Introduction to the standard model

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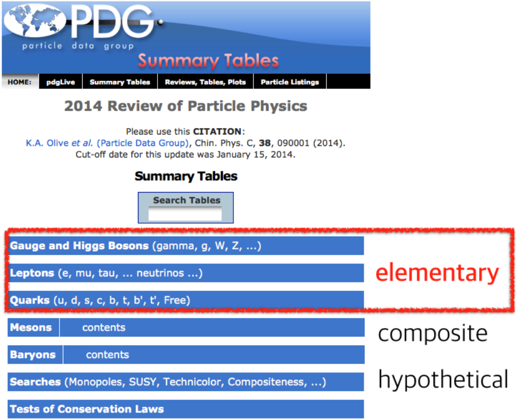
Overview

1. Lecture 1
   - SM in PDG
   - Why QFT?
   - Spacetime symmetry
   - Lagrangian
2. Lecture 2
   - Gauge symmetry(Abelian): QED
   - Gauge symmetry(non-Abelian, Yang-Mills:Weak, QCD)
   - Higgs mechanism
   - The standard model

Lecture #1

PDG: where you can find data



QUARKS

The u-, d-, and s-quark masses are estimates of so-called "current-quark masses," in a mass-independent subtraction scheme such as $\overline{\text{MS}}$ at a scale $\mu \approx 2$ GeV. The c- and b-quark masses are the "running" masses in the $\overline{\text{MS}}$ scheme. For the b-quark we also quote the 1S mass. These can be different from the heavy quark masses obtained in potential models.

u

$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

$m_u = 2.3^{+0.7}_{-0.5}$ MeV   Charge $= \frac{2}{3}\ e$ $I_z = +\frac{1}{2}$
$m_u/m_d = 0.38$–0.58

d

$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

$m_d = 4.8^{+0.5}_{-0.3}$ MeV   Charge $= -\frac{1}{3}\ e$ $I_z = -\frac{1}{2}$
$m_s/m_d = 17$–22
$\overline{m} = (m_u+m_d)/2 = 3.5^{+0.7}_{-0.2}$ MeV

s

$I(J^P) = 0(\frac{1}{2}^+)$

$m_s = 95 \pm 5$ MeV   Charge $= -\frac{1}{3}\ e$ Strangeness $= -1$
$m_s\ /\ ((m_u + m_d)/2) = 27.5 \pm 1.0$

c

$I(J^P) = 0(\frac{1}{2}^+)$

$m_c = 1.275 \pm 0.025$ GeV   Charge $= \frac{2}{3}\ e$ Charm $= +1$

b

$I(J^P) = 0(\frac{1}{2}^+)$

Charge $= -\frac{1}{3}\ e$ Bottom $= -1$

$m_b(\overline{\text{MS}}) = 4.18 \pm 0.03$ GeV
$m_b(1S) = 4.66 \pm 0.03$ GeV

t

$I(J^P) = 0(\frac{1}{2}^+)$

Charge $= \frac{2}{3}\ e$ Top $= +1$

Mass (direct measurements) $m = 173.21 \pm 0.51 \pm 0.71$ GeV [a,b]
Mass ($\overline{\text{MS}}$ from cross-section measurements) $m = 160^{+5}_{-4}$ GeV [a]
Mass (Pole from cross-section measurements) $m = 176.7^{+4.0}_{-3.4}$ GeV
$m_t - m_{\overline{t}} = -0.2 \pm 0.5$ GeV   (S = 1.1)
Full width $\Gamma = 2.0 \pm 0.5$ GeV
$\Gamma(W b)/\Gamma(W q(q = b, s, d)) = 0.91 \pm 0.04$

t-quark EW Couplings

$F_0 = 0.690 \pm 0.030$
$F_- = 0.314 \pm 0.025$
$F_+ = 0.008 \pm 0.016$
$F_{V+A} < 0.29$, CL = 95%

t DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$W q(q = b, s, d)$			–
$W b$			–
$\ell \nu_\ell$ anything	[c,d] (9.4±2.4) %		–
$\gamma q(q=u,c)$	[e] $< 5.9 \times 10^{-3}$	95%	–
$\Delta T = 1$ weak neutral current ($T1$) modes			
$Z q(q=u,c)$ $T1$	[f] $< 2.1 \times 10^{-3}$	95%	–

b' (4th Generation) Quark, Searches for

Mass $m > 190$ GeV, CL = 95%   ($p\overline{p}$, quasi-stable b')
Mass $m > 400$ GeV, CL = 95%   (pp, neutral-current decays)
Mass $m > 675$ GeV, CL = 95%   (pp, charged-current decays)
Mass $m > 46.0$ GeV, CL = 95%   (e^+e^-, all decays)

t' (4th Generation) Quark, Searches for

Mass $m > 782$ GeV, CL = 95%   (pp, neutral-current decays)
Mass $m > 700$ GeV, CL = 95%   (pp, charged-current decays)

Free Quark Searches

All searches since 1977 have had negative results.

LEPTONS

e $J = \frac{1}{2}$

Mass $m = (548.57990946 \pm 0.00000022) \times 10^{-6}$ u
Mass $m = 0.510998928 \pm 0.000000011$ MeV
$|m_{e^+} - m_{e^-}|/m < 8 \times 10^{-9}$, CL = 90%
$|q_{e^+} + q_{e^-}|/e < 4 \times 10^{-8}$
Magnetic moment anomaly
$(g-2)/2 = (1159.65218076 \pm 0.00000027) \times 10^{-6}$
$(g_{e^+} - g_{e^-}) / g_{\text{average}} = (-0.5 \pm 2.1) \times 10^{-12}$
Electric dipole moment $d < 10.5 \times 10^{-28}$ $e$cm, CL = 90%
Mean life $\tau > 4.6 \times 10^{26}$ yr, CL = 90% [a]

μ $J = \frac{1}{2}$

Mass $m = 0.1134289267 \pm 0.0000000029$ u
Mass $m = 105.6583715 \pm 0.0000035$ MeV
Mean life $\tau = (2.1969811 \pm 0.0000022) \times 10^{-6}$ s
$\tau_{\mu^+}/\tau_{\mu^-} = 1.00002 \pm 0.00008$
$c\tau = 658.6384$ m
Magnetic moment anomaly $(g-2)/2 = (11659209 \pm 6) \times 10^{-10}$
$(g_{\mu^+} - g_{\mu^-}) / g_{\text{average}} = (-0.11 \pm 0.12) \times 10^{-8}$
Electric dipole moment $d = (-0.1 \pm 0.9) \times 10^{-19}$ $e$cm

Decay parameters [b]

$\rho = 0.74979 \pm 0.00026$
$\eta = 0.057 \pm 0.034$
$\delta = 0.75047 \pm 0.00034$
$\xi P_\mu = 1.0009^{+0.0016}_{-0.0007}$ [c]
$\xi P_\mu \delta/\rho = 1.0018^{+0.0016}_{-0.0007}$ [c]
$\xi' = 1.00 \pm 0.04$
$\xi'' = 0.7 \pm 0.4$
$\alpha/A = (0 \pm 4) \times 10^{-3}$
$\alpha'/A = (-10 \pm 20) \times 10^{-3}$
$\beta/A = (4 \pm 6) \times 10^{-3}$
$\beta'/A = (2 \pm 7) \times 10^{-3}$
$\overline{\eta} = 0.02 \pm 0.08$

μ^- DECAY MODES		Fraction (Γ_i/Γ)		Confidence level	p (MeV/c)
$e^-\overline{\nu}_e\nu_\mu$		$\approx 100\%$			53
$e^-\overline{\nu}_e\nu_\mu\gamma$		[d] (1.4 ± 0.4) %			53
$e^-\overline{\nu}_e\nu_\mu e^+e^-$		[e] $(3.4\pm0.4) \times 10^{-5}$			53
Lepton Family number (*LF*) violating modes					
$e^-\nu_e\overline{\nu}_\mu$	*LF*	[f] < 1.2	%	90%	53
$e^-\gamma$	*LF*	< 5.7	$\times 10^{-13}$	90%	53
$e^-e^+e^-$	*LF*	< 1.0	$\times 10^{-12}$	90%	53
$e^-2\gamma$	*LF*	< 7.2	$\times 10^{-11}$	90%	53

τ $J = \frac{1}{2}$

Mass $m = 1776.82 \pm 0.16$ MeV
$(m_{\tau^+} - m_{\tau^-})/m_{\text{average}} < 2.8 \times 10^{-4}$, CL = 90%
Mean life $\tau = (290.3 \pm 0.5) \times 10^{-15}$ s
$c\tau = 87.03$ μm
Magnetic moment anomaly > -0.052 and < 0.013, CL = 95%
$\text{Re}(d_\tau) = -0.220$ to 0.45×10^{-16} $e$cm, CL = 95%
$\text{Im}(d_\tau) = -0.250$ to 0.0080×10^{-16} $e$cm, CL = 95%

Weak dipole moment

$\text{Re}(d_\tau^w) < 0.50 \times 10^{-17}$ $e$cm, CL = 95%
$\text{Im}(d_\tau^w) < 1.1 \times 10^{-17}$ $e$cm, CL = 95%

Weak anomalous magnetic dipole moment

$\text{Re}(\alpha_\tau^w) < 1.1 \times 10^{-3}$, CL = 95%
$\text{Im}(\alpha_\tau^w) < 2.7 \times 10^{-3}$, CL = 95%
$\tau^\pm \to \pi^\pm K_S^0 \nu_\tau$ (RATE DIFFERENCE) / (RATE SUM) = (-0.36 ± 0.25)%

GAUGE AND HIGGS BOSONS

γ

$I(J^{PC}) = 0,1(1^{--})$

Mass $m < 1 \times 10^{-18}$ eV
Charge $q < 1 \times 10^{-35}$ e
Mean life τ = Stable

g or gluon

$I(J^{P}) = 0(1^{-})$

Mass $m = 0$ [a]
SU(3) color octet

graviton

$J = 2$

Mass $m < 6 \times 10^{-32}$ eV

W

$J = 1$

Charge $= \pm 1$ e
Mass $m = 80.385 \pm 0.015$ GeV
$m_Z - m_W = 10.4 \pm 1.6$ GeV
$m_{W^+} - m_{W^-} = -0.2 \pm 0.6$ GeV
Full width $\Gamma = 2.085 \pm 0.042$ GeV
$\langle N_{\pi^\pm} \rangle = 15.70 \pm 0.35$
$\langle N_{K^\pm} \rangle = 2.20 \pm 0.19$
$\langle N_p \rangle = 0.92 \pm 0.14$
$\langle N_{\text{charged}} \rangle = 19.39 \pm 0.08$

W^- modes are charge conjugates of the modes below.

W^+ DECAY MODES		Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\ell^+\nu$	[b]	(10.86 ± 0.09) %		–
$e^+\nu$		(10.71 ± 0.16) %		40192
$\mu^+\nu$		(10.63 ± 0.15) %		40192
$\tau^+\nu$		(11.38 ± 0.21) %		40173
hadrons		(67.41 ± 0.27) %		–

Z

$J = 1$

Charge $= 0$
Mass $m = 91.1876 \pm 0.0021$ GeV [d]
Full width $\Gamma = 2.4952 \pm 0.0023$ GeV
$\Gamma(\ell^+\ell^-) = 83.984 \pm 0.086$ MeV [b]
$\Gamma(\text{invisible}) = 499.0 \pm 1.5$ MeV [e]
$\Gamma(\text{hadrons}) = 1744.4 \pm 2.0$ MeV
$\Gamma(\mu^+\mu^-)/\Gamma(e^+e^-) = 1.0009 \pm 0.0028$
$\Gamma(\tau^+\tau^-)/\Gamma(e^+e^-) = 1.0019 \pm 0.0032$ [f]

H^0

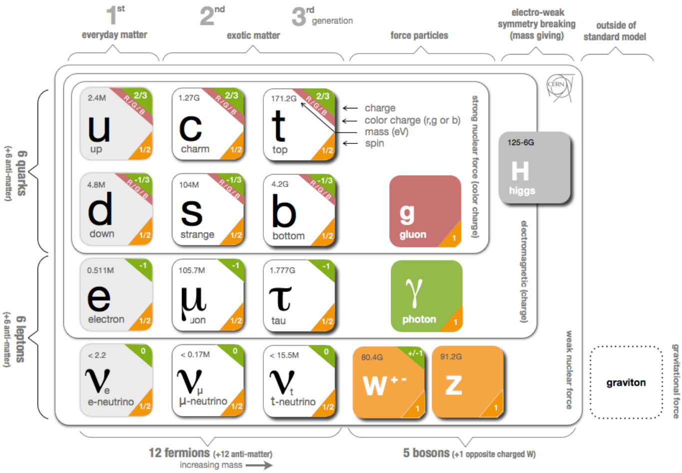
$J = 0$

Mass $m = 125.7 \pm 0.4$ GeV

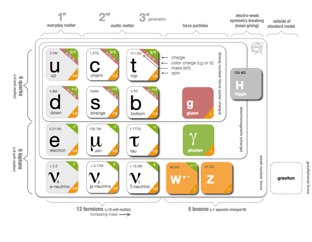
H^0 Signal Strengths in Different Channels

Combined Final States $= 1.17 \pm 0.17$ (S = 1.2)
$WW^* = 0.87^{+0.24}_{-0.22}$
$ZZ^* = 1.11^{+0.34}_{-0.28}$ (S = 1.3)
$\gamma\gamma = 1.58^{+0.27}_{-0.23}$
$b\bar{b} = 1.1 \pm 0.5$
$\tau^+\tau^- = 0.4 \pm 0.6$
$Z\gamma < 9.5$, CL = 95%

Elementary particles are well organized in the SM.



How organized?



Patterns found:

- $s = 1/2$ fermions: quarks and leptons
- $s = 1$ bosons: interaction mediators (force carriers)
- $s = 0$ boson: SSB and masses
- 3 generations

Units and conventions

- $c = \hbar = 1$ then $[L] = [T] = [M^{-1}]$. Schrödinger equation looks like

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + V\psi$$

where "quantization" rules are

$$H \to i\frac{\partial}{\partial t} \equiv i\partial_t, p_i \to i\frac{\partial}{\partial x^i} \equiv i\partial_i$$

.

- $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^i) = (t, x^i)$ in 4D spacetime. The Minkowski metric is mostly minus sign (=West coast, particle physics, energy-like convention)

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

so that $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = dt^2 - d\vec{x}\cdot d\vec{x}$.
[NOTE::: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla)$, $\partial^\mu = (\partial_t, -\nabla)$.]
[NOTE::: Energy has the correct sign in four momentum: $P^\mu = (E, \vec{p})$, $P_\mu = \eta_{\mu\nu}P^\nu = (E, -\vec{p})$, $P^2 = \eta_{\mu\nu}P^\mu P^\nu = P^\mu P_\mu = E^2 - \vec{p}^2 = m^2$]
[NOTE::: Quantization rule: $p_\mu \to i\partial_\mu$]

Some questions?

- Who decided $s = 0, \frac{1}{2}, 1$? Anything else? [**Q.** e.g., s=1/3?]
  [**Q.** What's the unit of spin, s?]
- How to describe interactions among particles? [**Q.** Any rule?]
- Particle or wave? [**Q.** Is there fundamental differences between electron and photon other than spin and mass?]

Short answers

- Who decided $s = 0, \frac{1}{2}, 1$? Ans: Spacetime symmetry= Lorentz (Poincaré) in SR
- How to describe interactions among particles? Ans: Gauge symmetry. [**NOTE**::: $SU(3) \times SU(2) \times U(1)$ in the SM]

  [**Q.** Symmetries (i.e. the spacetime and gauge symmetries) in Maxwell's EM ?]
- Particle or wave? Ans:They are all excitations of quantum fields.

SM is written in QFT.
(more precisely relativistic, gauge, quantum field theory.

(reminder) EM

- Relativity is required due to the fact that speed of light is constant

The wave equation with $c = 1$ is $(\partial_t^2 - \nabla^2)\phi(x^\mu) = 0$

- gauge symmetry

$$\nabla \cdot \vec{E} = \rho, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

is solved by $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$. but

$$\vec{A} \to \vec{A}_g = \vec{A} + \nabla\xi, \phi \to \phi_g = \phi - \partial_t \xi$$

do not change $\vec{E}, \vec{B}$.[**NOTE:::** $A^\mu = (\phi, \vec{A})$ with $A^\mu \to A^\mu_g = A^\mu - \partial^\mu \xi$.]

- Relativistic formulation of Maxwell's eqs

Field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ [**Q.** Show $F_{0i} = E_i, F_{ij} = \epsilon_{ijk} B_k$]forms the action

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \left(= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) - \rho\phi + \vec{j} \cdot \vec{A} \right)$$

, which provides

$$\partial_\mu F^{\mu\nu} = J^\nu \sim \nabla \cdot \vec{E} = \rho, \nabla \times \vec{B} - \partial_t \vec{E} = \vec{j}$$

The Bianchi identity, $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \sim \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} + \partial_t \vec{E} = 0$ completes the Maxwell's equations

QFT references

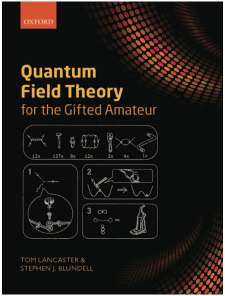
[**NOTE:::** QFT is the standard language of modern particle physics]

Refs

- Open KIAS school `http://workshop.kias.re.kr/KWS2013/?Program`
- PDG
  `http://pdg.lbl.gov/`

Recommendations for beginners:

- My recommendation for beginning students is this $\Longrightarrow$
- Tong, Lecture for Part-III (only QED but very clear exposition)
  `http://www.damtp.cam.ac.uk/user/tong/qft.html`
- A. Zee "Quantum field theory in a nutshell" (2nd, princeton 2010): intuitive, fun!
- L.Álvarez-Gaumé, M. Á. Vázquez-Mozo, "An invitation to quantum field theory" (springer 2012): clear exposition! recommended!



Standard texts:

- Peskin and Schroeder (1995), Schwartz (2014): standard of standard
- Weinberg I,II, III, Srednicki, Ramond, Ryder, many many others
- Müller-Kirsten, Wiedemann "Introduction to supersymmetry" (world scientific 2010): representation of Lorentz group +susy formalism

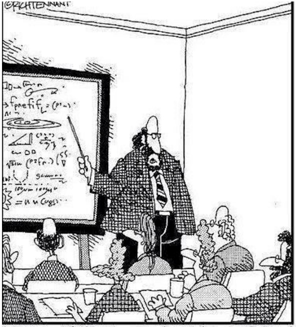
Particle physics references

Some readable texts: [NOTE::: QFT books typically have sections on the SM. More later.]

- Anchordoqui, Halzen "Lessons in Particle physics" (v4 2011 Dec.) 260 pages
  `http://arxiv.org/abs/0906.1271v4`
- M. Robinson "Symmetry and the Standard Model" (springer 2011): +math
- C.G.Tully "Elementary particle physics in a Nutshell" (princeton 2011)
- Barger, Phillips "Collider Physics": useful appendix!
- Dobado et.al. "Effective Lagrangians for the standard model" (springer 1997)

LHC!:

- CMS physics results
  `https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResults`
- ATLAS physics results
  `https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResults`



"Along with 'Antimatter,' and 'Dark Matter,' we've recently discovered the existence of 'Doesn't Matter,' which appears to have no effect on the universe whatsoever."

QFT-1

QFT = QM + Relativity.

- (1) $\Delta t \Delta E \gtrsim \hbar$: the energy can fluctuate wildly over a small interval of time
- (2) $E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$: energy can be converted into mass and vice versa
- (1)+(2) : mass (or particle) can be created/annihilated out of/into fluctuating energy! This is described by QFT!
- e.g. $e^+e^- \to \gamma\gamma$, $pp \to t\bar{t}$, $gg \to H \to b\bar{b}$ [**NOTE:::** # non-conserving phenomena could not be described by Schrödinger equation. [**Q.** are you sure?]]

QFT-2

QFT is a proper combination of Rel. and QM.

- particles and waves = excited states of quantum fields
- "all electrons look exactly same" because they are all excitations of the same field $\psi_e(x)$!
- $\psi_e(x) = \sum_p \sum_{s=\pm 1/2} \left(a_s(p) u_s(p) e^{-ipx} + b_s^\dagger(p) v_s(p) e^{ipx} \right)$
- This structure is generic:
  $Field = \sum_{p,\lambda} a_\lambda \times \text{polarization}_\lambda \times e^{-ipx} + (p^0 < 0)$

[**Q.** how about a scalar $\phi(x)$ and vector $A^\mu(x)$?.]

[**NOTE:::** $\sum_p = \int \frac{d^3p}{(2\pi)^3 2p^0} (= \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta^4(p^2 - m^2)\theta(p^0)$]

QFT-3

In QFT, # of particles is not conserved.

$$1 = |0\rangle\langle 0| + \sum_p |p\rangle\langle p| + \cdots \text{(Fock space)}$$

- vacuum: $a(p)|0\rangle = 0$
- one particle state: $a(p)^\dagger |0\rangle \propto |p\rangle$
- two particle state: $a(p)^\dagger a(k)^\dagger |0\rangle \propto |p, k\rangle$
- $\cdots$

QFT-4

Organizing principles of making a ‘QFT model’ (or Lagrangian)

- symmetry (spacetime, internal, super-, conformal, etc)… “group”
- matter contents and their transformation rules (i.e. quantum numbers)…“representation”

QFT-5

The SM is organized by

- Poincaré symmetry (Lorentz+translaion)
- Gauge symmetry: $G_{\rm SM} = SU(3)_c \times SU(2)_L \times U(1)_y$ associated with gauge bosons $A_\mu = (g_\mu^a, W_\mu^\pm, W_\mu^3, B_\mu)$
- Matter fields: $\psi = (\ell_L, e_R, Q_L, u_R, d_R)_i$, $i = 1, 2, 3$ and H
- $\psi \sim (1/2, 0)$ or $(0, 1/2)$, $H \sim (0, 0)$ and $A_\mu \sim (1/2, 1/2)$ representations of Lorentz group (or spinor, scalar and vectors).
- $\ell_L = \binom{\nu_L}{e_L} \sim (1, 2, -\frac{1}{2})$, $e_R \sim (1, 1, -1)$, $Q_L = \binom{u_L}{d_L} \sim (3, 2, \frac{1}{6})$, $u_R \sim (3, 1, \frac{2}{3})$, $d_R \sim (3, 1, -\frac{1}{3})$ representations of $G_{\rm SM}$.
- If you understand these, you can sleep now. :-)

Lorentz-1

Let's first understand $\Psi \sim (j, j')$ i.e. irreducible representations of Lorentz group.

- The constancy of speed of light demands $0 = dt^2 - d\vec{x} \cdot d\vec{x}$ with $c = 1$. [**NOTE:::** You can regard $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x} \cdot d\vec{x}$ as an infinitesimal length in spacetime. $x^\mu = (x^0, x^1, x^2, x^3) = (t, x^i)$ and $\eta = diag(1, -1, -1, -1)$]
- A linear transformation $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$ does not change ds^2 is called Lorentz transformation.
- Λ's form a group $L = \{\Lambda | \Lambda^T \eta \Lambda = \eta\}$, called Lorentz group.
  [**NOTE:::** Group: a closed system of operations with an identity]
  [**NOTE:::** Proper, time-direction-conserving transformation $L = SO(1, 3; R)$]
- There are 6 Lorentz transformations$=$ (3 boosts(η_x, η_y, η_z)), 3 rotations $(\theta_x, \theta_y, \theta_z)$), which keep ds^2 unchanged.
  [**NOTE:::** η_i are rapidities and θ_i are angles about i-axis]

Lorentz-2

Explicit form of Lorentz transformations:

- 3 rotations which keeps $d\vec{x}\cdot d\vec{x} = dx^2 + dy^2 + dz^2$ unchanged: R_x, R_y, R_z, respectively.

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_x & \sin\theta_x \\ & & -\sin\theta_x & \cos\theta_x \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_y & & -\sin\theta_y \\ & & 1 & \\ & \sin\theta_y & & \cos\theta_y \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix}$$

[**Q.** Show:$\sin^\theta + \cos^2\theta = 1$ guarantees that ds^2 is actually preserved.]

- 3 boosts which keeps $(dt^2 - dx^2)$, $(dt^2 - dy^2)$,$(dt^2 - dz^2)$ unchanged: B_x, B_y, B_z, respectively.

$$\begin{pmatrix} \gamma_x & -v_x\gamma_x & & \\ -v_x\gamma_x & \gamma_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \gamma_y & & -v_y\gamma & \\ & 1 & & \\ -v_y\gamma_y & & \gamma_y & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \gamma_z & & & -v_z\gamma_z \\ & 1 & & \\ & & 1 & \\ -v_z\gamma_z & & & \gamma_z \end{pmatrix}$$

where $\gamma_i = 1/\sqrt{1 - v_i^2} = \cosh\eta_i$ and $v_i\gamma_i = \sinh\eta_i$. [**Q.** Show: $\cosh^2\eta - \sinh^2\eta = 1$ guarantees that ds^2 is actually preserved.]

- $L = \{R_i, B_i\}$

Lorentz-3

To see the structure of the rotations, it is enough to analyze the infinitesimal changes ($\theta \ll 1$) from the origin (doing nothing=origin):

$$R_x = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_x & \sin\theta_x \\ & & -\sin\theta_x & \cos\theta_x \end{pmatrix} \approx 1 + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \theta_x \\ & & -\theta_x & 0 \end{pmatrix} = 1 + i\theta_x J_x$$

Similarly,

$$R_y = \begin{pmatrix} 1 & & & \\ & \cos\theta_y & & -\sin\theta_y \\ & & 1 & \\ & \sin\theta_y & & \cos\theta_y \end{pmatrix} \approx 1_4 + i\theta_y J_y, \quad R_z = \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix} \approx 1_4 + i\theta_z J_z$$

[**Q.** Find the explicit form of J_x, J_y, J_z.] [**NOTE:::** Note that they are nothing but the angular momentum operators generating rotations satisfying $[J_i, J_j] = i\epsilon_{ijk}J_k$. This means rotations form a group $SO(3) \simeq SU(2)$ as we know.]

Lorentz-4

Similarly, let's consider infinitesimal boosts ($\eta \ll 1$), by which $\cosh\eta \approx 1$ and $\sinh\eta \approx \eta$ and

$$B_x = \begin{pmatrix} \cosh\eta_x & -\sinh\eta_x & & \\ -\sinh\eta_x & \cosh\eta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\eta_x & & \\ -\eta_x & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = 1_4 + i\eta_x K_x$$

Similarly,

$$B_y \approx 1_4 + i\eta_y K_y, \;\; B_z \approx 1_4 + i\eta_z K_z$$

[**Q.** Find the explicit form of K_x, K_y, K_z.][**Q.** Show that $[K_i, K_j] = -i\epsilon_{ijk} J_k$, $[J_i, K_j] = i\epsilon_{ijk} K_k$. This means boosts do not form a group.]

Lorentz-5

- 6 generators of Lorentz group are found to satisfy Lie algebra: $[J_i, J_j] = i\epsilon_{ijk}J_k, [K_i, K_j] = -i\epsilon_{ijk}J_k, [J_i, K_j] = i\epsilon_{ijk}K_k$
- J_i form $SO(3)$ group but K_i do not. However, a clever combination of J_i and K_i are separate and form groups.
- $N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)$ then $[N_i^+, N_j^+] = i\epsilon_{ijk}N_k^+, [N_i^-, N_j^-] = i\epsilon_{ijk}N_k^-, [N_i^+, N_j^-] = 0$
- This means that Lorentz group is equivalent to product of two rotation groups:

$$L = SU(2) \times SU(2)^1$$

[1] More rigorously, $so(1,3;\mathbb{R})^{\mathbb{C}} \simeq su(2,\mathbb{C}) \times su(2,\mathbb{C})$

Lorentz-6

- A representation of L is labeled by an ordered numbers (j, j') where j and j' are eigenvalues of N^+ and N^- thus $j, j' = 0, 1/2, 1, 3/2, \cdots$.
- As $J_i = N_i^+ + N_i^-$, the total spin of (j, j') state is $j + j'$ by the rule of angular momentum addition.
- $(0, 0) : s = 0$,
  $(1/2, 0), (0, 1/2) : s = 1/2$ [**NOTE:::** left-handed and right-handed spinor representations]
  $(1/2, 1/2) : s = 1$.

Lorentz-7

- A scalar $\phi(x) \sim (0,0)$ is trivially transformed by Λ
- $\phi(x) \to \phi'(x') = \phi(\Lambda^{-1}x') = \phi(x)$

Lorentz-8

- A right-handed Weyl spinor $\psi_R \sim (1/2, 0)$
- $N^+ = 1/2, N^- = 0$ or $N_i^+ = \frac{\sigma_i}{2}$ and $N_i^- = 0$.
- $J_i = N_i^+ + N_i^- = \frac{\sigma_i}{2}$ and $K_i = -i\frac{\sigma_i}{2}$
- $\psi_R \to \psi_R' \approx (1 + i\theta_i J_i + i\eta_i K_i)\psi_R = (1 + (i\theta_i + \eta_i)\frac{\sigma_i}{2})\psi_R$
- $\psi_R \to \psi_R' = e^{(i\theta_i+\eta_i)\frac{\sigma_i}{2}}\psi_R$ for finite transformation.

[**Q.** Show $\psi_L \to \psi_L' = e^{(i\theta_i-\eta_i)\frac{\sigma_i}{2}}\psi_L$ i.e. the same in rotation but opposite in boost.]

[**Q.** Show $(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$ is a real scalar.]

[**Q.** Show $(\psi_R^\dagger\sigma^\mu\psi_R)$ and $(\psi_L^\dagger\overline{\sigma}^\mu\psi_L)$ are vectors i.e. transforms like x^μ, where $\sigma^\mu = (1_2, \vec{\sigma})$ and $\overline{\sigma}^\mu = (1_2, -\vec{\sigma})$.]

Lagrangian-1

- For a point particle, physics is conveniently described by Lagrangian. (a scalar function) $S[q] = \int dt L(q, \dot{q})$
- The classical behavior is obtained by the least action principle: $\delta S = 0$ or equivalenetly

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

- Quantum amplitude is obtained by path integral:

$$\mathcal{M}_{i\to f} = \int_i^f \mathcal{D}q e^{iS[q]}$$

Lagrangian-2

- For fields, physics is conveniently described by Lagrangian, too. But now Lagrangian density (a scalar distribution).
  $L = \int d^3x \mathcal{L}(\psi, \partial_\mu \psi)$, $S = \int d^4x \mathcal{L}$
  [NOTE::: $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \vec{\nabla})$, $\partial^\mu = (\partial_t, -\vec{\nabla})$]
- The classical behavior is obtained by the least action principle: $\delta S = 0$ or equivalenetly

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = \frac{\partial \mathcal{L}}{\partial \psi}$$

- Quantum amplitude is obtained by path integral:

$$\mathcal{M}_{i \to f} = \int_i^f \mathcal{D}\psi e^{iS[\psi]}$$

Lagrangian-3

- For a scalar field, Lagrangian density is almost trivially obtained
- $\mathcal{L} = \phi^*(x)(-\partial_\mu^2 - m^2)\phi$ and the equation of motion by $\delta\phi^*$ gives $(-\partial_\mu^2 - m^2)\phi(x) = 0$.

  [**NOTE:::** $H \to i\partial_t$, $\vec{p} \to -i\vec{\nabla}$ is collectively described by $p_\mu = (H, -\vec{p}) \to i\partial_\mu$. Thus the mass-shell condition $0 = p^2 - m^2$ is translated into $(-\partial^2 - m^2)\phi(x) = 0$, which is nothing but Klein-Gordon equation.]

  [**NOTE:::** $S = \int d^4x \phi^*(x)(-\partial_\mu^2 - m^2)\phi = \int d^4x \left(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\right) - \partial_\mu J^\mu$ where $J^\mu = \phi^*\partial^\mu\phi$. Since the last term is total divergence term, it does not affect local physics.]

  [**NOTE:::** $[\mathcal{L}] = 4$, $[\phi] = [\phi^*] = 1$ and $[m] = 1$ or mass.]

Lagrangian-4 (Weyl spinors)

For Weyl spinors, we use scalar combinations of ψ_L and ψ_R. There are three possible terms.

- $\mathcal{L}_L = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L$
- $\mathcal{L}_R = i\psi_R^\dagger \overline{\sigma}^\mu \partial_\mu \psi_R$
- $\mathcal{L}_{\rm mass} = -m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$

  [**NOTE:::** $[\psi_L] = [\psi_R] = \frac{3}{2}$ and $[m] = 1$ or mass]

  [**NOTE:::** When $m = 0$, two fields ψ_L and ψ_R are independent: $i\sigma^\mu \partial_\mu \psi_L = 0$ and $i\overline{\sigma}^\mu \partial_\mu \psi_R = 0$.]

  [**NOTE:::** One can regard m provides a physical 'mixing' between ψ_L and ψ_R. $i\sigma^\mu \partial_\mu \psi_L = m\psi_R$]

Lagrangian-5 (Dirac spinors)

- One can conveniently combine ψ_L and ψ_R into a 4 component Dirac spinor, $\psi_D = \binom{\psi_L}{\psi_R}$ with gamma matrices γ^μ satisfying
  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_4$(Cliffor algebra).
  [**NOTE:::** $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$ in Weyl representation. [**Q.** Show Clifford algebra.]]
- $\overline{\psi}_D = \psi_D^\dagger \gamma^0 = (\psi_R^\dagger, \psi_L^\dagger)$
  [**NOTE:::** $\overline{\psi}_D \psi_D = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$: scalar]
  [**NOTE:::** $\overline{\psi}_D \gamma^\mu \psi_D = \psi_R^\dagger \sigma^\mu \psi_R + \psi_L^\dagger \overline{\sigma}^\mu \psi_L$: a vector]
- $\mathcal{L} = \overline{\psi}_D \left(i\gamma^\mu \partial_\mu - m\right) \psi_D,$
  [**NOTE:::** The Dirac equation is derived by $\delta\overline{\psi}_D$:$\left(i\gamma^\mu \partial_\mu - m\right) \psi_D = 0$.]
  [**NOTE:::** A slash notation is useful: $\not{p} = \gamma^\mu p_\mu$.]
  [**Q.** $(i\not{\partial} + m) \times$ Dirac eq. $=$ KG eq.]

Lagrangian-6 (Gamma matrices)

- $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$
- $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\{\gamma^\mu, \gamma_5\} = 0$.
- Projection operators are $P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
- By projection $P_L\psi_D = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \equiv \Psi_L$ and $P_R\psi_D = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \equiv \Psi_R$
  [**NOTE:::** $\psi_D = \Psi_L + \Psi_R$]
  [**NOTE:::** $\gamma_5\Psi_L = -\Psi_L$ and $\gamma_5\Psi_R = \Psi_R$ thus $\Psi_{L/R}$ are eigenstates of γ_5.]

[**Q.** Show $\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\text{odd number of } \gamma\text{-matrices}) = 0$]

summary so far

QFT=SR+QM

$E = mc^2$ and $\Delta t \Delta E \gtrsim 1$: particles are created/ annihilated.

Organizing principles of QFT model

Spacetime symmetry (=Lorentz) and gauge symmetry

Representations of Lorentz group

$\phi \sim (0,0), \psi_R \sim (1/2,0), \psi_L \sim (0,1/2)$, $A_\mu \sim (1/2,1/2)$
No other states are available in nature!

Lagrangians

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \overline{\psi}_D (i\partial\!\!\!/ - m_\psi)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Quiz

Q1 Why there's no $s = 1/3$ state in nature?

Q2 What's the dimension of $(1/2, 0)$, the left-handed spinor representation?

Q3 Graviton has $s = 2$. What's the representation (j, j')?

Lecture #2

QED: $U(1)$ gauge theory(1)

Let's consider a free Dirac particle(=field, spinor, fermion ...) with m:

$$\mathcal{L}(\psi) \equiv \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

- You may regard ψ as the field for electron in low energy $(E \ll \langle h \rangle \simeq 246\mathrm{GeV})$
- $U(1)_{\rm global} : \psi \to e^{i\theta}\psi$ is a good symmetry. ($\mathcal{L}$ is invariant)
- $U(1)_{\rm local} : \psi \to e^{i\theta(x)}\psi$ is not a good symmetry. ($\mathcal{L}$ is not invariant)

$$\partial_\mu\psi \to \partial_\mu(e^{i\theta(x)}\psi(x)) = e^{i\theta(x)}\left(\partial_\mu\psi + i(\partial_\mu\theta)\psi\right),$$
$$\therefore \mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\mathcal{L} = \mathcal{L} - \bar{\psi}(\gamma^\mu\partial_\mu\theta)\psi \neq \mathcal{L}$$

QED: $U(1)$ gauge theory (2)

If you want local phase transition a good symmetry, you need to introduce gauge covariant derivative ($\partial_\mu \to D_\mu = \partial_\mu - igA_\mu$) to cancel out $\delta\mathcal{L}$, [**Q.** why do you want?]

$$\begin{aligned}\mathcal{L}(\psi) \to \mathcal{L}_{U(1)}(\psi, A_\mu) &\equiv \bar{\psi}\left(i(\partial\!\!\!/ - igA\!\!\!/) - m\right)\psi \\ &= \mathcal{L} + g\bar{\psi}\gamma^\mu\psi A_\mu = \mathcal{L} + gJ^\mu A_\mu\end{aligned}$$

- $D_\mu\psi \to e^{i\theta(x)}D_\mu\psi$ [**NOTE:::** $D_\mu\psi \to (\partial_\mu - igA'_\mu)(e^{i\theta(x)}\psi(x)) = e^{i\theta(x)}\left(\partial_\mu\psi + i(\partial_\mu\theta - gA'_\mu)\psi\right)$ thus $gA_\mu = gA'_\mu - \partial_\mu\theta$ or $A'_\mu = A_\mu + \frac{1}{g}\partial_\mu\theta$ guarantees the relationship. Note this is the same gauge transformation in Maxwell's equations.]
- The new gauge invariant Lagrangian $\mathcal{L}_{U(1)}$ contains $(current) \times (gauge\ field)$ type interaction. g describes the strength of the interaction.[**NOTE:::** In QED, $g = eQ$ where $e = \sqrt{4\pi\alpha}$ is the magnitude of the electron charge and $Q = -1$ for electron, $Q = +1$ for proton[**Q.** why the same magnitudes?]]
- The kinetic term for A_μ is given as $\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. [**NOTE:::** The gauge invariant field strength tensor for A_μ is $F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is 'curvature' in

QED: $U(1)$ gauge theory (3) practice!

Let's practice with a complex scalar particle(=field, boson ...) with $m = 0$:

$$\mathcal{L}(\phi) \equiv \eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi$$

- $U(1)_{\rm global} : \phi \to e^{i\theta}\phi, \phi^* \to e^{-i\theta}\phi^*$ is a good symmetry. ($\mathcal{L}$ is invariant)
- $U(1)_{\rm local} : \phi \to e^{i\theta(x)}\phi, \phi^* \to e^{-i\theta(x)}\phi^*$ is not a good symmetry. ($\mathcal{L}$ is not invariant)
- Recipe: $\partial_\mu \to D_\mu = \partial_\mu - igA_\mu$ with $A_\mu \to A'_\mu = A_\mu + \frac{\partial_\mu\theta}{g}$:

$$\mathcal{L}_{U(1)}(\phi, A_\mu) \equiv \eta^{\mu\nu}(\partial_\mu + igA_\mu)\phi^*(\partial_\nu - igA_\mu)\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

[**Q.** Show gauge invariance of the Lagrangian]

[**NOTE:::** $\mathcal{L}_{U(1)}(\phi, A_\mu) = \mathcal{L}(\phi) + gJ_\mu A^\mu + g^2 A_\mu A^\mu \phi^*\phi$. [**Q.** What is $J_\mu(\phi, \phi^*)$?]]

Non-Abelian: $SU(2)$ gauge theory(1)

- Historically proton and neuntron were known to form an isospin doublet $\binom{p}{n}$ i.e. they are regarded as up- and down-components of an isospin doublet state which transforms cordially. **[NOTE:::** indeed, we regard the doublet as a fundamental representation of $SU(2)$]
- In the SM, this doublet has more fundamental origin as $p = (uud)$ and $n = (udd)$ then fundamental doublet is $\binom{u}{d}$ rather than proton and neutron.
- Isospin symmetry is $SU(2)$ under which the doublet transforms as $\binom{u}{d} \to e^{i\theta^i T^i} \binom{u}{d}$ where generators for $SU(2)$ are Pauli matrices $T^i = (\frac{\sigma_i}{2})_{i=1,2,3}$ and θ^i are real valued parameters (angle of rotation in gauge space). **[NOTE:::** $U = e^{i\theta^i T^i}$ is unitary matrix with Hermitian generators T^i. $U^\dagger = U^{-1} = e^{-i\theta^i T^i}$]
- $[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}] = i\epsilon_{ijk}\frac{\sigma_k}{2}$ for $su(2)$ algebra.**[NOTE:::** can be generalized $[T^a, T^b] = if_{abc} T^c$ for a complact Lie algebra of non-Abelian group G where f_{abc} (structure constant) determines the algebraic structure.]

Non-Abelian: $SU(2)$ gauge theory(2)

Now let's think of an $G = SU(2)$ doublet, free Dirac field $\psi = \binom{u}{d}$ with m:

$$\mathcal{L}(\psi) \equiv \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

- This Lagrangian is invariant under global $SU(2)$: $\psi \to \psi' = U\psi$, $\bar{\psi} \to \bar{\psi}' = \bar{\psi}U^\dagger$ where $U = e^{i\vec{\theta}\cdot\vec{\sigma}/2}$. [**NOTE:::** $\bar{\psi}\psi = \bar{u}u + \bar{d}d$ is a singlet of $SU(2)$ (i.e. unchanged under transformation)]
- But, not invariant under local $SU(2)$ transformation with $\theta^i = \theta^i(x)$.
- Recipe: introduce covariant derivative $D_\mu = \partial_\mu - igT^iA^i_\mu$ with a proper rule for gauge transformation of $A_\mu \equiv T^iA^i_\mu$. [**NOTE:::** $A'_\mu = UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$] [**Q.** Check with $U(1)$ with $T = 1$] [**Q.** Check with infinitesimal transformation $U \simeq 1 + i\theta \cdot T$ that $A'^i_\mu = A^i_\mu + \frac{1}{g}\partial_\mu\theta^i - \epsilon^{ijk}\theta^j A^k_\mu$]

$$\mathcal{L}_{SU(2)}(\psi, A^i_\mu) \equiv \bar{\psi}\left(i(\partial\!\!\!/ - igT^i A\!\!\!/^i) - m\right)\psi$$

Non-Abelian: $SU(2)$ gauge theory (3) practice!

Let's practice with a complex doublet scalar $\phi = \binom{\phi_u}{\phi_d}$ with $m = 0$:

$$\mathcal{L}(\phi) \equiv \eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi = \partial_\mu\phi_u^\dagger\partial^\mu\phi_u + \partial_\mu\phi_d^\dagger\partial^\mu\phi_d$$

- $SU(2)_{\rm global}$: $\phi \to e^{i\vec{\theta}\cdot\vec{\sigma}/2}\phi$, $\phi^\dagger \to \phi^\dagger e^{-i\vec{\theta}\cdot\vec{\sigma}/2}$ is a good symmetry.
- $SU(2)_{\rm local}$: $\vec{\theta} = \vec{\theta}(x)$
- Recipe: $\partial_\mu \to D_\mu = \partial_\mu - ig\frac{\vec{\sigma}}{2}\cdot\vec{A}_\mu$ with $A'_\mu = UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$:

$$\mathcal{L}_{U(2)}(\phi, A_\mu) \equiv \eta^{\mu\nu}(\partial_\mu\phi - ig\frac{\vec{\sigma}}{2}\cdot\vec{A}_\mu\phi)^\dagger(\partial_\mu\phi - ig\frac{\vec{\sigma}}{2}\cdot\vec{A}_\mu\phi) - \frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}$$

[**Q.** Show gauge invariance of the Lagrangian]
[**NOTE:::** $\mathcal{L}_{SU(2)}(\phi, A^i_\mu) = \mathcal{L}(\phi) + gJ_\mu A^\mu + g^2 A_\mu A^\mu \phi^*\phi$. [**Q.** What is $J_\mu(\phi, \phi^*)$? Show, $D_\mu J^\mu = 0$]]
[**NOTE:::** $F_{\mu\nu} = F^i_{\mu\nu}T^i = \frac{i}{g}[D_\mu, D_\nu]$. This is 'curvature' in non-Abelian internal space.]
[**Q.** Show $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon_{ijk}A^j_\mu A^k_\nu$. There's no ϵ term in Abelian case. Due to this, there are self-interactions of non-Abelian gauge bosons!]

Non-Abelian (4): $SU(3)$ gauge theory

- Quarks are colored particles. That means a quark forms a triplet (i.e. fundamental representation) of $SU(3)_c$. Here c stands for 'color' gauge symmetry.

$$q = (q^a) = \begin{pmatrix} q^y \\ q^g \\ q^r \end{pmatrix} \tag{1}$$

[**NOTE:::** It is custom to call y, g, r as yellow, green, red but it does not mean anything to do with visible color.]

- There are 8 generators for $SU(3)$, $T^a = \lambda^a/2$ where λ^a's are Gell-Mann matrices with λ^3 and λ^8 diagonal. [**NOTE:::** For $SU(N)$, $N^2 - 1$.]

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}$$
$$\lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{2}$$

[**NOTE:::** Normalization is conventionally chosen to be $\mathrm{Tr}(T^a T^b) = \frac{\delta^{ab}}{2}$]

Non-Abelian (5): $SU(3)$ gauge theory

A gauge invariant Lagrangian is fairly easily constructed by prescription $\partial_\mu \to D_\mu = \partial_\mu - igG_\mu$ where 'gluon' is denoted as $G_\mu \equiv G^a_\mu \frac{\lambda^a}{2}$.

$$\mathcal{L}_{QCD} = \bar{q}(i\not{D} - m_q)q - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \tag{3}$$

[**NOTE:::** The Lagrangian includes interaction $\sim g\bar{q}_i \not{G}^a \frac{(\lambda^a)_{ij}}{2} q_j = gJ^a_\mu G^a_\mu$ and $D_\mu J^{a\mu} = 0$.]

[**Q.** Can you write down the action for a scalar-quark(squark) which is colored as usual quark?]

Non-Abelian (6): practice with $SU(N)$

A gauge invariant Lagrangian for an arbitrary gauge group $SU(N)$ is fairly easily constructed by prescription $\partial_\mu \to D_\mu = \partial_\mu - igW_\mu$ where 'gauge boson' is denoted as $W_\mu \equiv W_\mu^a T^a$ with $[T^a, T^b] = if^{abc}T^c$ and $\mathrm{Tr}(T^a T^b) = \frac{\delta_{ab}}{2}$:

$$\mathcal{L}_{SU(N)} = \bar{\psi}(i\not{D} - m_q)\psi - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} \tag{4}$$

[NOTE::: The Lagrangian includes interaction $\sim g\bar{\psi}_i \not{W}^a T^a_{ij} q_j = gJ^a_\mu W^a_\mu$ and $D_\mu J^{a\mu} = 0$.]

[NOTE::: $\psi = (\psi_i)_{i=1,2,\cdots,N}$ in fundamental representation of $SU(N)$.]

Non-Abelian (7): practice with $SU(N) \times SU(M)$

- Prescription $\partial_\mu \to D_\mu = \partial_\mu - ig_A A_\mu - ig_B B_\mu$ where 'gauge bosons' are denoted as $A_\mu \equiv A^a_\mu T^a_A$ with $[T^a_A, T^b_A] = if^{abc}_A T^c_A$ and $\mathrm{Tr}(T^a_A T^b_A) = \frac{\delta_{ab}}{2}$ and $B_\mu \equiv B^a_\mu T^a_B$ with $[T^a_B, T^b_B] = if^{abc}_B T^c_B$ and $\mathrm{Tr}(T^a_B T^b_B) = \frac{\delta_{ab}}{2}$, respectively.

$$\mathcal{L}_{SU(N)} = \bar{\psi}(i\not{D} - m_q)\psi - \frac{1}{4}A^a_{\mu\nu}A^{a\mu\nu} - \frac{1}{4}B^a_{\mu\nu}B^{a\mu\nu} \tag{5}$$

[NOTE::: The Lagrangian includes interaction $\sim g_A\bar{\psi}_i \not{A}^a T^a_{A\,ij}\psi_j = gJ^a_\mu A^a_\mu$ and similarly for B_μ.]

[NOTE::: $\psi = (\psi_{(i,j)})_{i=1,2,\cdots,N;j=1,2,\cdots,M}$ in fundamental representation of $SU(N)$ and $SU(M)$.]

Non-Abelian (8): finally $G_{\rm SM} = SU(3) \times SU(2) \times U(1)$

- Let's consider a left-handed quark $Q_L = P_L Q$ which is triplet of $SU(3)$, doublet of $SU(2)$ with a hypercharge y_Q. For now, let's assume it massless.
- Prescription

$$\partial_\mu \to D_\mu = \partial_\mu - i g_s g_\mu^a \frac{\lambda^a}{2} - i g W_\mu^i \frac{\sigma^i}{2} - i g' y_Q B_\mu$$

[**NOTE:::** Gauge bosons are 8 gluons (g_μ^a), 3 weak gauge bosons (W_μ^i) and hypercharge gauge boson (B_μ).] [**NOTE:::** 3 gauge couplings are g_s, g and g', respectively.]

[**NOTE:::** Leptons are $SU(3)$ singlets and does not interact with gluons.]

[**NOTE:::** The SM fermions are: $\ell_L = \binom{\nu_L}{e_L} \sim (1, 2, -\frac{1}{2})$, $e_R \sim (1, 1, -1)$, $Q_L = \binom{u_L}{d_L} \sim (3, 2, \frac{1}{6})$, $u_R \sim (3, 1, \frac{2}{3})$, $d_R \sim (3, 1, -\frac{1}{3})$ representations of $G_{\rm SM}$. [**Q.** How to write the gauge invariant Lagrangian for all particles?]]

Higgs mechanism(1): $U(1)$

- Write the Lagrangian for a scalar with a $U(1)$ charge q and a potential $V(\phi) = \lambda(\phi^*\phi - v^2)^2$. **[NOTE:::** The potential is $U(1)$ invariant.]

$$\mathcal{L} = [(\partial_\mu + iqA_\mu)\phi^*(\partial^\mu - iqA^\mu)\phi] - V(\phi^*\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- At the bottom of the potential, $|\phi| = v \neq 0$ Let's call v vacuum expectation value(VEV) because it is the value at the vacuum.
- Now an interesting thing happens! Let's see the physical fluctuation from the vacuum:
  $\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))e^{i\xi(x)/v} = \frac{1}{\sqrt{2}}(v + h + i\xi +$ quadratic and higher order terms). **[NOTE:::** One can use $U(1)$ symmetry to remove ξ then only physical degrees of freedom survives by a proper gauge choice.]
- The Lagrangian becomes
  $\mathcal{L} = -\frac{1}{4}FF + \frac{1}{2}[(\partial_\mu h)^2 - 4\lambda v^2 h^2] + \frac{1}{2}q^2v^2(A_\mu + \frac{\partial_\mu \xi}{qv})^2 +$ higher order
  [NOTE::: $A'_\mu = A_\mu + \frac{\partial_\mu \xi}{qv}$ (ξ is eaten!) or $\phi \to e^{-i\xi/v}\phi = (v + h)/\sqrt{2}$ makes ξ dissapear!]
  [NOTE::: $m'_A = qv$, $m_h^2 = 4\lambda v^2$]

Higgs mechanism(2): U(1)

What we have done? Starting from a gauge invariant action, we found a physical action near the non-zero vacuum with a massive gauge field! (by eating a Goldstone mode (here ξ along $U(1)$ direction)!

[**NOTE:::** Before eating ξ, A_μ was massless. Mass term ($\sim A_\mu A^\mu$) is forbidden by the gauge symmetry ($A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \theta$, right?).]

Higgs mechanism(3): $SU(2)$

Consider a $SU(2)$ doublet scalar $\phi = \begin{pmatrix} \phi_u \\ \phi_d \end{pmatrix}$. The Lagrangian density is

$$\mathcal{L} = |(\partial_\mu - ig\frac{\sigma^i}{2}W^i_\mu)\phi|^2 - V(\phi^\dagger\phi)$$

with

$$V = \lambda(|\phi|^2 - v^2/2)^2$$

.

- Take vev $\langle\phi\rangle = \frac{1}{\sqrt{2}}\binom{0}{v}$ and $\phi = \frac{1}{\sqrt{2}}\binom{0}{v+h}e^{i\theta_i T_i/v} \to \frac{1}{\sqrt{2}}\binom{0}{v+h}$. **[NOTE:::** Here we already choose a gauge where θ are hidden.]
- Find the Lagrangian at vacuum.(i.e. $\phi = \langle\phi\rangle$) What do you expect to happen?

Higgs mechanism(4): $SU(2)$

The Lagrangian at vacuum is

$$\begin{aligned} \langle \mathcal{L} \rangle &= \left| -i\frac{g}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^- \\ \sqrt{2}W_\mu^+ & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \right|^2 \\ &= \frac{g^2 v^2}{4} \left(W_\mu^+ W^{-\mu} + \frac{1}{2}(W_\mu^3)^2 \right) \\ &= m_{W^\pm}^2 |W_\mu^-|^2 + \frac{1}{2} m_{W^3}^2 (W_\mu^3)^2 \end{aligned} \tag{6}$$

[NOTE::: $W_\mu^\pm = \frac{W_\mu^1 \pm i W_\mu^2}{\sqrt{2}}$]

[NOTE::: $m_{W^\pm} = m_{W^3} = \frac{gv}{2}$]

Higgs mechanism(5): the SM! $SU(2) \times U(1)$

The Lagrangian at vacuum for $H \sim (1, 2, \frac{1}{2})$ is

$$\begin{aligned}\langle \mathcal{L} \rangle &= \left| -i\frac{1}{2} \begin{pmatrix} gW_\mu^3 + g'B_\mu & \sqrt{2}gW_\mu^- \\ \sqrt{2}gW_\mu^+ & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \right|^2 \\ &= \frac{v^2}{4}\left(2g^2 W_\mu^+ W^{-\mu} + \frac{1}{2}(gW_\mu^3 - g'B_\mu)^2 \right) \\ &= m_{W^\pm}^2 |W_\mu^-|^2 + \frac{1}{2} m_{Z^0}^2 (Z_\mu^0)^2 \end{aligned} \tag{7}$$

[**NOTE:::** $W_\mu^\pm = \frac{W_\mu^1 \pm iW_\mu^2}{\sqrt{2}}$, $Z_\mu^0 = \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2+g'^2}}$, $A_\mu = \frac{g'A_\mu + gB_\mu}{\sqrt{g^2+g'^2}}$]

[**NOTE:::** With $\sin\theta_w = \frac{g'}{g_Z}$, $\cos\theta_w = \frac{g}{g_Z}$ with $g_Z = \sqrt{g^2+g'^2}$, $\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$]

[**NOTE:::** $m_{W^\pm} = \frac{gv}{2}$, $m_Z = \frac{g_Z v}{2}$, $m_A = 0$]

[**NOTE:::** $\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = \frac{g_Z^2/m_Z^2}{g^2/m_W^2} = \frac{\text{NC fermion coupling}}{\text{CC Fermion coupling}} = 1$ with the Higgs doublet!]

[**Q.** Show $\rho = \frac{I(I+1) - I_3^2}{2I_3^2}$ with H weak isospin I and VEV direction I_3.
$\rho = 1$ is consistent with $(I, I_3) = (\frac{1}{2}, \pm\frac{1}{2})_{\rm SM}, (3, \pm 2), (\frac{25}{2}, \pm\frac{15}{2})$.]

SM(1) Leptons

- $$\ell_e = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \sim (1, 2, y_\ell), e_R \sim (1, 1, y_{e_R})$$
  $$\ell_\mu = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix} \sim (1, 2, y_\ell), \mu_R \sim (1, 1, y_{e_R})$$
  $$\ell_\tau = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \sim (1, 2, y_\ell), \tau_R \sim (1, 1, y_{e_R})$$
  [**NOTE:::** $y_\ell = -1/2, y_{e_R} = -1$: They have exactly same quantum numbers! The difference is in interaction with Higgs (thus mass).]
- $D_\mu \ell = (\partial_\mu - ig\frac{\sigma_i}{2} \cdot W^i_\mu - ig' y_\ell B_\mu)\ell_L$ [**NOTE:::** no strong interaction]
- $D_\mu e_R = (\partial_\mu - ig' y_{e_R} B_\mu) e_R$ [**NOTE:::** no strong, no $SU(2)$ interaction]
- $gT^i W^i_\mu + g' Y B_\mu = \frac{g}{\sqrt{2}}(T^+ W^+_\mu + T^- W^-_\mu) + gT^3 W^3_\mu + g' y B_\mu$ where $T^\pm = T_1 \pm iT^2$. [**NOTE:::** Diagonal part with $Q = T_3 + y$: $d = gT^3 W^3 + g'(Q - T^3)B = T^3(gW^3 - g'B) + g' QB$. $gW^3 - g'B = g_Z Z$ and $B = -s_w Z + c_w A$ provides $d = g_Z(T^3 - Q\sin^2\theta_w)Z_\mu + eQA_\mu$ where $e = g' \cos\theta_w$.]

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(T^+ W^+_\mu + T^- W^-_\mu) - ig_Z(T_3 - Q\sin^2\theta_w)Z_\mu - ieQA_\mu$$

SM(2) Quarks

- $$Q_1 = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (3, 2, y_Q),\, u_R \sim (3, 1, y_u),\, d_R \sim (3, 1, y_d)$$
  $$Q_2 = \begin{pmatrix} c_L \\ s_L \end{pmatrix} \sim (3, 2, y_Q),\, c_R \sim (3, 1, y_u),\, s_R \sim (3, 1, y_d)$$
  $$Q_3 = \begin{pmatrix} t_L \\ b_L \end{pmatrix} \sim (3, 2, y_Q),\, t_R \sim (3, 1, y_u),\, b_R \sim (3, 1, y_d)$$

  [**NOTE:::** $y_Q = \frac{1}{6}, y_u = \frac{2}{3}, y_d = -\frac{1}{3}$: They have exactly same quantum numbers! The difference is in interaction with Higgs (thus mass).]
- $Q_u = 2/3$, $Q_d = -1/3$ by $Q = T_3 + y$.
- $D_\mu = \partial_\mu - ig_s \frac{\lambda_a}{2} g^a_\mu - ig \frac{\sigma_i}{2} \cdot W^i_\mu - ig' y_\ell B_\mu$

$$D_\mu = \partial_\mu - ig_s \frac{\lambda_a}{2} g^a_\mu - i \frac{g}{\sqrt{2}} (T^+ W^+_\mu + T^- W^-_\mu) - ig_Z (T_3 - Q \sin^2 \theta_w) Z_\mu - ieQA_\mu$$

SM(3) Yukawa couplings

- $\mathcal{L}_{mass} \sim m\bar{\psi}_L \psi_R + h.c.$
- but! $\bar{e}_L e_R$ is not allowed [Q. why?]
- $\bar{\ell}_L H e_R$, $\bar{Q}_L \tilde{H} u_R$ and $\bar{Q}_L H d_R$ are allowed. $\tilde{H} = i\sigma_2 H^*$.
- General Yukawa interaction allows all the inter-generation mixings:

$$\mathcal{L}_{\rm yuk} = -y^e_{ij}\overline{\ell_i}_L H e_{jR} - y^u_{ij}\overline{Q_i}_L \tilde{H} u_{jR} - y^d_{ij}\overline{Q_i}_L H d_{jR} \tag{8}$$

- y^e, y^u, y^d are all complex valued matrices.

SM(4) Quark masses and mixings

Let's think of n generations of quarks $\{Q_{iL}, u_{iR}, d_{iR}\}_{i=1,2,\cdots,n}$

- Kinetic+gauge term for quarks is symmetric under $U(n) \times U(n) \times U(n)$ (global):

$$Q_L \to U_Q Q_L, \, u_R \to U_u u_R, \, d_R \to U_d d_R$$

[**NOTE:::** gauge bosons are blind of generations (universality)]

- Yukawa interactions break the symmetry

$$\mathcal{L}_{\text{yuk}} = -y^u_{ij} \overline{Q_i}_L \tilde{H} u_{jR} - y^d_{ij} \overline{Q_i}_L H d_{jR} + h.c. \tag{9}$$

[**NOTE:::** Yukawa's are the only source of symmetry breaking: Minimal flavor violation]

- y^u, y^d have $4n^2$ real parameters in total. But not all of them are physically observable. The symmetry of the kinetic term implies a kind of reparametrization invariance:

$$y^d \to U_Q^\dagger y_d U_d, \, y^u \to U_Q^\dagger y_u U_u$$

leaves the physics unchanged.

SM(5) Quark masses and mixings

- The $U(1)$ subgroup of $U(n)^3$:

$$U_Q = U_u = U_d = e^{i\theta}$$

does not change y_u and y_d. The effective reparametrization group is thus $U(n)^3/U(1)$ thus the space of physical parameters is

$$\mathbb{R}^4/\{U(n)^3/U(1)\}$$

with its dimension $= 4n^2 - (3n^2 - 1) = n^2 + 1$.

n^2+1 dim	$2n$ masses	$\frac{n(n-1)}{2}$ mixing angles	$\frac{(n-1)(n-2)}{2}$ phases(CPV)
$n=2$	4	1	0
$n=3$	6	3	1
$n=4$	8	6	3

SM(6) Quark masses and mixings

- Quark mass matrix

$$M_d = y_d \frac{v}{\sqrt{2}}, M_u = y_u \frac{v}{\sqrt{2}}$$

$$\mathcal{L}_m = -\bar{d}_L M_d d_R - \bar{u}_L M_u u_R$$

[**NOTE:::** Theorem: A complex $n \times n$ matrix M can be diagonalized by bi-similar transformation: $M = UDU'^{\dagger}$ where U and U' are unitary, D is diagonal, all elements ≥ 0.]

- Diagonalization

$$M_u = U_L M_u^{\rm diag} U_R^{\dagger}, M_d = V_L M_d^{\rm diag} V_R^{\dagger}$$

With mass eignstates $\hat{u}_{R/L} = U^{\dagger}_{R/L} u_{R/L}$ and $\hat{d}_{R/L} = V^{\dagger}_{R/L} d_{R/L}$,

$$\mathcal{L}_m = -\bar{\hat{u}}_L M_u^{\rm diag} \hat{u}_R - -\bar{\hat{d}}_L M_d^{\rm diag} \hat{d}_R + h.c.$$

SM(6) Quark masses and mixings

- Charged current interactions:

$$\begin{aligned}
\mathcal{L}_{CC} &= \frac{g}{\sqrt{2}}\overline{(u,d)_L}(T^+ W\!\!\!/^+ + T^- W\!\!\!/^-)\begin{pmatrix}u_L\\ d_L\end{pmatrix} \\
&= \frac{g}{\sqrt{2}}\left(\overline{u}_L W\!\!\!/^+ d_L + \overline{d}_L W\!\!\!/^- u_L\right) \\
&= \frac{g}{\sqrt{2}}\left(\overline{\hat{u}}_L U_L^\dagger V_L W\!\!\!/^+ \hat{d}_L + \overline{\hat{d}}_L V_L^\dagger U_L W\!\!\!/^- \hat{u}_L\right) \\
&= \frac{g}{\sqrt{2}}\left(\overline{\hat{u}}_L V_{\mathrm{CKM}} W\!\!\!/^+ \hat{d}_L + h.c.\right)
\end{aligned}$$

[NOTE::: CC interactions are flavor violating! $\overline{\hat{u}_L}\gamma_\mu \hat{d}'_L = \overline{\hat{u}_L}\gamma_\mu V_{\mathrm{CKM}}\hat{d}_L$]

- Neutral current interactions:

$$\mathcal{L}_{NC} \propto \overline{u_L}\gamma_\mu u_L, \overline{d_L}\gamma_\mu d_L \propto \overline{\hat{u}_L}\gamma_\mu \hat{u}_L, \overline{\hat{d}_L}\gamma_\mu \hat{d}_L \qquad (10)$$

[NOTE::: NC interactions are flavor diagonal! FCNC, GIM mechanism]

SM(7) Lepton masses

- Without having neutrino masses (or very small masses), only charged leptons got masses though Yukawa interactions and no CKM like mixings are allowed. (Lepton number conservation)
- Neutrinos however have masses possibly by a different mechanism other than conventional Higgs mechanism.
  $\sum_\nu m_\nu \lesssim 0.1\mathrm{eV} \ll m_e \ll m_t$! (Flavor hierarchy problem)

SM(7) Free parameters in the SM

- 3 gauge couplings: g_s, g, g
- 13: 9 fermion masses and 4 CKM mixings with a phase: $m_e, m_\mu, m_\tau, m_u, m_d, m_s, m_c, m_b, m_t$, 3 angles and 1 CPV phase.
- 1 Higgs vev produces 2 gauge boson masses : $(m_W = gv/2, m_Z = g_Z v/2) \sim v$ (or Higgs mass)
- 1 Higgs quartic coupling λ
- θ_{QCD} in $\theta G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a$

# of parameters	sector	parameters
3	gauge couplings	g_s, g, g'
9	fermion masses	$m_e, m_\mu, m_\tau, m_u, m_d, m_s, m_c, m_b, m_t$
4	CKM	3 angles + 1 phase
2	Higgs vev and quartic coupling	v, λ
(1)	QCD theta term in $\theta G^a_{\mu\nu}\tilde{G}^{\mu\nu}_a$	$\theta_{\rm QCD} \ll 1$
total 18(+1)		all measured!!

The End