Transversally harmonic and holomorphic maps on foliated manifolds

Seoung Dal Jung

Jeju National University,
Jeju 690-756, Korea

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Abstract & Keyword

- **Abstract**

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliated Riemannian manifold and let $\phi : M \to M'$ be a smooth foliated map, i.e., ϕ is a leaf-preserving map. Then we study the transversally holomorphic maps. In fact, a transversally holomorphic map is transversally harmonic map with the minimum energy in its foliated homotopy class.

- **Keyword**

Riemannian foliation, Kähler foliation, transversal energy, transversally holomorphic map, transversally harmonic map.

Harmonic function

Definition

Harmonic function on an open domain Ω of $\mathbb{R}^m$ is a solution of the Laplace equation

$$\Delta f = 0, \tag{1}$$

where $\Delta := -\frac{\partial^2}{(\partial x_1)^2} - \cdots - \frac{\partial^2}{(\partial x_m)^2}$ and $(x_1, \cdots, x_m) \in \Omega$. The operator Δ is called the **Laplace operator** or **Laplacian**.

- The harmonic functions are critical points of the Dirichlet functional

$$E_\Omega(f) = \frac{1}{2}\int_\Omega |df|^2 dx. \tag{2}$$

Tension field

- Let (M, g) and (N, h) be smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map.
- The **tension field** $\tau(\phi)$ of ϕ is defined by

$$\tau(\phi) := \mathrm{tr}_g \nabla d\phi = \mathrm{div}(d\phi) = \sum_{i=1}^{m} (\nabla_{e_i} d\phi)(e_i), \tag{3}$$

where $\{e_i\}$ is a local orthonormal frame field on M.

Harmonic map

Definition

Let $\phi : (M, g) \to (N, h)$ be a smooth map. Then ϕ is said to be **harmonic** if the tension field vanishes, i.e., $\tau(\phi) = 0$.

- Let $f : M \to \mathbb{R}$ be a smooth function. Then the Laplace-Beltrami operator Δ is given by

$$\Delta f = \delta df = -\mathrm{tr}(\nabla df) = -\tau(f). \tag{4}$$

- Hence $\Delta f = 0$ if and only if $\tau(f) = 0$. That is, $f : M \to \mathbb{R}$ is a harmonic function if and only if $\tau(f) = 0$.

Examples

Examples. (1) **Constant maps** and **identity maps**.
(2) **Geodesics**. For a unit speed curve $\gamma : I \to N$ $(I \subset \mathbb{R})$, the tension field $\tau(\gamma)$ is $\tau(\gamma) = \nabla_{\gamma'}\gamma'$, the acceleration vector; hence γ is harmonic if and only if it is a geodesic curve.
(3) **Isometric immersions**. Let $\phi : M^m \to N^n$ be an isometric immersion. Then $\tau(\phi) = mH$, where H is the mean curvature vector of M in N, so that ϕ is harmonic if and only if M is a minimal submanifold of N.
(4) **Gauss maps**. Let $G(k, n)$ be the Grassman manifold of k-planes in $\mathbb{R}^n$. Let $G : M^k \to G(k, n)$ be the Gauss map associated to an immersion $i : M \to \mathbb{R}^n$. Then the tension field $\tau(G)$ of G is $\tau(G) = \nabla H$, where H is the mean curvature vector field of M. Hence G is harmonic if and only if the mean curvature vector field is parallel.

Variation formula

- The **energy** of $\phi : M \to N$ over a compact domain Ω is

$$E_\Omega(\phi) = \frac{1}{2}\int_\Omega |d\phi|^2 dM. \tag{5}$$

Theorem(Variation formula)

The first variation is given by

$$\frac{d}{dt}E_\Omega(\phi_t)|_{t=0} = -\int_M < \tau(\phi), V > dM, \tag{6}$$

wgere $V = \frac{d\phi_t}{dt}|_{t=0}$ and $\{\phi_t\}$ be all smooth variations of ϕ.

- Hence a harmonic map ϕ is a critical point of $E_\Omega(\phi)$ over any compact domain Ω.

Riemannian foliation

- Let $(M, \mathcal{F})$ be a $(p+q)$-dimensional foliated Riemannian manifold with foliation $\mathcal{F}$ of codimension q.
- Let TM be the tangent bundle of M, L the tangent bundle of $\mathcal{F}$, and $Q = TM/L$ the corresponding normal bundle of $\mathcal{F}$.
- A foliation is **Riemannian** if there exists a Riemannian metric g satisfying $\theta(X)g = 0$ for all $X \in \Gamma L$. (This is called a bundle-like metric).
- Equivalently, a bundle-like metric means that all geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.
- Let R^Q and Ric^Q and be the transversal curvature and Ricci operator with respect to transversal Levi-Civita connection $\nabla^Q \equiv \nabla$ on Q, respectively.

Basic cohomology

- Let $\Omega_B^r(\mathcal{F})$ be a space of **basic form** ω, i.e., $i(X)\omega = 0$ and $i(X)d\omega = 0$ for any $X \in \Gamma L$. Locally, for a foliated coordinate (x_j, y_a),

$$\omega = \sum f_{a_1,\cdots,a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r}, \quad \frac{\partial f}{\partial x_j} = 0.$$

- Let $d_B = d|_{\Omega_B}$ and δ_B: the adjoint operator of d_B. Note that $\delta_B \neq \delta|_{\Omega_B}$.
- The **basic Laplacian** is defined by $\Delta_B = d_B\delta_B + \delta_B d_B$.
- **(de-Rham Hodge decomposition)** Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold M. Then

$$\Omega_B^r(\mathcal{F}) = \mathcal{H}_B^r \oplus \mathrm{im} d_B \oplus \mathrm{im} \delta_B, \tag{7}$$

  where $\mathcal{H}_B^r = \mathrm{Ker}\Delta_B$ is finite dimensional space.
- $\mathcal{H}_B^r \cong H_B^r$, where $H_B^r = \frac{\mathrm{Ker} d_B}{\mathrm{Im} d_B}$ is the de-Rham basic cohomology group.

Transversally harmonic maps

- Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be two foliated Riemannian manifolds and let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth leaf preserving map, i.e., $d\phi(L) \subset L'$.
- The differential map $d_T\phi : Q \to Q'$ is defined by

$$d_T\phi := \pi' \circ d\phi \circ \sigma, \tag{8}$$

  where $\sigma : Q \to L^\perp$ is an isomorphism with $\pi \circ \sigma = id$.
- Then $d_T\phi$ is a section in $Q^* \otimes \phi^{-1}Q'$.
- Let ∇^ϕ and $\tilde{\nabla}$ be the connections on $\phi^{-1}Q'$ and $Q^* \otimes \phi^{-1}Q'$, respectively.

Transversally harmonic maps

- A map ϕ is said to be **transversally totally geodesic** if

$$\tilde{\nabla}_{tr} d_T \phi = 0, \tag{9}$$

where $(\tilde{\nabla}_{tr} d_T \phi)(X, Y) = (\tilde{\nabla}_X d_T \phi)(Y)$ for any $X, Y \in Q$. This means that if γ is transversally geodesic, then $\phi \circ \gamma$ is also transversally geodesic.

- The **transversal tension field** $\tau_b(\phi)$ is defined by

$$\tau_b(\phi) = tr_Q \tilde{\nabla} d_T \phi = \sum_{a=1}^{q} (\tilde{\nabla}_{E_a} d_T \phi)(E_a), \tag{10}$$

where $\{E_a\}$ is a local orthonormal basis of Q.

Transversally harmonic map

Definition

A foliated map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be **transverally harmonic** if $\tau_b(\phi) = 0$.

- The **transversal energy** of ϕ on a compact domain $\Omega \subset M$ is defined by

$$E_B(\phi; \Omega) = \frac{1}{2} \int_\Omega |d_T \phi|^2 \mu_M, \tag{11}$$

where μ_M is the volume element of M.

The first variation formula I

Theorem (Jung-Jung(2012) [4])

Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Let $\{\phi_t\}$ be a smooth foliated variation of ϕ supported in a compact domain Ω. Then

$$\frac{d}{dt} E_B(\phi_t; \Omega)|_{t=0} = -\int_\Omega \langle V, \tau_b(\phi) - d_T\phi(\kappa_B^\sharp)\rangle \mu_M, \tag{12}$$

where $V(x) = \frac{d\phi_t}{dt}(x)|_{t=0}$ is the normal variation vector field of $\{\phi_t\}$.

- If $\mathcal{F}$ is minimal, then the transversally harmonic map is a critical point the transversal energy $E_B(\phi)$.

The transversal f-energy

Definition

Let f be a non-zero basic function on M. Then the **transversal f-energy** of ϕ on a compact domain Ω is defined by

$$E_f(\phi;\Omega) = \frac{1}{2}\int_\Omega |f d_T\phi|^2 \mu_M. \tag{13}$$

- When f is constant, $E_1(\phi) = E_B(\phi)$.

The first variation formula II

Theorem (Jung, 2013 [3])

Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. Let $\{\phi_t\}$ be a smooth foliated variation of ϕ supported in a compact domain Ω. Then, for any non-zero basic function f

$$\frac{d}{dt} E_f(\phi_t; \Omega)|_{t=0} = -\int_\Omega \langle V, \tau_b(\phi) - d_T\phi(\omega_f^\sharp)\rangle f^2 \mu_M, \qquad (14)$$

where $V(x) = \frac{d\phi_t}{dt}(x)|_{t=0}$ and

$$\omega_f = \kappa_B - d_B(\ln f^2). \qquad (15)$$

The first variation formula III

Theorem (Jung,2013 [3])

Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a smooth foliated map. If $\mathcal{F}$ is of positive transversal Ricci curvature or minimal, then

$$\frac{d}{dt} E_{f_k}(\phi_t; \Omega)|_{t=0} = -\int_{\Omega} \langle V, \tau_b(\phi)\rangle f_k^2 \mu_M, \tag{16}$$

where f_k is a solution of $\omega_f = 0$. Specially, any transversally harmonic map is a critical point of the transversal f_k-energy.

- The existence of the solution of $\omega_f = 0$ is partially assured on a foliated Riemannian manifold with positive transversal Ricci curvature. In fact, $H^1_B(\mathcal{F}) = 0$.

Kähler foliations

- Let $(M, g_M, J, \mathcal{F})$ be a Kähler foliation with a foliation $\mathcal{F}$ and a bundle-like metric g_M. Here $J : Q \to Q$ is a complex structure.
- Let $\omega \in \Omega_B^2(\mathcal{F})$ be a basic Kähler 2-form.
- Let $Q_C = Q \otimes \mathbb{C}$ be the complexified normal bundle.
- Then $Q_C = Q^{1,0} \oplus Q^{0,1}$, where

$$Q^{1,0} = \{X - iJX |\ X \in Q\}, \quad Q^{0,1} = \{X + iJX |\ X \in Q\}.$$

- And $Q_C{}^* = Q_{1,0} \oplus Q_{0,1}$, where

$$Q_{1,0} = \{\theta + iJ\theta |\ \theta \in Q^*\} \quad \text{and} \quad Q_{0,1} = \{\theta - iJ\theta |\ \theta \in Q^*\},$$

where $(J\theta)(X) = -\theta(JX)$ for any $X \in Q$.

Transversally holomorphic maps

- Let $(M, g_M, J, \mathcal{F})$ and $(M', g', \mathcal{F}', J')$ be two Riemannian manifolds with Kähler foliations $\mathcal{F}$ (codim $\mathcal{F} = 2n$) and $\mathcal{F}'$ (codim$\mathcal{F}' = 2n'$), respectively.
- Let $\phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J')$ be a smooth foliated map. We define the operators

$$\partial_T\phi : Q^{1,0} \to Q'^{1,0}, \quad \partial_T\bar{\phi} : Q^{1,0} \to Q'^{0,1},$$
$$\bar{\partial}_T\phi : Q^{0,1} \to Q'^{1,0}, \quad \bar{\partial}_T\bar{\phi} : Q^{0,1} \to Q'^{0,1}$$

by

$$d_T\phi|_{Q^{1,0}} = \partial_T\phi + \partial_T\bar{\phi}, \tag{17}$$
$$d_T\phi|_{Q^{0,1}} = \bar{\partial}_T\phi + \bar{\partial}_T\bar{\phi}. \tag{18}$$

Transversally holomorphic maps

- $\phi : (M, g, \mathcal{F}, J) \to (M', g', J', \mathcal{F}')$ is a **transversally holomorhic map (resp. anti holomorhic map)** if

$$d_T\phi \circ J = J' \circ d_T\phi \quad (\text{resp. } d_T\phi \circ J = -J' \circ d_T\phi). \tag{19}$$

- Note that

$$\bar\partial_T\phi = 0 \iff d_T\phi \circ J = J' \circ d_T\phi,$$
$$\partial_T\phi = 0 \iff d_T\phi \circ J = -J' \circ d_T\phi.$$

- Trivially, ϕ is transversally holomorphic (resp. anti-holomorphic) if and only if $\bar\partial_T\phi = 0$ (resp. $\partial_T\phi = 0$).

Transversally holomorphic maps

- Any transversally holomorphic(or anti-holomorhic) map $\phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J')$ is transversally harmonic. In fact, for J-basis $\{E_a, JE_a\}$,

$$\tau_b(\phi) = \sum_{a=1}^{n} \{\nabla^{\phi}_{E_a} d_T\phi(E_a) + J'\nabla^{\phi}_{E_a} J' d_T\phi(E_a)\}$$
$$= 0 \ (\phi : \ \text{holomorphic map})$$

- Transversal partial f-energies.

$$E_f^+(\phi) = \int_M |f\partial_T\phi|^2\mu_M, \quad E_f^-(\phi) = \int_M |f\bar{\partial}_T\phi|^2\mu_M.$$

- Trivially, $E_f(\phi) = E_f^+(\phi) + E_f^-(\phi)$.

Transversally holomorphic maps

- Define $K_f(\phi) := E_f^+(\phi) - E_f^-(\phi)$.
- For basic Kähler forms ω and ω' on $\mathcal{F}$ and $\mathcal{F}'$,

$$K_f(\phi) = \frac{1}{2}\int_M < \phi^*\omega', \omega > f^2\mu_M. \tag{20}$$

- Let $\{\phi_t\}$ be a foliated variation of ϕ.. Then $\frac{d}{dt}\phi_t^*\omega'$ is an exact form, i.e.,

$$\frac{d}{dt}\phi_t^*\omega' = d_B\theta_t. \tag{21}$$

- Then $K_{f_k}(\phi_t)$ is constant, where f_k is a solution of $\omega_f = 0$. In fact,

$$\frac{d}{dt}K_f(\phi_t) = \frac{1}{2}\int_M < \theta_t, \delta_T\omega + i(\omega_f^\sharp)\omega > f^2.$$

Transversally holomorphic map

Theorem (Jung-Jung,2014)

Let $\phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J')$ be a foliated map with M compact. If ϕ is transversally holomorphic, then ϕ is transversally harmonic with the minimum transversal f_k-energy in its foliated homotopy class.

Proof. Harmonicity is trivial. Since ϕ is transversally holomorphic, $E_f^-(\phi) = 0$. Then

$$\begin{aligned} E_{f_k}(\phi) &= E_{f_k}^+(\phi_0) - E_{f_k}^-(\phi_0) \\ &= K_{f_k}(\phi_0) = K_{f_k}(\phi_t) \quad \text{(constant)} \\ &\leqslant E_{f_k}(\phi_t). \end{aligned}$$

Generalized Jacobi fields

- The **generalized Jacobi field** V is the kernel of J^T_ϕ, where

$$J^T_\phi(V) = \left(\nabla^\phi_{tr}\right)^* \left(\nabla^\phi_{tr}\right)V - \nabla^\phi_\kappa V - tr_Q R^{Q'}(V, d_T\phi)d_T\phi.$$

- (Jung-Jung, 2014 [5]) Let M be a closed, oriented, connected Riemannian manifold and let $\phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J')$ be a transversally holomorphic map. Assume that $Ric^Q \geqslant 0$ and > 0 at some point or $\mathcal{F}$ is minimal. Then $V \in \phi^{-1}Q'$ is a generalized Jacobi field along ϕ if and only if V is a transversally holomorphic section.

Generalized Jacobi fields

- (Jung-Jung, 2014 [5]) Let $(M, g, \mathcal{F}, J)$ be a closed, oriented, connected Riemannian manifold with a Kähler foliation $\mathcal{F}$. If $\mathrm{Ric}^Q \geqslant 0$ and > 0 at some point or $\mathcal{F}$ is minimal, then for any $Y \in V(\mathcal{F})$, $\pi(Y)$ is a transversally holomorphic field, that is, $\theta(Y)J = 0$ if and only if $\pi(Y)$ is a generalized Jacobi field of $\mathcal{F}$, that is, $J^T_{id}(\pi(Y)) = 0$.
- When $\mathcal{F}$ is minimal, it is proved by S. Nishikawa and Ph. Tondeur [10].
- Let $(M, g, \mathcal{F}, J)$ be a compact Riemannian with a Kähler foliation. If $\mathrm{Ric}^Q \leqslant 0$ and < 0 at some point, then any transversally holomorphic field $\bar{Y}$ is trivial (Jung-Liu, 2012 [6]).

References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10(1992), 179-194.

[2] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86(1964), 106-160.

[3] S. D. Jung, *Variation formulas for transversally harmonic and bi-harmonic maps*, J. Geom. Phys. 70(2013), 9-20.

[4] M. J. Jung and S. D. Jung, *Transversally harmonic maps of foliated Riemannian manifolds.* J. Korean Math. Soc. 49(2012), 977-991.

[5] S. D. Jung and M. J. Jung, *Transversally holomorphic maps between Kähler foliations*, J. Math. Anal. Appl. 416(2014), 683-697.

[6] S. D. Jung and H. Liu, *Transversal infinitesimal automorphisms on Káhler foliations*, Bull. Aust. Math. Soc. 86 (2012), 405-415.
[7] J. Konderak and R. Wolak, *On transversally harmonic maps between manifolds with Riemannian foliations*, Quart. J. Math. Oxford ser. (2) 54(2003), 335-354.
[8] J. Konderak and R. Wolak, *Some remarks on transversally harmonic maps,* Glasgow Math. J. 50(2008), 1-16
[9] P. Molino, *Riemannian foliations*, Boston;Birkhäser, 1988.
[10] S. Nishikawa and Ph. Tondeur, *Transversal infinitesimal automorhisms for harmonic Kähler foliations*, Tohoku Math. J. 40 (1988), 599-611
[11] H. C. J. Sealey, *Harmonic maps of small energy*, Bull. London Math. Soc. V13(5), 1981), 405-408.

Thank You for your attention!