Detailed Balance Breaking and the Housekeeping Entropy Production in Continuous Stochastic Dynamics

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3rd East Asia Joint Seminar on Statistical Physics,
KIAS, 14-17 October, 2015

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Integral Fluctuation Theorem

- IFT for

$$\mathcal{R}[\mathbf{q}(t)] = \ln \frac{\mathcal{P}[\mathbf{q}(t)]}{\hat{\mathcal{P}}[\hat{\mathbf{q}}(t)]}$$

  - $\mathcal{P}[\mathbf{q}(t)]$: forward path probability density for a stochastic path $\mathbf{q}(t)$ for $0 \leq t \leq \tau$
  - $\hat{\mathcal{P}}[\hat{\mathbf{q}}(t)]$: path probability for the transformed path $\hat{\mathbf{q}}(t)$ with a specified time dependence.
  - $\mathbf{q}(t) \to \hat{\mathbf{q}}(t)$ has the Jacobian of unity.
  - $\langle \exp(-\mathcal{R}[\mathbf{q}]) \rangle = 1$, follows from the normalization of $\hat{\mathcal{P}}$.
  - $\langle \mathcal{R}[\mathbf{q}] \rangle \geq 0$ from Jensen's inequality
- Total entropy production:
  - $\hat{\mathbf{q}}(t) \to \boldsymbol{\epsilon}\mathbf{q}(\tau - t)$ [$\epsilon = \pm 1$ for even (odd) variables like position (momentum)]
  - $\hat{\mathcal{P}}$: reverse protocol
  - $\mathcal{R} \to \Delta S_{\rm tot}$

Steady State Thermodynamics

Even variables only

$$\Delta S_{\text{total}} = \Delta S_{\text{excess}} + \Delta S_{\text{housekeeping}}$$

- ΔS_{excess}
  - Hatano and Sasa (2001)
  - produced during transitions between stationary states
  - satisfies IFT on its own
  - transient component $\sim$ nonadiabatic EP (Esposito and Van den Broeck (2010))
- $\Delta S_{\text{housekeeping}}$
  - Speck and Seifert (2005)
  - necessary to maintain noneq. steady state
  - satisifes IFT on its own
  - adiabatic component
- $\hat{\mathcal{P}} \leftarrow$ Dual (Adjoint) Dynamics

In the presence of **odd** parity variables

- Spinney and Ford (2012):
  - $\Delta S_{\rm excess}$ can be separated out & shown to satisfy IFT
  - $\Delta S_{\rm housekeeping}$ does not satisfy IFT
  - $\Delta S_{\rm housekeeping} = \Delta S_2 + \Delta S_3$
    - ΔS_2 satisfies IFT; nontransient
    - ΔS_3 does not satisfy IFT; parity asymmetry of SSD; transient
- Lee, Kwon and Park (2013):
  - Master eq. for discrete variables
  - $\Delta S_{\rm housekeeping} = \Delta S_{\rm bDB} + \Delta S_{\rm as}$
  - $\Delta S_{\rm bDB}$ satisfies IFT; measures directly DB breakage; nontransient
  - $\Delta S_{\rm as}$ does not satisfy IFT; parity asymmetry of SSD; nontransient
- Question: Can we separate out $\Delta S_{\rm bDB}$ for continuous case?

Main Results

- ΔS_{bDB} for continuous variables is ill-defined & needs regularization
- Splitting of $\Delta S_{\mathrm{housekeeping}}$ is not unique
- It can be done in many ways (described by a parameter σ)
- Spinney & Ford $\leftrightarrow$ $\sigma = 0$

Setup

- Fokker-Planck equation for $\mathbf{q} = \{q_1, q_2, \ldots, q_n\}$

$$\partial_t \rho(\mathbf{q}, t) = [-\partial_i A_i(\mathbf{q}) + \partial_i \partial_j D_{ij}(\mathbf{q})]\, \rho(\mathbf{q}, t)$$
$$= \int d\mathbf{q}'\, \omega[\mathbf{q}, \mathbf{q}'] \rho(\mathbf{q}', t),$$

  where the transition rate is

$$\omega[\mathbf{q}, \mathbf{q}'] = [-\partial_i A_i(\mathbf{q}) + \partial_i \partial_j D_{ij}(\mathbf{q})]\, \delta(\mathbf{q} - \mathbf{q}')$$

- Parity under time-reversal:

$$\boldsymbol{\epsilon}\mathbf{q} = \{\epsilon_1 q_1, \epsilon_2 q_2, \ldots, \epsilon_n q_n\}, \qquad \epsilon_i = \pm 1$$

- Stationary state $\rho_{\rm s}(\mathbf{q})$ satisfies $\int d\mathbf{q}' \omega[\mathbf{q}, \mathbf{q}'] \rho_{\rm s}(\mathbf{q}') = 0$.

Example
Brownian Dynamics

- $\mathbf{q} = (\mathbf{x}, \mathbf{p}), \quad \epsilon\mathbf{q} = (\mathbf{x}, -\mathbf{p})$

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m},$$
$$\dot{\mathbf{p}} = -\mathrm{G}\frac{\mathbf{p}}{m} + \mathbf{f}(\mathbf{q};\lambda) + \boldsymbol{\xi}(t),$$

where $\mathrm{G} = \{G_{ij}\}$, $(\mathrm{G}\mathbf{p})_i = \sum_j G_{ij} p_j$, and

$$\langle \boldsymbol{\xi}(t) \rangle = 0, \qquad \langle \xi_i(t)\xi_j(t') \rangle = 2D_{ij}\delta(t-t')$$

- Fokker-Planck (Kramer's) Eq.

$$\partial_t \rho(\mathbf{q},t) = -\left[\partial_{\mathbf{x}}\frac{\mathbf{p}}{m} + \partial_{\mathbf{p}}\left(-\mathrm{G}\frac{\mathbf{p}}{m} + \mathbf{f}(\mathbf{q}) - \mathrm{D}\partial_{\mathbf{p}}\right)\right]\rho(\mathbf{q},t).$$

Review: $\Delta S_{\rm tot}$

$$\Delta S_{\rm tot} = \Delta S_{\rm sys} + \Delta S_{\rm env}, \tag{1}$$

where $\Delta S_{\rm sys} = -\ln \rho(\mathbf{q}(\tau), \tau) + \ln \rho(\mathbf{q}(0), 0)$ is the system entropy change. The environmental EP $\Delta S_{\rm env}$:

$$\Delta S_{\rm env} = \ln \frac{\Gamma[\mathbf{q}(\tau)|\mathbf{q}(0)]}{\Gamma^{\rm R}[\epsilon\mathbf{q}(0)|\epsilon\mathbf{q}(\tau)]}, \tag{2}$$

During an infinitesimal time interval,

$$dS_{\rm env} = \ln \frac{\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t]}{\Gamma[\epsilon\mathbf{q}, t+dt|\epsilon\mathbf{q}', t]}, \tag{3}$$

where

$$\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t] = \delta(\mathbf{q}' - \mathbf{q}) + (dt)\omega[\mathbf{q}', \mathbf{q}] \tag{4}$$

is the conditional probability from the state $\mathbf{q}$ to $\mathbf{q}'$ during dt.

$$\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t] = \frac{1}{(4\pi dt)^{N/2}|\det(\mathrm{D})|^{1/2}} \times \exp\Big[-\frac{dt}{4}h_i D_{ij}^{-1} h_j - \alpha(dt)\partial_i A_i + \alpha^2(dt)\partial_i\partial_j D_{ij}\Big], \tag{5}$$

where

$$h_i = \dot{q}_i - A_i + 2\alpha\partial_j D_{ij} \tag{6}$$

with $\dot{\mathbf{q}} = (\mathbf{q}' - \mathbf{q})/dt$ and α indicates the discretization scheme. Note that all the functions are evaluated at $\mathbf{q}^{(\alpha)} = \mathbf{q} + \alpha(\mathbf{q}' - \mathbf{q})$, where $\alpha = 0$ (1/2) corresponds to the Itô (Stratonovich) discretization.

To get a well-defined ΔS_{env}, we have to restrict to

$$\epsilon_i\epsilon_j D_{ij}(\boldsymbol{\epsilon}\mathbf{q}) = D_{ij}(\mathbf{q}). \tag{7}$$

Otherwise, we have $O(1)$ contribution from the determinant

$$
\begin{aligned}
dS_{\rm env} &= dt\Big(\dot{q}_i - A_i^{\rm rev}(\bar{\mathbf{q}})\Big)D_{ij}^{-1}(\bar{\mathbf{q}})\Big(A_j^{\rm ir}(\bar{\mathbf{q}}) - \partial_k D_{jk}(\bar{\mathbf{q}})\Big) \\
&\quad -dt\;\partial_i A_i^{\rm rev}(\bar{\mathbf{q}}),
\end{aligned} \tag{8}
$$

where $\bar{\mathbf{q}} = \mathbf{q} + \frac{1}{2}(\mathbf{q}' - \mathbf{q})$.

$$
A_i^{\rm rev}(\mathbf{q}) = \frac{1}{2}(A_i(\mathbf{q}) - \epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})), \qquad A_i^{\rm ir}(\mathbf{q}) = \frac{1}{2}(A_i(\mathbf{q}) + \epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})).
$$

➢ For the interpretation of $\Delta S_{\rm env}$ for Brownian dynamics, see C. Kwon et al, arXiv: 1506.02339.

➢ Average EP rates:

$$
\left\langle \frac{dS_{\rm tot}}{dt} \right\rangle = \int d\mathbf{q}\, \frac{j_i^{\rm ir}(\mathbf{q})\; D_{ij}^{-1}(\mathbf{q})\; j_j^{\rm ir}(\mathbf{q})}{\rho(\mathbf{q})}. \tag{9}
$$

$$
j_i^{\rm rev}(\mathbf{q}) = A_i^{\rm rev}(\mathbf{q})\rho(\mathbf{q}), \qquad j_i^{\rm ir}(\mathbf{q}) = (A_i^{\rm ir}(\mathbf{q}) - \partial_j D_{ij}(\mathbf{q}))\rho(\mathbf{q}).
$$

Review: $\Delta S_{\rm excess}$

The $*$-process:

$$\omega^*[\mathbf{q}', \mathbf{q}] \equiv \omega[\mathbf{q}, \mathbf{q}'] \frac{\rho^{\rm s}(\mathbf{q}')}{\rho^{\rm s}(\mathbf{q})}, \tag{10}$$

or $\Gamma^*[\mathbf{q}', t + dt|\mathbf{q}, t] \equiv \Gamma[\mathbf{q}, t + dt|\mathbf{q}', t]\rho^{\rm s}(\mathbf{q}')/\rho^{\rm s}(\mathbf{q})$.

$$dS_1 \equiv \ln \frac{\Gamma[\mathbf{q}', t + dt|\mathbf{q}, t]}{\Gamma^*[\mathbf{q}, t + dt|\mathbf{q}', t]} = \ln \frac{\rho^{\rm s}(\mathbf{q}')}{\rho^{\rm s}(\mathbf{q})} = -dt\ \dot{q}_i \partial_i \phi(\bar{\mathbf{q}}). \tag{11}$$

$$\Delta S_{\rm excess} = \Delta S_{\rm sys} + \int dS_1 \tag{12}$$

where $\rho^{\rm s}(\mathbf{q}) = \exp(-\phi(\mathbf{q}))$.
If the initial and final distributions are also $\rho^{\rm s}$,

$$\Delta S_{\rm excess} = \int_0^\tau dt\ \dot{\lambda}(t) \frac{\partial \phi}{\partial \lambda}, \tag{13}$$

which is exactly the familiar Hatano-Sasa expression.

➢ Average excess EP rate:

$$\frac{d}{dt}\langle \Delta S_{\rm excess}\rangle = -\int d\mathbf{q}\,\left(\partial_t \rho(\mathbf{q})\right) \ln \frac{\rho(\mathbf{q})}{\rho^{\rm s}(\mathbf{q})}, \tag{14}$$

$$= \int d\mathbf{q}\,\left\{\frac{j_i(\mathbf{q})}{\rho(\mathbf{q})} - \frac{j_i^{\rm s}(\mathbf{q})}{\rho^{\rm s}(\mathbf{q})}\right\} D_{ij}^{-1}(\mathbf{q}) \left\{\frac{j_j(\mathbf{q})}{\rho(\mathbf{q})} - \frac{j_j^{\rm s}(\mathbf{q})}{\rho^{\rm s}(\mathbf{q})}\right\}\rho(\mathbf{q}),$$

where the stationary state currents $\mathbf{j}^{\rm s} = \mathbf{j}^{\rm s,rev} + \mathbf{j}^{\rm s,ir}$ are defined similarly. One just replaces $\rho(\mathbf{q})$ by $\rho^{\rm s}(\mathbf{q})$ in the those expressions.

Main topic: Housekeeping EP

$$\Delta S_{\rm hk} = \Delta S_{\rm tot} - \Delta S_{\rm excess}$$

It cannot be expressed as a log-ratio of two probabilities: no IFT for $\Delta S_{\rm hk}$.

➤ Spinney and Ford [PRE **85**, 051113 (2012)]: a part of $\Delta S_{\rm hk}$

$$dS_2 \equiv \ln \frac{\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t]}{\Gamma^*[\epsilon\mathbf{q}', t+dt|\epsilon\mathbf{q}, t]}. \tag{15}$$

No appraent relation to DB breaking

Detailed Balance

In the presence of odd-parity variables, the DB condition reads

$$\omega[\mathbf{q}', \mathbf{q}]\rho_{\mathrm{s}}(\mathbf{q}) = \omega[\boldsymbol{\epsilon}\mathbf{q}, \boldsymbol{\epsilon}\mathbf{q}']\rho_{\mathrm{s}}(\boldsymbol{\epsilon}\mathbf{q}'), \tag{16}$$

In order to measure the departure from the DB, we define the adjoint process

$$\omega^{\dagger}[\mathbf{q}', \mathbf{q}] \equiv \frac{\omega[\boldsymbol{\epsilon}\mathbf{q}, \boldsymbol{\epsilon}\mathbf{q}']\rho_{\mathrm{s}}(\boldsymbol{\epsilon}\mathbf{q}')}{\rho_{\mathrm{s}}(\mathbf{q})} \tag{17}$$

such that the DB condition is equivalent to the condition

$$\omega[\mathbf{q}', \mathbf{q}] = \omega^{\dagger}[\mathbf{q}', \mathbf{q}].$$

This is a well-defined stochastic process as one can easily see that

$$\int d\mathbf{q}'\omega^{\dagger}[\mathbf{q}', \mathbf{q}] = 0$$

follows from the stationarity of ρ_{s}.

$$\omega^\dagger[\mathbf{q}',\mathbf{q}] = \left[-\partial_i' A_i^\dagger(\mathbf{q}') + \partial_i'\partial_j' D_{ij}^\dagger(\mathbf{q}')\right]\delta(\mathbf{q}'-\mathbf{q}) \tag{18}$$

$$A_i^\dagger(\mathbf{q}) = -\epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})e^{\phi_\mathrm{A}(\mathbf{q})} + \frac{2\epsilon_i\epsilon_j}{\rho_\mathrm{s}(\mathbf{q})}\partial_j D_{ij}(\boldsymbol{\epsilon}\mathbf{q})\rho_\mathrm{s}(\boldsymbol{\epsilon}\mathbf{q}) \tag{19}$$

$$D_{ij}^\dagger(\mathbf{q}) = \epsilon_i\epsilon_j D_{ij}(\boldsymbol{\epsilon}\mathbf{q})e^{\phi_\mathrm{A}(\mathbf{q})}, \tag{20}$$

where

$$\boxed{\phi_\mathrm{A}(\mathbf{q}) \equiv \phi(\mathbf{q}) - \phi(\boldsymbol{\epsilon}\mathbf{q})}, \qquad \rho_\mathrm{s}(\mathbf{q}) = \exp(-\phi(\mathbf{q})). \tag{21}$$

Recall that

$$\epsilon_i\epsilon_j D_{ij}(\boldsymbol{\epsilon}\mathbf{q}) = D_{ij}(\mathbf{q})$$

Then we have

$$A_i^\dagger(\mathbf{q}) = -\epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})e^{\phi_\mathrm{A}(\mathbf{q})} + \frac{2}{\rho_\mathrm{s}(\mathbf{q})}\partial_j(D_{ij}(\mathbf{q})\rho_\mathrm{s}(\boldsymbol{\epsilon}\mathbf{q})), \tag{22}$$

$$D_{ij}^\dagger(\mathbf{q}) = D_{ij}(\mathbf{q})e^{\phi_\mathrm{A}(\mathbf{q})} \tag{23}$$

DB condition

The DB condition can be obtained by setting $A_i^\dagger = A_i$ and $D_{ij}^\dagger = D_{ij}$.

$$\boxed{\phi_{\rm A}(\mathbf{q}) = 0}, \tag{24}$$

$$\frac{1}{2}\Big(A_i(\mathbf{q})\rho_{\rm s}(\mathbf{q}) + \epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})\rho_{\rm s}(\boldsymbol{\epsilon}\mathbf{q})\Big) - \partial_j\big(D_{ij}(\mathbf{q})\rho_{\rm s}(\boldsymbol{\epsilon}\mathbf{q})\big) = 0. \tag{25}$$

Using the first condition (parity symmetry), one can rewrite the second one as

$$\boxed{j_i^{\rm s,ir}(\mathbf{q}) \equiv \frac{1}{2}\big(A_i(\mathbf{q}) + \epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})\big)\rho_{\rm s}(\mathbf{q}) - \partial_j\big(D_{ij}(\mathbf{q})\rho_{\rm s}(\mathbf{q})\big) = 0}, \tag{26}$$

which gives the vanishing irreversible current in the stationary state.
Define

$$A_i^{\rm ir}(\mathbf{q}) \equiv \frac{1}{2}\big(A_i(\mathbf{q}) + \epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q})\big)$$

$\Delta S_{\rm bDB}$

➢ Lee, Kwon and Park [PRL **110**, 050602 (2013)]: a part of $\Delta S_{\rm hk}$

$$dS_{\rm bDB} \equiv \ln \frac{\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t]}{\Gamma^\dagger[\mathbf{q}', t+dt|\mathbf{q}, t]}.$$

- Directly measures the brekage of DB
- Constructed for discrete jumping process described by Master equation.

Problem of $\Delta S_{\rm bDB}$ / Continuous Case

- Recall

$$D^\dagger_{ij}(\mathbf{q}) = D_{ij}(\mathbf{q})e^{\phi_{\rm A}(\mathbf{q})}$$

  The ratio

$$\left[\frac{\det(e^{\phi_{\rm A}(\bar{\mathbf{q}})}\mathbb{D}(\bar{\mathbf{q}}))}{\det(\mathbb{D}(\bar{\mathbf{q}}))}\right]^{\frac{1}{2}}$$

  is not of O(dt).

- In particular, even if we start from an additive noise in the forward process, the dagger process becomes one with the multiplicative noise. In such cases the path probability ratio during the time interval dt results in $O(1)$ rather than $O(dt)$.
- This in turn makes the entropy production $\Delta S_{\rm bDB}$ for a finite time interval diverge.

Regularization

- We introduce another stochastic process for which the path probability ratio taken with the forward process for a finite time interval is well defined.
- We also want this entropy production to represent the departure from the DB as closely as possible.
- Introduce a stochastic process $\omega_h^\ddagger$ defined as

$$\omega_h^\ddagger[\mathbf{q}', \mathbf{q}] \equiv \omega[\epsilon\mathbf{q}, \epsilon\mathbf{q}'] \frac{h(\mathbf{q}')}{h(\mathbf{q})} - \delta(\mathbf{q} - \mathbf{q}') \frac{1}{h(\mathbf{q})} \int d\mathbf{q}'' \omega[\epsilon\mathbf{q}, \epsilon\mathbf{q}''] h(\mathbf{q}'').$$

  where $h(\mathbf{q})$ is an arbitrary function to be specified later.
- The last term should be there to enforce the stochasticity, $\int d\mathbf{q}' \omega_h^\ddagger[\mathbf{q}', \mathbf{q}] = 0$.
- The choice of $h(\mathbf{q}) = \rho_{\mathrm{s}}(\epsilon\mathbf{q})$ was made by Spinney and Ford. In that case, the second term on the right hand side vanishes.

- We have

$$A_i^{\ddagger}(\mathbf{q}) = -\epsilon_i A_i(\boldsymbol{\epsilon}\mathbf{q}) + \frac{2}{h(\mathbf{q})}\partial_j(D_{ij}(\mathbf{q})h(\mathbf{q})), \tag{27}$$

$$D_{ij}^{\ddagger}(\mathbf{q}) = D_{ij}(\mathbf{q}). \tag{28}$$

- Because of Eq. (28), $\ln \Gamma/\Gamma^{\ddagger}$ will now be $O(dt)$.
- As a part of ΔS_{hk}, we now study the EP from

$$dS_h \equiv \ln \frac{\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t]}{\Gamma_h^{\ddagger}[\mathbf{q}', t+dt|\mathbf{q}, t]}. \tag{29}$$

- ΔS_h measures the breakage of $\omega = \omega_h^{\ddagger}$ condition
- We want this to be a measure of DB breakage

Special Choice of h

- Let us consider for some number σ,

$$h(\mathbf{q}) = h_\sigma(\mathbf{q}) \equiv (\rho^{\mathrm{s}}(\mathbf{q}))^\sigma \left(\rho^{\mathrm{s}}(\epsilon\mathbf{q})\right)^{1-\sigma}. \tag{30}$$

- $\sigma = 0 \to$ Spinney and Ford's dS_2.
- For general values of σ, $\omega = \omega_h^\ddagger$ implies

$$(1-2\sigma) D_{ij}(\mathbf{q}) \partial_j \phi_{\mathrm{A}}(\mathbf{q}) = 0. \tag{31}$$

and

$$j_i^{\mathrm{s,ir}}(\mathbf{q}) = 0, \tag{32}$$

- For $\sigma \neq 1/2$, one can show that Eq. (31) gives the parity symmetry, $\phi_{\mathrm{A}} = 0$ if D^{-1} exists at least in the subspace spanned by the odd-parity variables (e.g. Brownian dynamics).
- Therefore, $\omega = \omega_h^\ddagger$ is equivalent to DB.

Regularized ΔS_{bDB}

$$dS_\sigma \equiv \ln \frac{\Gamma[\mathbf{q}', t+dt|\mathbf{q}, t]}{\Gamma^\ddagger_{h_\sigma}[\mathbf{q}', t+dt|\mathbf{q}, t]}.$$

$$\begin{aligned} dS_\sigma = & \; dt \left(\dot{q}_i - A_i^{\mathrm{rev}}(\bar{\mathbf{q}}) + D_{ik}(\bar{\mathbf{q}})(\partial_k \psi_\sigma(\bar{\mathbf{q}}))\right) D^{-1}_{ij}(\bar{\mathbf{q}}) \\ & \times \left(A_j^{\mathrm{ir}}(\bar{\mathbf{q}}) - \partial_l D_{jl}(\bar{\mathbf{q}}) + D_{jl}(\bar{\mathbf{q}})\partial_l \psi_\sigma(\bar{\mathbf{q}})\right) \\ & - dt\, \partial_i \left(A_i^{\mathrm{ir}}(\bar{\mathbf{q}}) - \partial_k D_{ik}(\bar{\mathbf{q}}) + D_{ik}(\bar{\mathbf{q}})\partial_k \psi_\sigma(\bar{\mathbf{q}})\right), \end{aligned} \tag{33}$$

where we define

$$\psi_\sigma(\mathbf{q}) \equiv -\ln h_\sigma(\mathbf{q}) = \phi(\epsilon\mathbf{q}) + \sigma\phi_{\mathrm{A}}(\mathbf{q}). \tag{34}$$

Average EP rates

$$\left\langle \frac{dS_\sigma}{dt} \right\rangle = \int d\mathbf{q} \left\{ \frac{\epsilon_i j_i^{\mathrm{s,ir}}(\boldsymbol{\epsilon}\mathbf{q})}{\rho^{\mathrm{s}}(\boldsymbol{\epsilon}\mathbf{q})} + \sigma D_{ik}(\mathbf{q}) \partial_k \phi_{\mathrm{A}}(\mathbf{q}) \right\}$$
$$\times D_{ij}^{-1}(\mathbf{q}) \left\{ \frac{\epsilon_j j_j^{\mathrm{s,ir}}(\boldsymbol{\epsilon}\mathbf{q})}{\rho^{\mathrm{s}}(\boldsymbol{\epsilon}\mathbf{q})} + \sigma D_{jl}(\mathbf{q}) \partial_l \phi_{\mathrm{A}}(\mathbf{q}) \right\} \rho(\mathbf{q}), \qquad (35)$$

or

$$\left\langle \frac{dS_\sigma}{dt} \right\rangle = \int d\mathbf{q} \left\{ \frac{j_i^{\mathrm{s,ir}}(\mathbf{q})}{\rho^{\mathrm{s}}(\mathbf{q})} - (1-\sigma) D_{ik}(\mathbf{q}) \partial_k \phi_{\mathrm{A}}(\mathbf{q}) \right\}$$
$$\times D_{ij}^{-1}(\mathbf{q}) \left\{ \frac{j_j^{\mathrm{s,ir}}(\mathbf{q})}{\rho^{\mathrm{s}}(\mathbf{q})} - (1-\sigma) D_{jl}(\mathbf{q}) \partial_l \phi_{\mathrm{A}}(\mathbf{q}) \right\} \rho(\mathbf{q}). \qquad (36)$$

The Remaining ΔS_{hk}

- $$\Delta S_{\mathrm{tot}} = \Delta S_{\mathrm{excess}} + \Delta S_{\sigma} + \Delta S^{\sigma}_{\mathrm{as}} \tag{37}$$
- $$dS^{\sigma}_{\mathrm{as}} = \ln\left[\frac{\Gamma^*[\mathbf{q},t+dt|\mathbf{q}',t]}{\Gamma[\mathbf{q}',t+dt|\mathbf{q},t]}\frac{\Gamma^{\ddagger}_{h_\sigma}[\mathbf{q}',t+dt|\mathbf{q},t]}{\Gamma[\boldsymbol{\epsilon}\mathbf{q},t+dt|\boldsymbol{\epsilon}\mathbf{q}',t]}\right]. \tag{38}$$
- This cannot be expressed as the log-ratio of the two conditional probabilities, and therefore there is no IFT for $\Delta S^{\sigma}_{\mathrm{as}}$.
- The average rate of this EP is

$$\left\langle \frac{dS^{\sigma}_{\mathrm{as}}}{dt} \right\rangle = \int d\mathbf{q}\, \Big[\phi_{\mathrm{A}}(\mathbf{q})\partial_t\rho(\mathbf{q}) \tag{39}$$

$$-\sigma(\partial_i\phi_{\mathrm{A}}(\mathbf{q}))\left\{\frac{2\epsilon_i j_i^{\mathrm{s,ir}}(\boldsymbol{\epsilon}\mathbf{q})}{\rho^{\mathrm{s}}(\boldsymbol{\epsilon}\mathbf{q})} + \sigma D_{ij}(\mathbf{q})\partial_j\phi_{\mathrm{A}}(\mathbf{q})\right\}\rho(\mathbf{q})\Big].$$

- It vanishes when $\phi_{\mathrm{A}} = 0$.

Example

As an example, we consider the one dimensional system driven by a constant force F. We have $\dot{x} = p/m$ and

$$\dot{p} = -\frac{G}{m}p + F + \xi(t) \tag{40}$$

with $\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$ and $D = GT$. Suppose that the system has reached the stationary state described by the distribution

$$\rho^{\mathrm{s}}(p) = \sqrt{\frac{1}{2\pi m T}} \exp\left[-\frac{(p - \langle p \rangle_{\mathrm{s}})^2}{2mT} \right], \tag{41}$$

where

$$\langle p \rangle_{\mathrm{s}} = m\frac{F}{G}. \tag{42}$$

In this case, we have

$$\phi_{\mathrm{A}}(p) = -\frac{2\langle p \rangle_{\mathrm{s}} p}{mT}, \tag{43}$$

$$D\partial_p \phi_{\mathrm{A}}(p) = -2\frac{G}{m}\langle p\rangle_{\mathrm{s}}. \tag{44}$$

The combination in Eq. (35) is given by

$$\frac{j_p^{\mathrm{s,ir}}(-p)}{\rho^{\mathrm{s}}(-p)} = \frac{G}{m}p - D\partial_p\phi(-p) = -\frac{G}{m}\langle p\rangle_{\mathrm{s}}. \tag{45}$$

We have the EP rate in the steady state as

$$\left\langle \frac{dS_\sigma}{dt} \right\rangle_{\mathrm{s}} = (2\sigma-1)^2\frac{G^2}{Dm^2}\langle p\rangle_{\mathrm{s}}^2 = (2\sigma-1)^2\frac{F^2}{D}. \tag{46}$$

Recall that we are considering the case where $\sigma \neq 1/2$. From Eq. (39), we have

$$\left\langle \frac{dS_{\mathrm{as}}^\sigma}{dt} \right\rangle_{\mathrm{s}} = -4\sigma(\sigma-1)\frac{F^2}{D}, \tag{47}$$

and $\langle dS_{\mathrm{hk}}/dt\rangle_{\mathrm{s}} = \langle d(S_\sigma + S_{\mathrm{as}}^\sigma)/dt\rangle_{\mathrm{s}} = F^2/D$.

Summary

- We have shown how $\Delta S_{\rm hk}$ for the continuous stochastic dynamics in the presence of odd-parity variables can be separated into two parts with one part satisfying the IFT.
- There is one-parameter family of the EPs belonging to $\Delta S_{\rm hk}$, all of which satisfying the IFT.
- $\Delta S_{\rm bDB}$ In the continuous variable cases turns out to be ill-defined.
- By considering a generalized adjoint process, we were able to find the quantity ΔS_σ, which depends on a range of values of the parameter σ.
- When $\sigma \neq 1/2$, we may interpret all these EPs as those directly related to the breakage of DB. ($\sigma = 0 \rightarrow$ Spinney and Ford)