbb{P}Z$ and L and π from the Springer map of $T^*(G/P) \to \bar{\mathcal{O}}_P$.

Recall our second problem:

Problem

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Question: Why generic singularities?

Motivated by representation theory (modular Springer correspondence...)

This gives isolated symplectic singularities...

Isolated symplectic singularities

Let (Y, L) be a projective contact manifold with $b_2 = 1$, then L^* can be contracted along the zero section, giving $f : L^* \to W$ with an isolated symplectic singularity. As $L^\times$ has a symplectic form, hence f becomes an isolated symplectic singularity.

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Motivated by LeBrun-Salamon conjecture,

Speculation

A normal isolated symplectic singularity is analytically locally isomorphic to a finite quotient of either $\mathbb{C}^{2n}$ or of $\overline{\mathcal{O}}_{min}$.

Evidences:

(i) [Beauville] OK if the projective tangent cone is smooth.

(ii) A conjecture of Kollár implies $\pi_1(W_{reg})$ is finite.

Slodowy slice

Theorem (Jacobson-Morozov)

Let $e \in \mathcal{N}$ be a nilpotent element, then there exists an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, i.e. find h and f such that

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f$$

We denote by $\mathfrak{g}^f$ the centralizer of f in $\mathfrak{g}$. One can show that the affine subspace

$$S_e := e + \mathfrak{g}^f$$

is transverse to the orbit $\mathcal{O}_e$ of e in $\mathfrak{g}$.

The variety $S_e \cap \mathcal{N}$ carries *all essential* information of singularities of $\mathcal{N}$ to $\mathcal{O}_e$.

Generic singularities

Definition

$\mathcal{O}' \subset \bar{\mathcal{O}}$ is a **minimal degeneration** *if there is no other orbits lies between $\mathcal{O}$ and $\mathcal{O}'$.*

Take an element $e \in \mathcal{O}'$, the singularity of $S_e \cap \bar{\mathcal{O}}$ at e is called **generic singularity** of $\bar{\mathcal{O}}$ along $\mathcal{O}'$, denoted by $\mathrm{Sing}(\mathcal{O}, \mathcal{O}')$. Each irreducible componente of $\mathrm{Sing}(\mathcal{O}, \mathcal{O}')$ is an isolated symplectic singularity of dimension $= codim_{\bar{\mathcal{O}}}(\mathcal{O}')$.

Problem

Determine/recognize *generic singularities of nilpotent orbit closures.*

Theorems of Brieskorn

The regular nilpotent orbit $\mathcal{O}_{\text{reg}}$ is dense in $\mathcal{N}$, and there is a a unique subregular nilpotent orbit $\mathcal{O}_{\text{subreg}}$ open in $\mathcal{N} - \mathcal{O}_{\text{reg}}$. It is of codimension 2.

Theorem (Brieskorn, 1970)

Suppose that $\mathfrak{g}$ is of *ADE* type Γ. Then

$$\text{Sing}(\overline{\mathcal{O}}_{\text{reg}}, \mathcal{O}_{\text{subreg}}) = \Gamma$$

is a simple surface singularity of the same type, and Grothendieck's simultaneous resolution restricted to the slice gives a versal deformation.

Non simply-laced cases

Non simply-laced cases

Slodowy explained what happens for non simply-laced types:

$$\begin{array}{lcl} B_n = A_{2n-1}^+ & := & A_{2n-1} \text{ with } \mathfrak{S}_2\text{-action} \\ C_n = D_{n+1}^+ & := & D_{n+1} \text{ with } \mathfrak{S}_2\text{-action} \\ F_4 = E_6^+ & := & E_6 \text{ with } \mathfrak{S}_2\text{-action} \\ G_2 = D_4^{++} & := & D_4 \text{ with } \mathfrak{S}_3\text{-action} \end{array}$$

Results of Kraft and Procesi for $\mathfrak{sl}_n$

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Theorem (Kraft-Procesi, case of $\mathfrak{sl}_n$)

If the r first lines and the s first columns of λ et μ are identical, and if $\hat{\lambda}$ and $\hat{\mu}$ are the partitions obtained by removing those common lines and columns, then

$$\mathrm{Sing}(\overline{\mathcal{O}}_\lambda, \mathcal{O}_\mu) = \mathrm{Sing}(\overline{\mathcal{O}}_{\hat{\lambda}}, \mathcal{O}_{\hat{\mu}})$$

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It follows that all minimal degenerations are

- either A_k (codimension 2)
- or a_k (codimension > 2)!

Example: type A_5

$$
\begin{tikzcd}
 & (6) \arrow[d, dash, "A_5"] & & 30 \\
 & (5,1) \arrow[d, dash, "A_3"] & & 28 \\
 & (4,2) \arrow[dl, dash, "A_1"'] \arrow[dr, dash, "A_1"] & & 26 \\
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 & (2,1^4) \arrow[d, dash, "a_5"] & & 10 \\
 & (1^6) & & 0
\end{tikzcd}
$$

Other classical groups

Theorem (Kraft-Procesi)

In other classical types, if $\mathcal{O}' < \mathcal{O}$ a minimal degeneration, then $\mathrm{Sing}(\overline{\mathcal{O}}, \mathcal{O}')$ is equivalent to one of the following:

$$A_{2k-1}$$
$$D_k$$
$$A_{2k-1} \cup A_{2k-1}$$

$$b_k$$
$$c_k$$
$$d_k$$

Observation for classical types

Assume $\mathfrak{g}$ is of classical type and $\mathcal{O}' < \mathcal{O}$ a minimal degeneration.

- Sing$(\mathcal{O}, \mathcal{O}')$ is isom. to some $\bar{\mathcal{O}}_{min}$ if dim ≥ 4.
- Sing$(\mathcal{O}, \mathcal{O}')$ is non-normal $\Leftrightarrow$ it is isom. to $2A_{2k-1}$.
- The non-normality of $\bar{\mathcal{O}}$ can be detected by its generic singularites of dim. 2.

Upshot of this work

Many funny phenomena appear when $\mathfrak{g}$ is of exceptional type.

Notations for singularities: dim=2

The component group $A(\mathcal{O}')$ acts on $Sing(\mathcal{O}, \mathcal{O}')$. We keep track of this action.

- A_n, D_n, E_n (no action);
- A_{2n-1} with $\mathfrak{S}_2$-action, denoted B_n;
- D_{n+1} with $\mathfrak{S}_2$-action, denoted C_n;
- E_6 with $\mathfrak{S}_2$-action, denoted F_4;
- D_4 with $\mathfrak{S}_3$-action, denoted G_2;
- A_2 and A_4 with $\mathfrak{S}_2$-action, denoted A_2^+ and A_4^+;
- $\mathrm{Sing}(\tilde{A}_1, A_1)$ in G_2 is denoted by m.

Notations for singularities: dim ≥ 4

- the minimal orbit closure a_n, b_n, c_n, d_n, g_2, e_n, f_4 with trivial action of $A(\mathcal{O}')$;
- the minimal orbit closure in d_n with $\mathfrak{S}_2$-action, denoted d_n^+;
- the minimal orbit closure in d_4, with $\mathfrak{S}_3$-action, denoted d_4^{++};
- the minimal orbit closure in a_2, a_3, a_4, or a_5 with $\mathfrak{S}_2$-action, denoted a_n^+.

Main theorem in short

Theorem (FJLS)

We determined all general singularities of nilpotent orbit closures in exceptional Lie algebras (up to normalization for a few surface cases in E_7, E_8).

Main theorem in more details

Theorem (FJLS)

Let $\mathfrak{g}$ be a simple exceptional Lie algebra and $\mathcal{O}' < \mathcal{O}$ a minimal degeneration. Let $S = \mathrm{Sing}(\mathcal{O}, \mathcal{O}')$. Then

(a) *If* $\dim S = 2$, *then S is a union of surfaces of one of the following types*

- $A_k (k = 1, 2, 5), A_k^+ (k = 2, 4)$,
- $B_3, C_k (k = 2, 3, 4, 5, 6), D_6$,
- $E_k (k = 6, 7, 8), F_4, G_2$,
- m

(b) *If* $\dim S \geq 4$, *then it is equivalent to one of the following*

- $a_k (k = 1, 2, 5), a_k^+ (k = 2, 3, 4)$,
- $b_k (k = 2, 3, 4, 5, 6), c_k (k = 3, 4), d_6, d_4^{++}$,
- $e_k (k = 6, 7, 8), e_6^+$,
- *one of four exceptional singularities of codimension 4, each occuring only once.*

Main Theorem itself: G_2 (Kraft)

$$
\begin{array}{cc}
\mathbf{G_2} & 12 \\
\Big| \, G_2 & \\
\mathbf{G_2(a_1)} & 10 \\
\Big| \, a_1 & \\
\mathbf{\tilde{A}_1} & 8 \\
\Big| \, m & \\
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\mathbf{0} & 0
\end{array}
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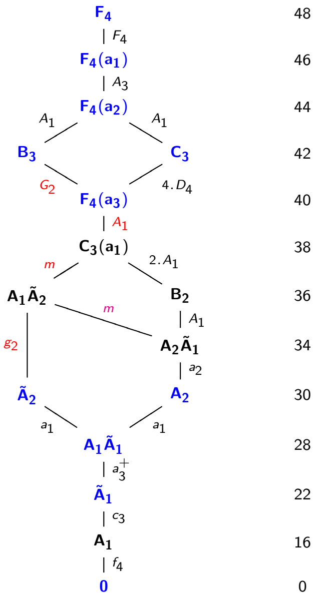
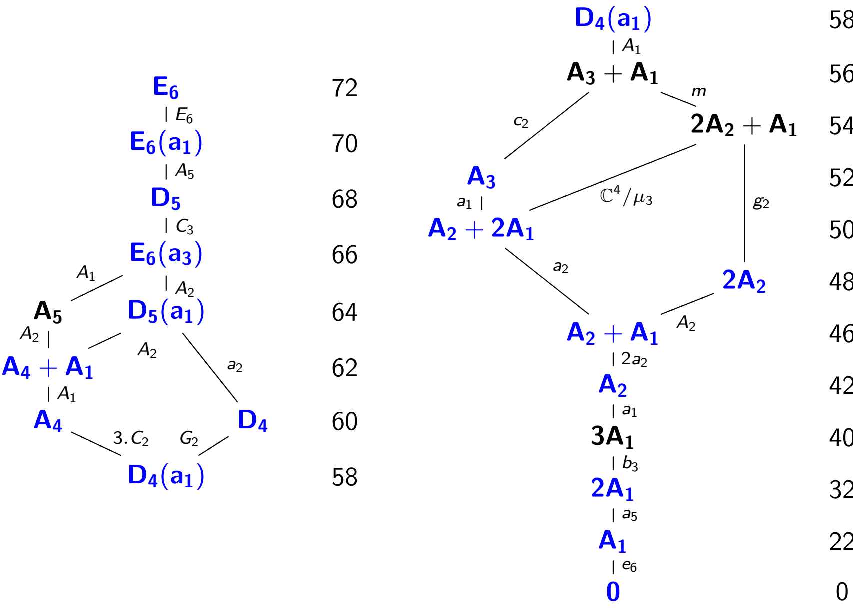
\end{array}
$$

All orbit closures are normal except for $\tilde{A}_1$, for which the normalization is a homeomorphism.

Main Theorem itself: F_4



Main Theorem itself: E_6



Main Theorem itself: E_7, E_8

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The margin is too narrow to contain$\cdots$

The singularity m

Proposition

The singularity m can be described as the image of the morphism:

$$\begin{array}{ccc} \mathbb{C}^2 & \to & \mathbb{C}^7 = \mathbb{C}^3 \oplus \mathbb{C}^4 \\ (u, v) & \mapsto & (u^2, uv, v^2; \quad u^3, u^2 v, uv^2, v^3) \end{array}$$

The exceptional codimension 4 singularities

$\mathrm{Sing}(2A_2 + A_1, A_2 + 2A_1)_{E_6} = \mathbb{C}^4/\mu_3$

Here μ_3 acts on $\mathbb{C}^4$ by $\xi \cdot (z_1, z_2, z_3, z_4) = (\xi z_1, \xi z_2, \xi^{-1} z_3, \xi^{-1} z_4)$

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It is a pinched $\mathbb{C}^4$, in the same way as m is a pinched $\mathbb{C}^2$.

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$\mathrm{Sing}(A_4 + A_3, A_4 + A_2 + A_1)_{E_8}$

Consider the dihedral group Γ of order 10 acting on its reflection representation V. Our singularity is the blow-up of $(V \oplus V^*)/\Gamma$ at the singular locus.

Immediate Corollaries

Corollary

The isolated symplectic singularities coming from generic singularities of nilpotent orbit closures are finite quotient of either $\bar{\mathcal{O}}_{min}$ or $\mathbb{C}^{2n}$, except **possibly** *the case τ' in E_8.*

Conjecture (Andreatta-Wisniewski, Arxiv.1101.4884)

There is no symplectic resolution $\pi : Z \to X$ with a codimension 2 locus of A_{2n} singularities of X and a non-trivial numerical equivalence in Z of curves in a general fiber of π over this locus.

Corollary

The conjecture of Andreatta-Wisniewski is false. We have an example of A_2^+ and A_4^+ in E_7 and E_8.

Recall: Observation for classical types

Assume $\mathfrak{g}$ is of classical type and $\mathcal{O}' < \mathcal{O}$ a minimal degeneration.

- $\mathrm{Sing}(\mathcal{O}, \mathcal{O}')$ is isom. to some $\bar{\mathcal{O}}_{min}$ if dim ≥ 4.
- $\mathrm{Sing}(\mathcal{O}, \mathcal{O}')$ is non-normal $\Leftrightarrow$ it is isom. to $2A_{2k-1}$.
- The non-normality of $\bar{\mathcal{O}}$ can be detected by its generic singularites of dim. 2.

Observation for exceptional types

$\mathfrak{g}$ is of exceptional type and $S = \mathrm{Sing}(\mathcal{O}, \mathcal{O}')$ is a generic singularity of a nilpotent orbit closure.

- A completely new non-normal surface singularity m appears, whose normalization is $\mathbb{C}^2$. It appears in each type several times.

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- A quotient singularity $\mathbb{C}^4/\mathbb{Z}_3$ appears.
- A quotient of $\bar{\mathcal{O}}_{min} \in \mathfrak{sl}_3$ by $\mathbb{Z}_2$ appears in E_7.
- An isolated symplectic singularity appears which **looks like** not to be a finite quotient of $\bar{\mathcal{O}}_{min}$ or $\mathbb{C}^{2n}$.

Normality of orbit closures

[Kraft-Procesi]+[Sommers]: in the classical groups, the non-normality of $\bar{\mathcal{O}}$ is detected by its generic singularities

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- it is branched at a minimal degeneration (e.g. $3.a_1$ or $2.g_2$).
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Actually, these are not all: the orbits $B_2, \tilde{A}_2$ in F_4 are non-normal while not of the above cases.

Three methods to determine generic singularities

Three methods are applied to accomplish the proof.

- Use the slice of reductive centralizers
  This determines most of singularities with dim ≥ 4 or those of type A_1.
- Use geometrical method via minimal resolutions
  This determines all surface singularities (up to normalization)
- Use computer-aided ad hoc methods
  This deals with the remaining hard cases, which also removes "up to normalization" in several cases.

Geometric method in more details

- (Namikawa, Fu) Every nilpotent orbit closure has an explicit minimal model given by generalized Springer maps or by normalizations.
- The preimage over the slice gives the minimal resolution of the surface singularity.
- We can use Springer correspondence and a formula of Borho-MacPherson to compute the number of $\mathbb{P}^1$'s in the minimal resolution and also the monodromy action.

Move to the Geometry of Special Pieces

Special nilpotent orbits

Springer correspondence:
$W=$ Weyl Group.

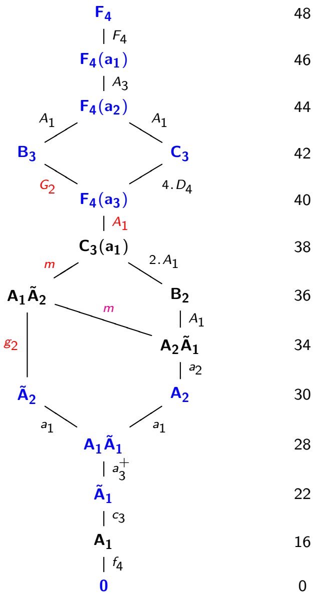
$$\{irreducible \quad W-modules\} \hookrightarrow$$

$$\{(\mathcal{O},\phi)|\mathcal{O} \quad nilp. \quad orbit, \phi \quad irredu. \quad representation \quad of \quad A(\mathcal{O})\}.$$

A nilp. orbit $\mathcal{O}$ is called **special** if the irred. rep. $\rho_{(\mathcal{O},1)}$ is a special W-representation.
Special nilpotent orbits play a key role in several problems in representation theory:

- Classification of irred. complex rep. of a reductive gp over a finite field
- classification of primitive ideals in the enveloping alg. of a semi-simple Lie algebra

Special orbits in F_4



Special pieces

Definition

For $\mathcal{O}$ a special orbit, the special piece of $\mathcal{O}$ is

$$\mathcal{P}(\mathcal{O}) = \overline{\mathcal{O}} - \bigcup_{\mathcal{O}' < \mathcal{O} special} \overline{\mathcal{O}'}$$

- The special pieces form a partition of the nilpotent cone. $\mathcal{N}$ (Spaltenstein).
- Lusztig conjectured in 1981 that every special piece is rationally smooth (i.e. for any $x \in \mathcal{P}(\mathcal{O})$, we have $IH^i_x(\mathcal{P}(\mathcal{O}), \mathbb{Q}) = \mathbb{Q}$ if $i = 0$ and zero otherwise). This has been proved by Kraft-Procesi (for classical types), Beynon-Spaltenstein (for E_n), Shoji (for F_4) and Lusztig (for G_2).

A conjecture of Lusztig

In 1997, Lusztig formulated the following conj. to explain the rationally smoothness:

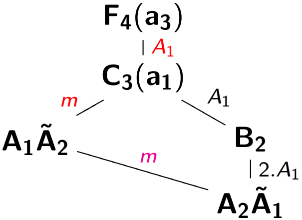
Conjecture (Lusztig)

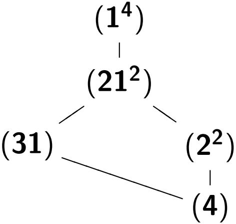
Every special piece $\mathcal{P}(\mathcal{O})$ is a finite quotient of a smooth variety P/H, and the orbits in $\mathcal{P}(\mathcal{O})$ correspond to the images of points in P whose H-stabilizer are in the same conjugacy classes of H.

- Known for classical Lie algebras by Kraft-Procesi (1989).
- Lusztig has predicted the group H and the correspondence between conj. classes in H and nilpotent orbits contained in $\mathcal{P}(\mathcal{O})$.

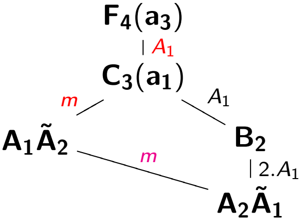
Conjecture (Achar-Sage, 2009)

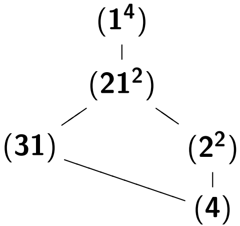
Every special piece is normal.


$\mathbf{F_4(a_3)}$
A_1
$\mathbf{C_3(a_1)}$
m
A_1
$\mathbf{A_1\tilde{A}_2}$
$\mathbf{B_2}$
m
$2.A_1$
$\mathbf{A_2\tilde{A}_1}$


(1^4)
(21^2)
(31)
(2^2)
(4)

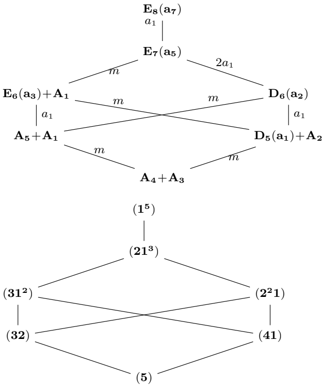
The special piece of $F_4(a_3)$





Observation: The right-hand diagram is nothing but the Hasse diagram of strata in $W := (\mathbb{C}^3 \oplus (\mathbb{C}^3)^*)/\mathfrak{S}_4$. If S is a transverse slice to the special piece, then S and W are of the same dimension, both symplectic with the same Hasse diagram...

A special piece in E_8



Brave guess

Summing up the observations so far:

- The group H is predicted to be $\mathfrak{S}_k, k = 2, 3, 4, 5$.
- The slice S of a special piece looks like very similar to $W := (\mathbb{C}^{k-1} \oplus (\mathbb{C}^{k-1})^*)/\mathfrak{S}_k$.

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Could it be that S is isomorphic to W?
Not always! There are examples of special pieces consisting of two orbits, with generic singularities being c_n. Note that $c_n = \mathbb{C}^{2n}/\pm 1$, which can be written as $(\mathbb{C} \oplus \mathbb{C})^{\oplus n}/\mathfrak{S}_2$.

Main theorem on special pieces

Theorem (FJLS)

Consider a special piece in a simple Lie algebra. A Slodowy transverse slice to the minimal orbit in the piece is isomorphic to

$$(\mathfrak{h}_n \oplus \mathfrak{h}_n^*)^k / \mathfrak{S}_{n+1}$$

where k and n are (uniquely determined) integers and $\mathfrak{h}_n$ is the n-dimensional reflection representation of the symmetric group $\mathfrak{S}_{n+1}$.

Immediate consequences

Going back to Lusztig's conjecture, we obtain

Corollary

For every special piece $\mathcal{P}(\mathcal{O})$, there exists a vector space V endowed with an action of $\mathfrak{S}_{n+1}$ (for some n uniquely determined by $\mathcal{O}$) such that we have a surjective smooth morphism

$$G \times V/\mathfrak{S}_{n+1} \to \mathcal{P}(\mathcal{O}).$$

Going back to the conjecture of Achar-Sage, we have

Corollary

Every special piece is rationally smooth and normal.

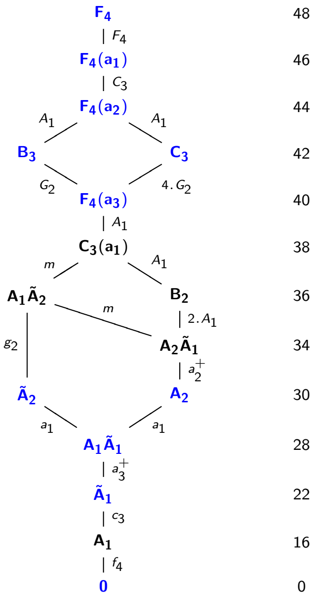
Words about the proof

- For classical types, this can be proved based on previous work of Kraft-Procesi.
- If the special piece consists of two orbits, then the result follows from our previous result on generic singularities of nilpotent orbit closures. (all of type c_k)
- For the special piece $[G_2(a_1), \tilde{A}_1, A_1]$ in G_2, a slice to it is isomorphic to $(\mathfrak{h}_2 \oplus \mathfrak{h}_2^*)/\mathfrak{S}_3$. This solves several others in E_6, E_7, E_8.
- The remaining cases are : $[D_4(a_1) + A_1, 2A_2 + 2A_1]_{E_8}$ is equivalent to $(\mathfrak{h}_2 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_2)/S_3$; $[F_4(a_3), A_2 + \tilde{A}_1]_{F_4}$ is equivalent to $(\mathfrak{h}_3 \oplus \mathfrak{h}_3)/S_4$ and $[E_8(a_7), A_4 + A_3]_{E_8}$ is equivalent to $(\mathfrak{h}_4 \oplus \mathfrak{h}_4)/S_5$.

Summarise

- A geometrical characterization of nilpotent orbits with symplectic resolutions.
- We have determined all general singularities of nilpotent orbits in exceptional types (up to normalization for a handful surface cases in E_7, E_8).
- A transverse slice to any special piece is isomorphic to a natural finite quotient.
- Duality between general singularities in nilpotent orbit closures.

Type F_4


F4
F4
F4(a1)
C3
F4(a2)
A1
A1
B3
C3
G2
4.G2
F4(a3)
A1
C3(a1)
m
A1
A1Ã2
B2
m
2.A1
A2Ã1
g2
Ã2
A2
a1
a1
A1Ã1
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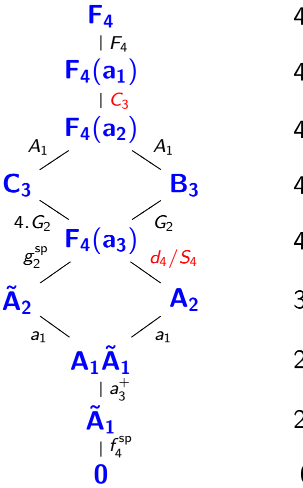
F4
F4
F4(a1)
C3
F4(a2)
A1
A1
C3
B3
4.G2
G2
F4(a3)
d4/S4
Ã2
A2
a1
a1
A1Ã1
Ã1
0


Thanks!