Minimal Legendrian surfaces in the 5-dimensional Heisenberg group

joint work with Reiko Aiyama (Univ. of Tsukuba)

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Contents

References

[AA] (with R. Aiyama), *Minimal Legendrian surfaces in the five-dimensional Heisenberg group*, in preparation.

[AAK] (with R. Aiyama and Y. Kawakami), *The Gauss map of complete minimal Lagrangian surfaces in the complex two-space*, in preparation.

- minimal surface in $\mathbb{R}^3$: very Rich, Deep, Beautiful !

One of reasons

$\exists$Wererstrass Representation (holomorphic description)

For Gauss data $(g, \omega = h(z)dz)$

$g : M^2 \to S^2 = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: holo. map

ω :holo. 1-form on M

$$\rightsquigarrow \quad f(z) = Re\left(\int^z \frac{1}{2}(1 - g^2)\omega, \int^z \frac{i}{2}(1 + g^2)\omega, \int^z g\omega\right) : M \to \mathbb{R}^3$$

conformal minimal immersion

or

For $F : M^2 \to \mathbb{C}^3$: null holo. map ($\Leftrightarrow \quad < dF, dF >= 0$)

$$\rightsquigarrow \quad f(z) = Re\, F(z) : M \to \mathbb{R}^3 \ : \text{conformal minimal immersion}$$

Generalization : Kenmotsu Representation

Kenmotsu Representation   For $H \in C^\infty(M)$,

Solve $\boxed{H(g_{z\bar{z}} - \frac{2\bar{g}}{1+|g|^2} g_z g_{\bar{z}}) = H_z g_{\bar{z}}}$ for $g \in C^\infty(M, \widehat{\mathbb{C}})$

Assume $\omega := \frac{-2(\bar{g})_z}{H(1+|g|^2)^2} dz$: C^∞ 1-form on M

$\rightsquigarrow$ $f(z) = Re\left(\int^z \frac{1}{2}(1-g^2)\omega, \int^z \frac{i}{2}(1+g^2)\omega, \int^z g\omega\right) : M \to \mathbb{R}^3$
conf. imm. with mean curvature H & Gauss map g

- $ds^2 = (1+|g|^2)^2|\omega|^2$
- $K = H^2\left(1 - |g_z|^2/|g_{\bar{z}}|^2\right)$

Note

- *W-rep. for CMC surfaces in $\mathbb{H}^3$ / $\mathbb{S}^3_1$*
  $\cdots$ Bryant, Umehara-Yamada / Aiyama-Ak
- *K-rep. for surfaces in $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{S}^3_1$, $\mathbb{H}^3_1$ $\cdots$ Aiyama-Ak*

Analogue in Symplectic Riemannian Geometry

- $\mathbb{C}^2$: Flat symplectic Riem. mfd
  $\cup$
  M^2 : Lagrangian surface $\quad (\Leftrightarrow \ J(T_xM) \perp T_xM$ for $x \in M$)

Chen-Morvan '87

$M \subset \mathbb{C}^2$: *minimal Lag. surfaces*

$\Updownarrow$ *1 to 1*

$S \subset \mathbb{C}^2$: *(nondeg.) complex curves*

$\rightsquigarrow$ Generalization (K-type rep.) by Hélein-Ramon '00, Aiyama '01

$(\phi, \psi) \in \Gamma\big((M \times \mathbb{C}^2) \oplus (K_M^{-1} \oplus K_M)\big)$: spinor of $\mathrm{spin}^{\mathbb{C}}$-bundle

In terms of plus spinor $\phi \in \Gamma(M \times \mathbb{C}^2)$ $\cdots$ Aiyama

minus spinor $\psi \in \Gamma(K_M^{-1} \oplus K_M)$ $\cdots$ Hélein-Ramon

Wererstrass Rep. for minimal Lagrangian surfaces in $\mathbb{C}^2$

Chen-Morvan, Aiyama

$z \in M^2$: Riem. surface, $F = (F_1, F_2) : M \to \mathbb{C}^2$ *: holo. with* $|S_1|^2 + |S_2|^2 \neq 0$
$S_1 := (F_2)_z, \quad S_2 := -(F_1)_z$

For const. β,

$\rightsquigarrow$ $\boxed{f(z) = \frac{1}{\sqrt{2}} e^{\mathbf{i}\beta/2} (F_1 - \mathbf{i}\overline{F_2},\ F_2 + \mathbf{i}\overline{F_1}) : M \to \mathbb{C}^2}$ *: conf. minimal Lag. imm.*

with const. Lag. angle β & *Gauss map* $g := [-S_2 : S_1] = \frac{-S_2}{S_1} : M \to \widehat{\mathbb{C}}$

- $ds^2 = (|S_1|^2 + |S_2|^2)|dz|^2$
- $K = -2\frac{|S_1(S_2)_z - S_2(S_1)_z|^2}{(|S_1|^2+|S_2|^2)^3} \leq 0$

Note

Regards f as $f : M \to \mathbb{R}^4 = \mathbb{C}^2$

$\rightsquigarrow$ *Its generalized Gauss map* $\mathcal{G} : M \to S^2(1) \times S^2(1) \subset \mathbb{R}^3 \times \mathbb{R}^3$
$\mathcal{G} = (\mathcal{G}_-, \mathcal{G}_+) = (g, (e^{\mathbf{i}\beta/2}, 0))$

Note

For Gauss data $(g, \omega = h(z)dz := S_1 dz)$

$\rightsquigarrow$ • $ds^2 = (1+|g|^2)|\omega|^2$ • $K = -2\left(\frac{|g_z|}{|h|(1+|g|^2)^{3/2}}\right)^2 \le 0$

$$\boxed{f(z) = \frac{1}{\sqrt{2}} e^{\mathbf{i}\beta/2}\left(\int^z g\omega - \mathbf{i}\left(\overline{\int^z \omega}\right), \int^z \omega + \mathbf{i}\left(\overline{\int^z g\omega}\right)\right)} \cdots \textit{Hélein-Ramon}$$

Case of minimal surfaces in $\mathbb{R}^3$

• $d\tilde{s}^2 = \frac{1}{4}(1+|g|^2)^2|\omega|^2$ • $\widetilde{K} = -\left(\frac{4|g_z|}{|h|(1+|g|^2)^2}\right)^2 \le 0$

Remark Value distribution of Gauss map of complete minimal Lagrangian surfaces in $\mathbb{C}^2$ is also very interesting ! See Ai-Ak-Kawakami [AAK]

§ 2 5-dim. Heisenberg group $\mathfrak{H}^5 = \mathbb{C}^2 \times \mathbb{R}$

Motivation

- $\mathfrak{H}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$: $(2n+1)$-dim. Heisenberg group as Contact Riem. mfd corresponding to $\mathbb{E}^n$
  $\cup$
  M^n : Legendrian submanifold ($\Leftrightarrow$ $T_xM \subset H_x := \mathrm{Ker}\,\eta_x$ for $x \in M$)

Question $\exists$? W-rep. for minimal Leg. surfaces in $\mathfrak{H}^5$

Answer Yes !

Def. of 5-dim. Heisenberg group $\mathfrak{H}^5$

Definition

- $\mathfrak{H}^5 := \mathbb{R}^5 = \mathbb{C}^2 \times \mathbb{R} = \{(\mathbf{z} = (z^1, z^2), t) \mid z^j = x^j + \sqrt{-1}y^j \in \mathbb{C}, t \in \mathbb{R}\}$
- $(\mathbf{z}, t) \cdot (\mathbf{z}', t') = (\mathbf{z} + \mathbf{z}', t + t' + 2\,\mathrm{Im}(\mathbf{z} \cdot \overline{\mathbf{z}'}))$ $\cdots$ *group str.*
- $\eta = dt - \frac{1}{2}(\mathbf{x} \cdot d\mathbf{y} - \mathbf{y} \cdot d\mathbf{x})$ $\cdots$ *contact 1-form* $(\Leftrightarrow \eta \wedge (d\eta)^2 \neq 0)$

  Note $(R_p)^*\eta = \eta$ *for* $p = (\mathbf{z}, t) \in \mathfrak{H}^5$
- $H := \mathrm{Ker}\,\eta$ $\cdots$ *contact str.* $\leftarrow$ *codim.* 1 *non-integrable subbundle of* $T\mathfrak{H}^5$

Note

- $H = \mathbb{R}\{T_j := \partial_{x^j} - \frac{1}{2}y^j\partial_t, T_{2+j} := \partial_{y^j} + \frac{1}{2}x^j\partial_t \mid j = 1, 2\}$
- ξ $\cdots$ *Reeb v.f.* $(\Leftrightarrow \eta(\xi) = 1$ & $d\eta(\xi, \cdot) = 0)$ $\rightsquigarrow$ $\xi = \partial_t$
- $\boxed{T\mathfrak{H}^5 = H \oplus \mathbb{R}\xi}$ $\cdots$ *natural decomposition*

Note

- $H = \mathbb{R}\{T_j := \partial_{x^j} - \frac{1}{2}y^j\partial_t,\ T_{2+j} := \partial_{y^j} + \frac{1}{2}x^j\partial_t \mid j = 1, 2\}$, $\quad \xi = \partial_t$
- $\boxed{T\mathfrak{H}^5 = H \oplus \mathbb{R}\xi}$ $\cdots$ *natural decomposition*
- $J \curvearrowright H$ $\cdots$ *almost complex str.* by $J(T_j) = T_{2+j}$, $J(T_{2+j}) = -T_j$
- $g_H := d\eta \circ (J \otimes \mathbf{1})$ $\cdots$ *inner product on H*
- $\boxed{g_\eta := \pi_H^* g_H + \eta \otimes \eta}$ $\cdots$ *standard Sasakian/Webster metric on* $\mathfrak{H}^5$

$\pi_H : H \oplus \mathbb{R}\xi \to H$

$\rightsquigarrow$ $\boxed{g_\eta = |d\mathbf{x}|^2 + |d\mathbf{y}|^2 + \left(dt - \frac{1}{2}(\mathbf{x} \cdot d\mathbf{y} - \mathbf{y} \cdot d\mathbf{x})\right)^2}$ $\cdots$ *not flat*

- *triple* (η, g_H, J) $\cdots$ *contact Riem. str.*

§ 3 W-rep. for minimal Legendrian surfaces in $\mathfrak{H}^5$

Prop. (cf. Ekholm-Entyre-M. Sullivan)

- $\pi_{\mathbb{C}^2} : \mathfrak{H}^5 = \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2$

(1) $M^2 \subset \mathfrak{H}^5$: *Legendrian* $\Rightarrow$ $\check{M} := \pi_{\mathbb{C}^2}(M) \subset \mathbb{C}^2$: *Lagrangian*

(2) $\Sigma^2 \subset \mathbb{C}^2$: *Lag.* $\Rightarrow$ $f(z) := (z,\ t(z) := \frac{1}{2}\int^z (\mathbf{x}\cdot d\mathbf{y} - \mathbf{y}\cdot d\mathbf{x})) : \widetilde{\Sigma^2} \to \mathfrak{H}^5$
Leg. conf. imm.

Key Prop. (Aiyama-Ak)

- $M^2 \subset \mathfrak{H}^5$: *Leg. surface* ($\rightsquigarrow$ $\check{M} \subset \mathbb{C}^2$: *Lag. surface*)

(1) $\pi_{\mathbb{C}^2}|_M : (M, ds^2) \to (\check{M}, d\check{s}^2)$: *isometric*

(2) $\{e_1, e_2, e_3, e_4, e_5 := \xi = \partial_t\}$: *adapted contact ONF on* $U \subset M$
($\Leftrightarrow$ $(e_1)_x, (e_2)_x \in T_xM$, $(e_3)_x, (e_4)_x \in T_x^{\perp}M$ *for* $x \in U$)

Set $\underline{\check{e}_\ell := d\pi_{\mathbb{C}^2}(e_\ell)}$ $\rightsquigarrow$ $\{\check{e}_1, \check{e}_2, \check{e}_3, \check{e}_4\}$: *adapted ONF on* $\check{U} \subset \check{M}$

$\underline{(A^\ell_{jk}) = A^\ell(e_j, e_k)e_{2+\ell}}$: 2^{nd} *f.f.of* M, $\underline{(\check{A}^\ell_{jk}) = \check{A}^\ell(\check{e}_j, \check{e}_k)\check{e}_{2+\ell}}$: 2^{nd} *f.f.of* $\check{M}$

$\rightsquigarrow$ $A^3_{jk} \equiv 0$, $\boxed{A^\ell_{jk} \equiv \check{A}^\ell_{jk}}$ $(j, k, \ell = 1, 2)$

Key Prop. (Aiyama-Ak)

- $M^2 \subset \mathfrak{H}^5$: *Leg. surface* ($\rightsquigarrow$ $\check{M} \subset \mathbb{C}^2$: *Lag. surface*)

(1) $\pi_{\mathbb{C}^2}|_M : (M, ds^2) \to (\check{M}, d\check{s}^2)$: *isometric*

(2) $\{e_1, e_2, e_3, e_4, e_5 := \xi = \partial_t\}$: *adapted contact ONF on* $U \subset M$

$$(\Leftrightarrow \; (e_1)_x, (e_2)_x \in T_x M, \; (e_3)_x, (e_4)_x \in T_x^{\perp} M \;\; \textit{for } x \in U)$$

Set $\underline{\check{e}_\ell := d\pi_{\mathbb{C}^2}(e_\ell)}$ $\rightsquigarrow$ $\{\check{e}_1, \check{e}_2, \check{e}_3, \check{e}_4\}$: *adapted ONF on* $\check{U} \subset \check{M}$

$\underline{(A^\ell_{jk}) = A^\ell(e_j, e_k) e_{2+\ell}}$: 2^{nd} *f.f.of* M, $\underline{(\check{A}^\ell_{jk}) = \check{A}^\ell(\check{e}_j, \check{e}_k)\check{e}_{2+\ell}}$: 2^{nd} *f.f.of* $\check{M}$

$$\rightsquigarrow \quad A^3_{jk} \equiv 0, \quad \boxed{A^\ell_{jk} \equiv \check{A}^\ell_{jk}} \quad (j, k, \ell = 1, 2)$$

Cor.

$M^2 \subset \mathfrak{H}^5$: *minimal Leg. surface* $\Leftrightarrow$ $\check{M} := \pi_{\mathbb{C}^2}(M) \subset \mathbb{C}^2$: *minimal Lag. surface*

Cor.

$M^2 \subset \mathfrak{H}^5$: *minimal Leg. surface* $\Leftrightarrow$ $\check{M} := \pi_{\mathbb{C}^2}(M) \subset \mathbb{C}^2$: *minimal Lag. surface*

$\Downarrow$ Apply W-rep. for mini. Lag. in $\mathbb{C}^2$ to mini. Leg. in $\mathfrak{H}^5$

Weierstrass Representation (Aiyama-Ak)

$z \in M^2$: *(simply conn.) Riem. surface*

$F = (F_1, F_2) : M \to \mathbb{C}^2$: *holo. with* $|S_1|^2 + |S_2|^2 \neq 0$, $S_1 := (F_2)_z$, $S_2 := -(F_1)_z$

For const. β, *Define* $f : M \to \mathfrak{H}^5 = \mathbb{C}^2 \times \mathbb{R}$ $\rightsquigarrow$

$$\boxed{f(z) = \left(\frac{e^{i\beta/2}}{\sqrt{2}}(F_1 - i\overline{F_2}),\ \frac{e^{i\beta/2}}{\sqrt{2}}(F_2 + i\overline{F_1}),\ t(z) := -\frac{1}{2}\mathrm{Re}\int^z (F_1 S_1 + F_2 S_2)dz\right)}$$

conf. minimal Leg. imm. from M *to* $\mathfrak{H}^5$

For Gauss data $(g := \frac{-S_2}{S_1}, \omega = hdz := S_1 dz)$

- $ds^2 = (|S_1|^2 + |S_2|^2)|dz|^2 = (1 + |g|^2)|\omega|^2$
- $K = -2\frac{|S_1(S_2)_z - S_2(S_1)_z|^2}{(|S_1|^2+|S_2|^2)^3} = -2\left(\frac{|g_z|}{|h|(1+|g|^2)^{3/2}}\right)^2 \leq 0$

Example (Legendrian Catenoid)

Catenoid Set $F(z) = (\mathbf{i}\sinh z,\ \cosh z,\ z) : \mathbb{C} \to \mathbb{C}^3$: null holo.

$\rightsquigarrow$ $f(z) = \mathrm{Re}F(z) = (-\cosh x \sin y,\ \cosh x \cos y,\ x) : \mathbb{C} \to \mathbb{R}^3$: catenoid

- $(x^1)^2 + (x^2)^2 = \cosh^2 t$

Legendrian Catenoid Set $F(z) := (\sqrt{2}\sinh z,\ \sqrt{2}\cosh z) : \mathbb{C} \to \mathbb{C}^2$: holo.

$\rightsquigarrow$

$$f(z) = \left(\frac{e^{\mathbf{i}\beta/2}}{\sqrt{2}}(F_1 - \mathbf{i}\overline{F_2}),\ \frac{e^{\mathbf{i}\beta/2}}{\sqrt{2}}(F_2 + \mathbf{i}\overline{F_1}),\ -\frac{1}{2}\mathrm{Re}\int^z (F_1 S_1 + F_2 S_2)dz \right) \in \mathfrak{H}^5$$

$$= \Big(\sqrt{2}\sinh x \sin(y - \tfrac{\pi}{4}),\ \sqrt{2}\cosh x \sin(y - \tfrac{\pi}{4}),$$
$$\sqrt{2}\sinh x \cos(y - \tfrac{\pi}{4}),\ \sqrt{2}\cosh x \cos(y - \tfrac{\pi}{4}),\ x\Big)$$: Legendrian catenoid

- $(y^1)^2 + (y^2)^2 = 2\cosh^2 t$
- $F(\mathbb{C}) \simeq S^1 \times \mathbb{R}$

Summary

Note

$$\boxed{\textit{Gauss data of } M^2 \subset \mathfrak{H}^5 : (\textit{minimal}) \textit{ Legendrian surface}}$$

$$\|$$

$$\boxed{\textit{Gauss data of } \check{M}^2 := \pi_{\mathbb{C}^2}(M^2) \subset \mathbb{C}^2 : (\textit{minimal}) \textit{ Lagrangian surface}}$$

$$\Downarrow$$

Value distribution theory of Gauss map of complete mini. Leg. surfaces in $\mathfrak{H}^5$

& the one of complete mini. Lag. surfaces in $\mathbb{C}^2$ are almost same !

$$\Downarrow$$

Note

$$\boxed{\textit{Geometry of } M^2 \subset \mathfrak{H}^5 : (\textit{minimal}) \textit{ Legendrian surface}}$$

$$\|$$

$$\boxed{\textit{Geometry of } \check{M}^2 \subset \mathbb{C}^2 : (\textit{mini.}) \textit{ Lag. surface} \quad \oplus \quad \underline{\textit{Information of } t(z)}}$$

Kenmotsu Representation (Aiyama-Ak)

$z \in M^2$: (simply conn.) Riem. surface Given $\beta \in C^\infty(M, \mathbb{R}/(4\pi\mathbb{Z}))$

Solve Dirac eq. $F = (F_1, F_2) : M \to \mathbb{C}^2$ $\left(\text{Regards } (F_1, \overline{F_2}) \in \Gamma(M \times \mathbb{C}^2)\right)$

$$\begin{cases} (F_1)_{\bar{z}} = -\frac{1}{2}\beta_{\bar{z}}\overline{F_2} \\ (F_2)_{\bar{z}} = \frac{1}{2}\beta_{\bar{z}}\overline{F_1} \end{cases}$$

Define $(S_1, S_2) : M \to \mathbb{C}^2$ by

$$\begin{cases} S_1 := (F_2)_z - \frac{1}{2}\beta_z\overline{F_1} \\ S_2 := -(F_1)_z - \frac{1}{2}\beta_z\overline{F_2} \end{cases}$$

Assume $|S_1|^2 + |S_2|^2 \neq 0$ on M **Define** $f : M \to \mathfrak{H}^5 = \mathbb{C}^2 \times \mathbb{R}$ by

$$\boxed{f(z) = \left(\frac{e^{\mathrm{i}\beta/2}}{\sqrt{2}}(F_1 - \mathrm{i}\overline{F_2}),\ \frac{e^{\mathrm{i}\beta/2}}{\sqrt{2}}(F_2 + \mathrm{i}\overline{F_1}),\ t(z)\right)}$$

conf. Legendrian imm.

$$t(z) := -\frac{1}{2}\mathrm{Re}\int^z (F_1 S_1 + F_2 S_2)dz$$

Define $f : M \to \mathfrak{H}^5 = \mathbb{C}^2 \times \mathbb{R}$ by

$\boxed{f(z) = \left(\check{f}(z),\ t(z)\right)}$ conf. Legendrian imm.

$$\check{f}(z) := \frac{e^{i\beta/2}}{\sqrt{2}}(F_1 - \mathbf{i}\overline{F_2}),\ \frac{e^{i\beta/2}}{\sqrt{2}}(F_2 + \mathbf{i}\overline{F_1}), \quad t(z) := -\frac{1}{2}\mathrm{Re}\int^z (F_1 S_1 + F_2 S_2) dz$$

- β : Lagrangian angle of $\check{f}$
- $\mathbf{H} = -\frac{1}{2}(|S_1|^2 + |S_2|^2)^{-1}\left(\mathbf{i}\beta_{\bar{z}}\check{f}_z + \mathbf{i}\beta_z\check{f}_{\bar{z}}\right)|dz|^2$: mean curvature of f

For Gauss data $(g := \frac{-S_2}{S_1}, \omega = hdz := S_1 dz)$

- $ds_f^2 = (|S_1|^2 + |S_2|^2)|dz|^2 = (1 + |g|^2)|\omega|^2$
- $K_f = -2\frac{|S_1(S_2)_z - S_2(S_1)_z|^2}{(|S_1|^2 + |S_2|^2)^3} = -2\left(\frac{|g_z|}{|h|(1+|g|^2)^{3/2}}\right)^2 \leq 0$

§ 4 An Application (Halfspace Theorem)

Original Halfspace Theorem

$M^2 \subset \mathbb{R}^3$: properly immersed complete minimal surface

Assume $M \subset \exists$ "halfspace", **then** $M =$ "2-plane" !

Set $\mathfrak{H}^5_{\geq t_0} := \{p \in \mathfrak{H}^5 \mid t(p) \geq t_0\}$, $\mathfrak{H}^5_{\leq t_0} := \{p \in \mathfrak{H}^5 \mid t(p) \leq t_0\}$

Halfspace Thm (Aiyama-Ak)

$M^2 \subset \mathfrak{H}^5$: *properly immersed complete minimal Legendrian surface*

Assume $M \subset \mathfrak{H}^5_{\geq t_0}$ (or $\mathfrak{H}^5_{\leq t_0}$), **then** $M \subset \{p \in \mathfrak{H}^5 \mid t(p) = t_1\} = \mathbb{C}^2 \times \{t_1\}$

Moreover $M \subset \mathbb{C}^2 = \mathbb{C}^2 \times \{t_1\}$: *Lagrangian plane !*

Potential Theoretical Proof of Halfspace Theorem

Def. Riem. surface M $\underline{\text{with } \partial M = \emptyset \text{ or } \neq \emptyset}$: parabolic

$\Updownarrow$ **def.**

$\forall$ bounded harmonic fct u $\underline{\text{with } u|_{\partial M} \equiv \mathrm{const}}$ $\Rightarrow$ $u \equiv \mathrm{const}$

- $M^2 \subset \mathfrak{H}^5$: properly immersed complete minimal Legendrian surface

Fact coord. fcts $z^1 = x^1 + \mathbf{i}y^1$, $z^2 = x^2 + \mathbf{i}y^2$, t : harmonic on M

May Assume $M \subset \mathfrak{H}^5_{\geq 0}$ & $\inf t(M) = 0$ ($\rightsquigarrow$ $\nexists \min t(M)$)

By using properly immersedness & Key Lemma below

Key Lemma $r := \sqrt{|z^1|^2 + |z^2|^2}$ $\Rightarrow$ $\boxed{\Delta_M \log r = \dfrac{2|\nabla^M t|^2}{r^4}}$ for $r > 0$

Claim $\forall \delta > 0$, $M(\delta) := \{p \in M \mid 0 \leq t(p) \leq \delta\}$: parabolic !

May Assume $M \subset \mathfrak{H}^5_{\geq 0}$ & $\inf t(M) = 0$ ($\rightsquigarrow$ $\nexists \min t(M)$)

Claim $\forall \delta > 0$, $M(\delta) := \{p \in M \mid 0 \leq t(p) \leq \delta\}$: parabolic !

Proof of Halfspace Thm

Suppose $t|_M \not\equiv \mathrm{const}$ ($\rightsquigarrow$ $t(\exists p_0) > 0$)

Then $\exists \varepsilon > 0$ **s.t.** $M(\varepsilon) \neq \emptyset$

By Claim $M(\varepsilon)$: parabolic !

$t \equiv \varepsilon$ on $\partial M(\varepsilon)$ $\Rightarrow$ $\boxed{t \equiv \varepsilon \text{ on } M(\varepsilon)}$ $\leftarrow$ contradicts to $\inf t(M) = 0$

$\rightsquigarrow$ $t|_M \equiv \mathrm{const}$ $\rightsquigarrow$ $t|_M \equiv 0$ $\rightsquigarrow$ $M \subset \mathbb{C}^2 \times \{0\}$: Lag. plane **QED**

Thank you so much for your attention !