Fibred varieties and new surfaces with $p_g = q$.

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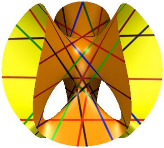

Outline

1. Fujita's theorems
2. Answer to Fujita's question
3. Hermitian curvature
4. Sketch of proof of Fujita's theorem
5. Hypergeometric integrals leading to a unitary flat bundle Q of infinite order
6. Surfaces with $p_g = q = 1$
7. New surfaces

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Fujita's first theorem

An important progress in classification theory was stimulated by a theorem of Fujita, who showed
(*On Kähler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), no. 4, 779–794*):

Theorem

If X is a compact Kähler manifold and $f : X \to B$ is a fibration onto a projective curve B (i.e., f has connected fibres), then the direct image sheaf

$$V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B))$$

is a nef vector bundle on B.
This means that each quotient bundle Q of V has degree $\deg(Q) \geq 0$*; sometimes, instead of the word nef, one uses the terminology 'V is numerically semipositive'.*

Kawamata's theorem

Soon afterwards, using Griffihts' results on Variation of Hodge Structures, since the fibre of $V := f_* \omega_{X|B}$ over a point $b \in B$ such that $X_b := f^{-1}(b)$ is smooth is the vector space $V_b = H^0(X_b, \Omega^{n-1}_{X_b})$, Kawamata improved on Fujita's result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

$$Kod(X) \geq Kod(B) + Kod(F),$$

(here F is a general fibre) showing the semipositivity also for the direct image of higher powers of the relative dualizing sheaf

$$W_m := f_*(\omega_{X|B}^{\otimes m}) = f_*(\mathcal{O}_X(m(K_X - f^* K_B))).$$

Much later, Kawamata extended his result to the case where the dimension of the base variety B is > 1.

Fujita's second theorem

In the note *The sheaf of relative canonical forms of a Kähler fiber space over a curve* Proc. Japan Acad. Ser. A Math. Sci. 54 (1978), no. 7, 183–184, Fujita announced the following stronger result, sketching the idea of proof, but referring to a forthcoming article concerning the positivity of the so-called local exponents (this article was never written).

Theorem

(Fujita 's second theorem)

Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf

$$V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B)).$$

Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.

Ample, semiample, nef

Let V be a holomorphic vector bundle over a projective curve B.

Definition

Let $p : \mathbb{P} := Proj(V) = \mathbb{P}(V^\vee) \to B$ be the associated projective bundle, and let H be a hyperplane divisor (s.t. $p_*(\mathcal{O}_\mathbb{P}(H)) = V$). Then V is said to be:
(NP) numerically semi-positive if and only if every quotient bundle Q of V has degree $deg(Q) \geq 0$,
(NEF) nef if and only if H is nef on $\mathbb{P}$,
(A) ample if and only if H is ample on $\mathbb{P}$
(SA) semi-ample if and only H is semi-ample on $\mathbb{P}$ (there is a positive multiple mH yielding a morphism).
Recall that (A) $\Rightarrow$ (SA) $\Rightarrow$ (NEF) $\Leftrightarrow$(NP).

Flat and unitary flat bundles

Definition

A flat holomorphic vector bundle on a complex manifold M is a holomorphic vector bundle $\mathcal{H} := \mathcal{O}_M \otimes_{\mathbb{C}} \mathbb{H}$, where $\mathbb{H}$ is a local system of complex vector spaces associated to a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$,

$$\mathbb{H} := (\tilde{M} \times \mathbb{C}^r)/\pi_1(M),$$

$\tilde{M}$ being the universal cover of M (so that $M = \tilde{M}/\pi_1(M)$). We say that $\mathcal{H}$ is unitary flat if it is associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$.

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Fujita's question

Recall Fujita's second theorem, for which a complete proof was given in our joint work with Michael Dettweiler (arXiv 1311.3232 and CRAS Ser. I, 352 (2014), 241-244)

Theorem

(Fujita 's second theorem)
Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B. Then $V := f_* \omega_{X|B} = f_*(\mathcal{O}_X(K_X - f^* K_B))$ splits as $V = A \oplus Q$, with A an ample vector bundle and Q a unitary flat bundle.

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Fujita posed in 1982 (Proceedings of the 1982 Taniguchi Conference) the following

Question

(Fujita) *Is the direct image $V := f_*\omega_{X|B}$ semi-ample ?*

Fujita's theorem and Fujita's question

The following result is due to Hartshorne:

Proposition

A vector bundle V on a curve is nef if and only it is numerically semi-positive, i.e., if and only if every quotient bundle Q of V has degree $deg(Q) \geq 0$, and V is ample if and only if every quotient bundle Q of V has degree $deg(Q) > 0$.

Then there is a technical result we established, which clarifies how Fujita's question is related to Fujita's II theorem

Theorem

Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold M, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semi-ample if and only if $Im(\rho)$ is finite.

Answer to Fujita's question

This is the main new result in our joint work with Dettweiler:

Theorem

There exist surfaces X of general type endowed with a fibration $f : X \to B$ onto a curve B of genus ≥ 3, and with fibres of genus 6, such that $V := f_* \omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where A is an ample rank-2 vector bundle, and the flat unitary rank-2 summands Q_1, Q_2 have infinite monodromy group (i.e., the image of ρ_j is infinite). In particular, V is not semi-ample.

Thus Fujita's question has a negative answer in general.

Cases where V is semiample.

Corollary

Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B. Then $V := f_* \omega_{X|B}$ is a direct sum $V = A \bigoplus (\oplus_{i=1}^h Q_i)$, with A ample and each Q_i unitary flat without any nontrivial degree zero quotient. Moreover,
(I) if Q_i has rank equal to 1, then it is a torsion bundle ($\exists$ m such that $Q_i^{\otimes m}$ is trivial) (Deligne)
(II) if the curve B has genus 1, then rank $(Q_i) = 1$, $\forall i$.
(III) In particular, if B has genus at most 1, then V is semi-ample.

(I) This was proven by Deligne (and by Simpson using the theorem of Gelfond-Schneider)
(II) Follows since $\pi_1(B)$ is abelian,if B has genus 1: hence every representation splits as a direct sum of 1-dimensional ones.

Flat versus unitary flat

While a unitary flat bundle is nef, the same does not hold for a flat bundle.

Theorem (C-Dettweiler) *Let $f : X \to B$ be a Kodaira fibration, i.e., X is a surface and all the fibres of f are smooth curves of genus $g \geq 2$ not all isomorphic to each other. Then $V := f_* \omega_{X|B}$ has strictly positive degree, hence $\mathcal{H} := R^1 f_*(\mathbb{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not nef.*

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Proof 1) Since all the fibres of f are smooth, $V = f_*(\Omega^1_{X|B})$ and we have an exact sequence

$$0 \to V \to \mathcal{H} \to V^\vee \to 0,$$

and it suffices to show that the degree of the quotient bundle $V^\vee$ is strictly negative, or, equivalently, $\deg(V) > 0$.

Flat versus unitary flat, cont.

We want to show that $\deg(V) > 0$, as proven by Kodaira (this follows also from the results of Kawamata and Arakelov).
We have that

$$12\deg(V) = K_X^2 - 8(b-1)(g-1),$$

where g is the genus of the fibres of f, and b is the genus of B, since f is a differentiable fibre bundle, and we have for the Euler- Poincaré characteristic of X

$$e(X) = 4(b-1)(g-1).$$

Kodaira proved that for such fibrations the topological index $\sigma(X)$(signature of the intersection form on $H^2(X, \mathbb{R})$) is positive. By the index theorem we have

$$0 < 3\sigma(X) = c_1^2(X) - 2c_2(X) = K_X^2 - 2e(X) = \deg(V).$$

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Curvature decreases in subbundles?

The example of Kodaira fibrations produces subbundles of a flat bundle (they have zero curvature) which are positively curved. Does this contradict the slogan above? Not really, the correct principle is (see the book by Griffiths and Harris): **curvature decreases in Hermitian subbundles.** The above principle is the first ingredient in the proof of the theorem mentioned above.

Theorem

Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold M, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semi-ample if and only if $Im(\rho)$ is finite.

Since $\mathcal{H}$ is unitary flat, $\mathcal{H}$ is a Hermitian holomorphic bundle, and by the principle 'curvature decreases in Hermitian subbundles' each subbundle has degree ≤ 0 and each quotient bundle W of $\mathcal{H}$ has degree ≥ 0, hence $\mathcal{H}$ is nef.

Unitary flat bundles

If $\mathcal{H}$ is unitary flat, $\mathcal{H}$ is a Hermitian holomorphic bundle, and by the principle ‘curvature decreases in Hermitian subbundles’ each subbundle has degree ≤ 0 and each quotient bundle W of $\mathcal{H}$ has degree ≥ 0, hence $\mathcal{H}$ is nef.
Moreover, by Lefschetz ’ theorem, we can reduce to the case where M is a curve. Let B be a projective curve, let $\rho : \pi_1(B) \to U(r, \mathbb{C})$ be a unitary representation, and let $\mathcal{H}_\rho$ be the associated flat holomorphic bundle. Since ρ is unitary, it is a direct sum of irreducible unitary representations $\rho_j, j = 1, \ldots k$.
Accordingly, we have a splitting

$$\mathcal{H}_\rho = \oplus_{j=1}^{k} \mathcal{H}_{\rho_j}.$$

Narasimhan and Seshadri have proven that each $\mathcal{H}_{\rho_j}$ is a stable degree zero holomorphic bundle on B. This result plays another crucial role in the proof of the above theorem.

Curvature and numerical positivity

Definition

Let (E, h) be a Hermitian vector bundle on a complex manifold M. Take the canonical Chern connection associated to the Hermitian metric h, and denote by $\Theta(E, h)$ the associated Hermitian curvature, which gives a Hermitian form on the complex vector bundle bundle $T_M \otimes E$.
Then one says that E is Nakano positive (resp.: semi-positive) if there exists a Hermitian metric h such that the Hermitian form associated to $\Theta(E, h)$ is strictly positive definite (resp.: semi-positive definite).

Remark

Umemura proved that a vector bundle V over a curve B is positive (i.e., Griffiths positive, or equivalently Nakano positive) if and only if V is ample.

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Idea of proof in the case of no singular fibres

V is a holomorphic subbundle of the holomorphic vector bundle $\mathcal{H}$ associated to the local system $\mathbb{H} := \mathcal{R}^m f_*(\mathbb{Z}_X), \ m = \dim(X) - 1$ (i.e., $\mathcal{H} = \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_B$).
The bundle $\mathcal{H}$ is flat, hence the curvature $\Theta_{\mathcal{H}}$ associated to the flat connection satisfies $\Theta_{\mathcal{H}} \equiv 0$.
We view V as a holomorphic subbundle of $\mathcal{H}$, while

$$V^\vee \cong R^m f_* \mathcal{O}_X, \ m = \dim(X) - 1$$

is a holomorphic quotient bundle of $\mathcal{H}$.
By the curvature formula for subbundles we obtain

$$\Theta_V = \Theta_{\mathcal{H}}|_V + \bar{\sigma}\ {}^t\sigma = \bar{\sigma}\ {}^t\sigma,$$

and Griffiths proves that the curvature of $V^\vee$ is semi-negative, since its local expression is of the form $ih'(z) d\bar{z} \wedge dz$, where $h'(z)$ is a semi-positive definite Hermitian matrix.

The case of no singular fibres

In particular we have that the curvature Θ_V of V is semipositive and, moreover, that the curvature vanishes identically if and only if the second fundamental form σ vanishes identically, i.e., if and only if V is a flat subbundle.
However, by semi-positivity, we get that the curvature vanishes identically if and only its integral, the degree of V, equals zero.
Hence V is a flat bundle if and only if it has degree 0.
The same result then holds true, by a similar reasoning, for each holomorphic quotient bundle Q.

The general case

In the general case we use:
1) The semistable reduction theorem (a base change $B' \to B$ such that all fibres of the pull-back $X' \to B'$ are reduced with normal crossings)
2) A comparison of the pull-back of V with the analogously defined V'
3) Some crucial estimates given by Zucker (using Schmid's asymptotics for Hodge structures) for the growth of the norm of sections of the L^2-extension of Hodge bundles, and

The general case, cont.

4) A lemma by Kawamata

Lemma

Let L be a holomorphic line bundle over a projective curve B, and assume that L admits a singular metric h which is regular outside of a finite set S and has at most logarithmic growth at the points $p \in S$.
Then the first Chern form $c_1(L, h) := \Theta_h$ is integrable on B, and its integral equals $deg(L)$.

This shows that in the semistable case singularities are ininfluent, and the argument runs as in the case of no singular fibres.

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Symmetry by a cyclic group of order 7

Proposition

Let $f : X \to B$ be a semistable fibration of a surface X onto a projective curve, such that the group $G = \mu_7 \cong \mathbb{Z}/7$ acts on this fibration inducing the identity on B. Assume that the general fibre F has genus 6 and that G has exactly 4 fixed points on F, with tangential characters $(1, 1, 1, 4)$.
Then if we split $V = f_*(\omega_{X|B})$ into eigensheaves, then the eigensheaves V_1, V_2 are unitary flat rank 2 bundles.

Symmetry by a cyclic group of order 7, cont.

Idea of proof:

- we show that V_1, V_2 have rank 2, V_3, V_4 have rank 1, $V_5 = V_6 = 0$
- Let $\mathcal{H}_j := \mathbb{H}_j \otimes \mathcal{O}_B$: for $j = 1, 2$ we have that $(\bar{V})_j = \overline{V_{7-j}} = 0$, hence $V_j = \mathcal{H}_j$ over $B^* = B \setminus S$, S being the set of critical values of f.
- We saw that the norm of a local frame of V_j has at most logarithmic grow at the points $p \in S$. This shows that V_j is a subsheaf of $\mathcal{H}_j$: by semipositivity we conclude that we have equality $V_j = \mathcal{H}_j$.

Symmetry by a cyclic group of order 7, cont.

Idea of proof:

- we show that V_1, V_2 have rank 2, V_3, V_4 have rank 1, $V_5 = V_6 = 0$
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Since the fibration is semistable, the local monodromies are unipotent: on the other hand, they are unitary, hence they must be trivial. This implies that the local systems $\mathbb{H}_1^*$ and $\mathbb{H}_2^*$ have respective flat extensions to local systems $\mathbb{H}_1$ and $\mathbb{H}_2$ on the whole curve B.

The examples

The equation

$$z_1^7 = y_1 y_0 (y_1 - y_0)(x_0 y_1 - x_1 y_0)^4 x_0^3.$$

describes a singular surface Σ' which is a cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $G := \mathbb{Z}/7$.
Let Y be a minimal resolution of singularities of Σ: Y admits a fibration $\varphi : Y \to \mathbb{P}^1$ with fibres curves of genus 6.
We let X be the minimal resolution of the fibre product of $\varphi : Y \to \mathbb{P}^1$ with $\psi : B \to \mathbb{P}^1$, where ψ is the G-Galois cover branched on $\infty = \{x_0 = 0\}, 0 = \{x_1 = 0\}, 1 = \{x_1 = x_0\}$, and with local characters $(1, 1, -2)$. In particular B has genus 3 by Hurwitz' formula.

Properties of the example

Theorem

The above surface X is a surface of general type endowed with a fibration $f : X \to B$ onto a curve B of genus 3, and with fibres of genus 6, such that $V := f_* \omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where A is an ample rank-2 vector bundle, and the unitary flat rank-2 summands Q_1, Q_2 have infinite monodromy.

The last assertion is a consequence of the classification by Schwarz of the cases where the monodromy of hypergeometric integrals is finite, as we now see.

Hypergeometric integrals

Another example is given by the equation

$$z_1^7 = y_1 y_0^4 (y_1 - y_0)(y_1 - x y_0), \quad x \in \mathbb{C} \setminus \{0, 1\}$$

which gives another family of curves. It is similar to the previous family, except that here V_1 is generated by

$$\eta := y^{-\frac{6}{7}} (y-1)^{-\frac{6}{7}} (y-x)^{-\frac{6}{7}} dy, \text{and by } y \cdot \eta.$$

Varying x, we obtain a rank-2 local system over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which is equivalent, in view of the Riemann-Hilbert correspondence, to a second order differential equation with regular singular points. Indeed, using results of Deligne-Mostow and Kohno, we see that we have a Gauss hypergeometric equation, and we can see that the local monodromies have order 7, hence we are not in the Schwarz list and the monodromy is infinite (and irreducible).

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Surfaces fibred over elliptic curves

Let now $X = S$ be a surface and $B = E$ be an elliptic curve, and assume we have a fibration $f : S \to E$. Then, by Atiyah's classification of vector bundles on elliptic curves

$$V = (\oplus_j A_j) \bigoplus (\mathcal{O}_E^{q-1}) \bigoplus (\oplus_i Q_i),$$

where A_j is ample and indecomposable, Q_i is a nontrivial torsion line bundle, and $q := q(S) = h^1(\mathcal{O}_S)$.

Surfaces with $q = 1$

Let $X = S$ be a surface with $q := q(S) = 1$ so that the Albanese map yields a fibration onto an elliptic curve E. Then

$$V = (\oplus_j A_j) \bigoplus (\oplus_i Q_i),$$

where A_j is ample and indecomposable, Q_i is a nontrivial torsion line bundle, and $p_g := p_g(S) = h^0(V) = \sum_j h^0(A_j)$. If moreover $p_g = 1$, then

$$V = A_1 \bigoplus (\oplus_i Q_i),$$

where A_1 is indecomposable of degree 1 and uniquely determined by its rank, up to tensoring with a line bundle. With Ciliberto we proved many years ago:

Surfaces with $p_g = q = 1$

Theorem

(C. - Ciliberto) Let S be a surface with $p_g = q = 1$, let $f : S \to E$ the Albanese map and set $V = A_1 \bigoplus (\oplus_i^\lambda Q_i)$. Then the projectivization of the first summand is a symmetric product of the elliptic curve

$$\mathbb{P}(A_1^\vee) = E^{(\iota)}, \ \iota := g - \lambda$$

and the natural (rational map)

$$S \to E^{(\iota)}$$

is the paracanonical map associating to $x \in S$ the $\{t \in B\}$ such that $x \in C_t$ (here C_t is, for general t, the unique curve in $|K + t|$, and is called a paracanonical curve).

Status of the classification of surfaces with $p_g = q = 1$

For these surfaces $K^2 \in \{2, 3, \dots, 9\}$ (Miyaoka-Yau inequality).

1. $K^2 = 2$: an irreducible moduli space (C. 1977, Horikawa 1978), S is a double cover of $E^{(2)}$, g=2
2. $K^2 = 3$: $g \leq 3$, there are exactly 5 irreducible connected components of the moduli space, one with $g = 3$ (C.-Ciliberto, 1989), 4 with $g = 2$ (C.- Pignatelli, 2004)
3. $K^2 = 4, 5$ exist, as I showed (98) with examples having $g = 2$; later Pignatelli showed: for $K^2 = 4, g = 2$ get more than 8 connected components !
4. $K^2 = 4, 6, 8$ exist by Polizzi, Rito, Frapporti-Pignatelli (quotients of products of curves) ($g = 3, 4, 5, 7$)
5. $K^2 = 7$ exist by Lei Zhang, Rito
6. $K^2 = 9$ existence shown by Cartwright and Steger

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New surfaces with $p_g = q$

Theorem (Bauer, C., Frapporti)

There are 16 irreducible families of generalized Burniat type surfaces with $K_S^2 = 6$, $0 \leq p_g(S) = q(S) \leq 3$. Those with $p_g(S) = q(S) = 1$ are summarized in the following table, 5)-10) form 6 connected components of the moduli space, 11) and 12) are contained in a unique irreducible connected component of the moduli space.

Table: $p_g = q = 1$

	p_g	$H_1(S,\mathbb{Z})$	dim	
5)	1	$(\mathbb{Z}/2)^3 \times \mathbb{Z}^2$	3	
6)	1	$(\mathbb{Z}/2)^2 \times \mathbb{Z}^2$	3	
7)	1	$\mathbb{Z}/4 \times \mathbb{Z}^2$	3	
8)	1	$(\mathbb{Z}/2)^2 \times \mathbb{Z}^2$	3	
9)	1	$(\mathbb{Z}/2 \times \mathbb{Z}/4) \times \mathbb{Z}^2$	3	
10)	1	$(\mathbb{Z}/2)^2 \times \mathbb{Z}^2$	3	
11)	1	$(\mathbb{Z}/2)^3 \times \mathbb{Z}^2$	3	
12)	1	$(\mathbb{Z}/2)^3 \times \mathbb{Z}^2$	3	

Sicilian surfaces

Definition

A *Sicilian surface* is a minimal surface S of general type with $K_S^2 = 6$, $p_g = q = 1$ such that

- there exists an unramified double cover $\hat{S} \to S$ with $q(\hat{S}) = 3$, and
- such that the Albanese morphism $\hat{\alpha} \colon \hat{S} \to A$ is birational onto its image Z, which is a divisor in A with $Z^3 = 12$.

Remark

A generalized Burniat type surface S is a Sicilian surface if and only if S is in one of the families 11 or 12.

Theorem (Bauer, C. , Frapporti)

Sicilian surfaces have an irreducible four dimensional moduli space, and the general fibre of their Albanese map $\alpha\colon S \to A_1$ is a non hyperelliptic curve of genus $g = 3$.
Moreover, any surface homotopically equivalent to a Sicilian surface is a Sicilian surface.

Method used: the theory of Inoue type varieties and Bagnera de Franchis varieties, introduced in joint work with Ingrid Bauer.