Bertini irreducibility theorems over finite fields

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ICM Satellite Conference, Daejeon, August 9, 2014

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  If X has some property (smooth, geometrically irreducible, $\ldots$), then so does a sufficiently general hyperplane section of X, i.e. the set of such hyperplane sections contains a Zariski dense open subset of $\check{\mathbb{P}}^n_k$.
- Over an infinite field, every nonempty open subset of $\check{\mathbb{P}}^n_k$ has a rational point, so we can actually find such a hyperplane section.

Failure over finite fields

- Bertini theorems can fail over finite fields.

Proposition (Katz)

Let X be the smooth hypersurface defined by

$$\Sigma_{i=1}^{n+1}(X_i Y_i^q - X_i^q Y_i) = 0$$

in $\mathbb{P}^{2n+1}_{\mathbb{F}_q}$. Then any hyperplane section of X is singular.

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- Suggestion of Katz: consider hypersurfaces of larger degree !

A Bertini smoothness theorem over finite fields

Let k be a finite field. For $d \leq 1$, let S_d be the set of hypersurfaces of degree d in $\mathbb{P}^n_k$ and $S = \bigcup_d S_d$. If E is a subset of S, the **density** of E is

$$\mu(E) := \lim_{d \to \infty} \frac{|E \cap S_d|}{|S_d|}$$

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Let k be a finite field, and let X be a smooth quasiprojective subscheme of $\mathbb{P}_k^{2n+1}$. Then the set of hypersurfaces H of $\mathbb{P}_k^n$ such that $H \cap X$ is smooth has positive density.

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- In particular, there exists a smooth hypersurface in X defined over k.
- The density cannot be equal to 1: given a closed point P of X, the probability that a hypersurface is singular at P is positive.

Sketch of proof

Let m be the dimension of X, q the cardinal of k.

- Given P a point of degree r, the probability that a hypersurface $H \cap X$ is singular at P is $q^{-(m+1)r}$ – write the Taylor expansion at P.

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- Related, independent results of Gabber.

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Theorem (C., Poonen)

Let X be a geometrically irreducible subscheme of $\mathbb{P}^n_k$. Assume the dimension of X is at least 2. Then the set of hypersurfaces H of $\mathbb{P}^n_k$ such that $H \cap X$ is geometrically irreducible has density 1.

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This statement is closer to the usual Bertini theorems: most hypersurfaces satisfy the conclusion.

Variants and general remarks

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- Suitably modified, the result holds for arbitrary maps $X \to \mathbb{P}^n_k$ and preimages of hypersurfaces. In order to get density 1 statements, one has to ignore the components that are contracted.
- Singularities of X are a problem. We will reduce to surfaces – using a weak version of the theorem – and use resolution of singularities.

Applications and a counterexample

Using density 1, we get:

Corollary

Let X be a geometrically irreducible variety of dimension $m \geq 2$ over a field k. Let F be a finite set of closed points in X. Then there exists a geometrically irreducible variety of dimension $m - 1$ $Y \subset X$ containing F.

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Take X that contains all lines defined over k!

An easy case

- Special case: X is smooth projective of dimension at least 3. Then

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- However, for most H, the singular locus of $H \cap X$ is finite by the sieving techniques above.
- This is the only case where the statement can actually be checked locally analytically. Even in dimension at least 3, it is less clear in the presence of singularities.

Reduction to surfaces

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- Idea of the reduction: start with $X \subset \mathbb{P}^n_k$, smooth, irreducible, $\dim X = m \geq 3$. Find a single irreducible hypersurface J of X that is also irreducible, and does not contain any positive-dimensional irreducible component of $\overline{X} \setminus X$.

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- This is dangerously close to what we were trying to prove ! However being in dimension at least 3 and not requiring density 1 turns out to make things easier.
- Show that for most hypersurfaces H,

$$H \cap J \, \text{irreducible} \implies H \cap X \, \text{irreducible}.$$

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  with π generically finite.
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- Claim : for most irreducible $H \subset \mathbb{P}^m_k$, $J = \pi^{-1}(H)$ is irreducible.
- Reduce to a statement about finite étale morphisms.

An application of the Chebotarev density theorem

Lemma

Let $\pi : X \to Y$ be a finite étale morphism of irreducible schemes of dimension at least 2, with $Y \subset \mathbb{P}^n_k$. Then for a density 1 set of $H \subset \mathbb{P}^n_k$,

$$H \cap Y \, \text{irreducible} \implies \pi^{-1}(H \cap Y) \, \text{irreducible}.$$

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Proof.

Can assume that the cover is Galois with group G. $F_P \subset G$ the conjugacy class of Frobenius at a closed point P.

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Chebotarev density theorem (Lang): given C, the number of $\mathbb{F}_{q^e}$-points P of Y such that $F_P = C$ is about $\frac{|C|}{|G|}\frac{1}{e}q^{me}$. $\square$

End of the proof

Proof (continued).

Probability that a given hypersurface misses all these points is about

$$(1 - q^{-e})^{cq^{me}/e} \to 0$$

as $e \to \infty$. This holds because $m \geq 2$.

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Proof (continued).

Probability that a given hypersurface misses all these points is about

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as $e \to \infty$. This holds because $m \geq 2$.
For most hypersurfaces H of $\mathbb{P}^n_k$, the various Frobenius at closed points of $H \cap Y$ meet all the conjugacy classes of G, so the étale cover $\pi^{-1}(H \cap Y) \to H \cap Y$ is irreducible. $\square$

Smooth surfaces and the Hodge index theorem

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- We can assume both C and C' have high degree. Consequence: the singular locus of a reducible curve has large length.
- We can use a local analytic analysis again to conclude that this does not happen outside of a density zero subset of the set of hypersurfaces.

Treating the singular case

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- Start with X projective, and consider a resolution of singularities $\pi : \widetilde{X} \to X$. We want to work on $\widetilde{X}$.
- Problem : hypersurfaces in $\widetilde{X}$ coming from X form a density zero subset of the set of hypersurfaces in $\widetilde{X}$. We need better estimates.

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  To pass from numerical equivalence classes to classes of divisors, just multiply by the number of rational points of Pic^τ.

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$$\leq \frac{\deg X}{2} d^2 - \frac{n(d \deg X - n)}{\deg X} + O(d).$$
- Final estimate: the number of decompositions $\pi^{-1}(H) = D + D'$ is at most
$$\Sigma_{n=d_0}^{d \deg X - d_0} O(n^\rho) q^{\frac{\deg X}{2} d^2 - \frac{n(d \deg X - n)}{\deg X} + O(d)}.$$

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- We are done !
- Remark: this is a variant of the Hodge index argument in the smooth case.