Transversally harmonic and holomorphic maps on foliated manifolds

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Abstract

Let \((M, \mathcal{F})\) and \((M', \mathcal{F}')\) be two foliated Riemannian manifold and let \(\phi : M \to M'\) be a smooth foliated map, i.e., \(\phi\) is a leaf-preserving map. Then we study the transversally holomorphic maps. In fact, a transversally holomorphic map is transversally harmonic map with the minimum energy in its foliated homotopy class.

Keyword

Riemannian foliation, Kähler foliation, transversal energy, transversally holomorphic map, transversally harmonic map.
Harmonic function

Definition

Harmonic function on an open domain \( \Omega \) of \( \mathbb{R}^m \) is a solution of the Laplace equation

\[
\Delta f = 0, \tag{1}
\]

where \( \Delta := -\frac{\partial^2}{(\partial x_1)^2} - \cdots - \frac{\partial^2}{(\partial x_m)^2} \) and \( (x_1, \cdots, x_m) \in \Omega \). The operator \( \Delta \) is called the Laplace operator or Laplacian.

- The harmonic functions are critical points of the Dirichlet functional

\[
E_\Omega(f) = \frac{1}{2} \int_\Omega |df|^2 dx. \tag{2}
\]
Let $(M, g)$ and $(N, h)$ be smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map.

The tension field $\tau(\phi)$ of $\phi$ is defined by

$$\tau(\phi) := \text{tr}_g \nabla d\phi = \text{div}(d\phi) = \sum_{i=1}^{m} (\nabla_{e_i} d\phi)(e_i), \quad (3)$$

where $\{e_i\}$ is a local orthonormal frame field on $M$. 
Definition
Let $\phi : (M, g) \rightarrow (N, h)$ be a smooth map. Then $\phi$ is said to be **harmonic** if the tension field vanishes, i.e., $\tau(\phi) = 0$.

- Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Then the Laplace-Beltrami operator $\Delta$ is given by
  \[
  \Delta f = \delta df = -\text{tr}(\nabla df) = -\tau(f). \tag{4}
  \]

- Hence $\Delta f = 0$ if and only if $\tau(f) = 0$. That is, $f : M \rightarrow \mathbb{R}$ is a harmonic function if and only if $\tau(f) = 0$. 
Examples. (1) **Constant maps** and **identity maps**.
(2) **Geodesics**. For a unit speed curve $\gamma : I \to N$ ($I \subset \mathbb{R}$), the tension field $\tau(\gamma)$ is $\tau(\gamma) = \nabla_{\gamma'} \gamma'$, the acceleration vector; hence $\gamma$ is harmonic if and only if it is a geodesic curve.
(3) **Isometric immersions**. Let $\phi : M^m \to N^n$ be an isometric immersion. Then $\tau(\phi) = mH$, where $H$ is the mean curvature vector of $M$ in $N$, so that $\phi$ is harmonic if and only if $M$ is a minimal submanifold of $N$.
(4) **Gauss maps**. Let $G(k, n)$ be the Grassman manifold of $k$-planes in $\mathbb{R}^n$. Let $G : M^k \to G(k, n)$ be the Gauss map associated to an immersion $i : M \to \mathbb{R}^n$. Then the tension field $\tau(G)$ of $G$ is $\tau(G) = \nabla H$, where $H$ is the mean curvature vector field of $M$. Hence $G$ is harmonic if and only if the mean curvature vector field is parallel.
Variation formula

The energy of \( \phi : M \to N \) over a compact domain \( \Omega \) is

\[
E_{\Omega}(\phi) = \frac{1}{2} \int_{\Omega} |d\phi|^2 dM. \tag{5}
\]

Theorem (Variation formula)

The first variation is given by

\[
\frac{d}{dt} E_{\Omega}(\phi_t)|_{t=0} = -\int_{M} <\tau(\phi), V > dM, \tag{6}
\]

where \( V = \frac{d\phi_t}{dt}|_{t=0} \) and \( \{\phi_t\} \) be all smooth variations of \( \phi \).

Hence a harmonic map \( \phi \) is a critical point of \( E_{\Omega}(\phi) \) over any compact domain \( \Omega \).
Let $(M, F)$ be a $(p + q)$-dimensional foliated Riemannian manifold with foliation $F$ of codimension $q$.

Let $TM$ be the tangent bundle of $M$, $L$ the tangent bundle of $F$, and $Q = TM/L$ the corresponding normal bundle of $F$.

A foliation is **Riemannian** if there exists a Riemannian metric $g$ satisfying $\theta(X)g = 0$ for all $X \in \Gamma L$. (This is called a bundle-like metric).

Equivalently, a bundle-like metric means that all geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.

Let $R^Q$ and $\text{Ric}^Q$ and be the transversal curvature and Ricci operator with respect to transversal Levi-Civita connection $\nabla^Q \equiv \nabla$ on $Q$, respectively.
Basic cohomology

- Let $\Omega^r_B(\mathcal{F})$ be a space of **basic form** $\omega$, i.e., $i(X)\omega = 0$ and $i(X)d\omega = 0$ for any $X \in \Gamma L$. Locally, for a foliated coordinate $(x_j, y_a)$,

$$\omega = \sum f_{a_1, \ldots, a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r}, \quad \frac{\partial f}{\partial x_j} = 0.$$ 

- Let $d_B = d|_{\Omega_B}$ and $\delta_B$: the adjoint operator of $d_B$. Note that $\delta_B \neq \delta|_{\Omega_B}$.

- The **basic Laplacian** is defined by $\Delta_B = d_B \delta_B + \delta_B d_B$.

- **(de-Rham Hodge decomposition)** Let $\mathcal{F}$ be a Riemannian foliation on a closed manifold $M$. Then

$$\Omega^r_B(\mathcal{F}) = \mathcal{H}^r_B \oplus \text{Im} d_B \oplus \text{Im} \delta_B,$$

where $\mathcal{H}^r_B = \text{Ker} \Delta_B$ is finite dimensional space.

- $\mathcal{H}^r_B \cong \mathcal{H}^r_B$, where $H^r_B = \frac{\text{Ker} d_B}{\text{Im} \delta_B}$ is the de-Rham basic cohomology group.
Transversally harmonic maps

- Let \((M, g, \mathcal{F})\) and \((M', g', \mathcal{F}')\) be two foliated Riemannian manifolds and let \(\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')\) be a smooth leaf preserving map, i.e., \(d\phi(L) \subset L'\).
- The differential map \(d_T\phi : Q \to Q'\) is defined by
  \[
d_T\phi := \pi' \circ d\phi \circ \sigma, \tag{8}
\]
  where \(\sigma : Q \to L^\perp\) is an isomorphism with \(\pi \circ \sigma = \text{id}\).
- Then \(d_T\phi\) is a section in \(Q^* \otimes \phi^{-1}Q'\).
- Let \(\nabla^\phi\) and \(\tilde{\nabla}\) be the connections on \(\phi^{-1}Q'\) and \(Q^* \otimes \phi^{-1}Q'\), respectively.
A map $\phi$ is said to be **transversally totally geodesic** if

$$\tilde{\nabla}_{\text{tr}} d_T \phi = 0, \quad (9)$$

where $(\tilde{\nabla}_{\text{tr}} d_T \phi)(X, Y) = (\tilde{\nabla}_X d_T \phi)(Y)$ for any $X, Y \in Q$. This means that if $\gamma$ is transversally geodesic, then $\phi \circ \gamma$ is also transversally geodesic.

The **transversal tension field** $\tau_b(\phi)$ is defined by

$$\tau_b(\phi) = \text{tr}_Q \tilde{\nabla} d_T \phi = \sum_{a=1}^{q} (\tilde{\nabla}_{E_a} d_T \phi)(E_a), \quad (10)$$

where $\{E_a\}$ is a local orthonormal basis of $Q$. 
A foliated map \( \phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}') \) is said to be **transversally harmonic** if \( \tau_b(\phi) = 0 \).

The **transversal energy** of \( \phi \) on a compact domain \( \Omega \subset M \) is defined by

\[
E_B(\phi; \Omega) = \frac{1}{2} \int_{\Omega} |d_T \phi|^2 \mu_M, \tag{11}
\]

where \( \mu_M \) is the volume element of \( M \).
Theorem (Jung-Jung(2012) [4])

Let \( \phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}') \) be a smooth foliated map. Let \( \{\phi_t\} \) be a smooth foliated variation of \( \phi \) supported in a compact domain \( \Omega \). Then

\[
\frac{d}{dt} E_B(\phi_t; \Omega) \big|_{t=0} = -\int_{\Omega} \langle V, \tau_b(\phi) - d_T \phi(\kappa^B) \rangle \mu_M, \tag{12}
\]

where \( V(x) = \frac{d\phi_t}{dt}(x) \big|_{t=0} \) is the normal variation vector field of \( \{\phi_t\} \).

- If \( \mathcal{F} \) is minimal, then the transversally harmonic map is a critical point the transversal energy \( E_B(\phi) \).
The transversal f-energy

**Definition**

Let $f$ be a non-zero basic function on $M$. Then the **transversal f-energy** of $\phi$ on a compact domain $\Omega$ is defined by

$$E_f(\phi; \Omega) = \frac{1}{2} \int_{\Omega} |f d_T \phi|^2 \mu_M. \quad (13)$$

- When $f$ is constant, $E_1(\phi) = E_B(\phi)$. 
The first variation formula II

Theorem (Jung, 2013 [3])

Let \( \phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}') \) be a smooth foliated map. Let \( \{\phi_t\} \) be a smooth foliated variation of \( \phi \) supported in a compact domain \( \Omega \). Then, for any non-zero basic function \( f \)

\[
\frac{d}{dt} E_f(\phi_t; \Omega)|_{t=0} = -\int_\Omega \langle V, \tau_b(\phi) - d_T \phi(\omega_f^\#) \rangle f^2 \mu_M, \tag{14}
\]

where \( V(x) = \frac{d\phi_t}{dt}(x)|_{t=0} \) and

\[
\omega_f = \kappa_B - d_B(\ln f^2). \tag{15}
\]
The first variation formula III

**Theorem (Jung,2013 [3])**

Let \( \phi : (M, g, \mathcal{F}) \rightarrow (M', g', \mathcal{F}') \) be a smooth foliated map. If \( \mathcal{F} \) is of positive transversal Ricci curvature or minimal, then

\[
\frac{d}{dt}E_{f_k}(\phi_t; \Omega)|_{t=0} = -\int_{\Omega} \langle V, \tau_b(\phi) \rangle f_k^2 \mu_M, \quad (16)
\]

where \( f_k \) is a solution of \( \omega_f = 0 \). Specially, any transversally harmonic map is a critical point of the transversal \( f_k \)-energy.

- The existence of the solution of \( \omega_f = 0 \) is partially assured on a foliated Riemannian manifold with positive transversal Ricci curvature. In fact, \( H^1_B(\mathcal{F}) = 0 \).
Kähler foliations

- Let $(\mathcal{M}, g_{\mathcal{M}}, J, \mathcal{F})$ be a Kähler foliation with a foliation $\mathcal{F}$ and a bundle-like metric $g_{\mathcal{M}}$. Here $J : Q \to Q$ is a complex structure.
- Let $\omega \in \Omega^2_B(\mathcal{F})$ be a basic Kähler 2-form.
- Let $Q_C = Q \otimes \mathbb{C}$ be the complexified normal bundle.
- Then $Q_C = Q^{1,0} \oplus Q^{0,1}$, where

  $Q^{1,0} = \{X - iJX| \ X \in Q\}$, \quad $Q^{0,1} = \{X + iJX| \ X \in Q\}$.

- And $Q_C^* = Q_{1,0} \oplus Q_{0,1}$, where

  $Q_{1,0} = \{\theta + iJ\theta| \ \theta \in Q^*\}$ \quad and \quad $Q_{0,1} = \{\theta - iJ\theta| \ \theta \in Q^*\}$,

  where $(J\theta)(X) = -\theta(JX)$ for any $X \in Q$. 
Transversally holomorphic maps

- Let \((M, g_M, J, \mathcal{F})\) and \((M', g', \mathcal{F}', J')\) be two Riemannian manifolds with Kähler foliations \(\mathcal{F}\) (codim \(\mathcal{F} = 2n\)) and \(\mathcal{F}'\) (codim \(\mathcal{F}' = 2n'\)), respectively.

- Let \(\phi : (M, g, \mathcal{F}, J) \rightarrow (M', g', \mathcal{F}', J')\) be a smooth foliated map. We define the operators

\[
\begin{align*}
\partial_T \phi & : Q^{1,0} \rightarrow Q'^{1,0}, & \partial_T \bar{\phi} & : Q^{1,0} \rightarrow Q'^{0,1}, \\
\bar{\partial}_T \phi & : Q^{0,1} \rightarrow Q'^{1,0}, & \bar{\partial}_T \bar{\phi} & : Q^{0,1} \rightarrow Q'^{0,1}
\end{align*}
\]

by

\[
\begin{align*}
d_T \phi|_{Q^{1,0}} &= \partial_T \phi + \partial_T \bar{\phi}, \quad (17) \\
d_T \phi|_{Q^{0,1}} &= \bar{\partial}_T \phi + \bar{\partial}_T \bar{\phi}. \quad (18)
\end{align*}
\]
Transversally holomorphic maps

- \( \phi : (M, g, F, J) \rightarrow (M', g', J', F') \) is a transversally holomorphic map (resp. anti holomorphic map) if

\[
d_T \phi \circ J = J' \circ d_T \phi \quad (\text{resp. } d_T \phi \circ J = -J' \circ d_T \phi). \tag{19}
\]

- Note that

\[
\bar{\partial}_T \phi = 0 \iff d_T \phi \circ J = J' \circ d_T \phi,
\]

\[
\partial_T \phi = 0 \iff d_T \phi \circ J = -J' \circ d_T \phi.
\]

- Trivially, \( \phi \) is transversally holomorphic (resp. anti-holomorphic) if and only if \( \bar{\partial}_T \phi = 0 \) (resp. \( \partial_T \phi = 0 \)).
**Transversally holomorphic maps**

- Any transversally holomorphic (or anti-holomorphic) map \( \phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J') \) is transversally harmonic. In fact, for \( J \)-basis \( \{E_a, JE_a\} \),

\[
\tau_b(\phi) = \sum_{a=1}^{n} \left\{ \nabla^\phi_{E_a} d_T \phi(E_a) + J' \nabla^\phi_{E_a} J' d_T \phi(E_a) \right\} = 0 \quad (\phi: \text{holomorphic map})
\]

- Transversal partial \( f \)-energies.

\[
E^+_f(\phi) = \int_M |f \bar{\partial} T \phi|^2 \mu_M, \quad E^-_f(\phi) = \int_M |f \bar{\partial} T \phi|^2 \mu_M.
\]

- Trivially, \( E_f(\phi) = E^+_f(\phi) + E^-_f(\phi) \).
Transversally holomorphic maps

- Define $K_f(\phi) := E_f^+(\phi) - E_f^-(\phi)$.
- For basic Kähler forms $\omega$ and $\omega'$ on $\mathcal{F}$ and $\mathcal{F}'$,

$$K_f(\phi) = \frac{1}{2} \int_M <\phi^* \omega', \omega > f^2 \mu_M.$$  \hfill (20)

- Let $\{\phi_t\}$ be a foliated variation of $\phi$. Then $\frac{d}{dt} \phi_t^* \omega'$ is an exact form, i.e.,

$$\frac{d}{dt} \phi_t^* \omega' = d_B \theta_t.$$  \hfill (21)

- Then $K_{f_k}(\phi_t)$ is constant, where $f_k$ is a solution of $\omega_f = 0$.

In fact,

$$\frac{d}{dt} K_f(\phi_t) = \frac{1}{2} \int_M <\theta_t, \delta_T \omega + i(\omega_f^\sharp)\omega > f^2.$$
Theorem (Jung-Jung, 2014)

Let \( \phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J') \) be a foliated map with \( M \) compact. If \( \phi \) is transversally holomorphic, then \( \phi \) is transversally harmonic with the minimum transversal \( f_k \)-energy in its foliated homotopy class.

**Proof.** Harmonicity is trivial. Since \( \phi \) is transversally holomorphic, \( E_f^-(\phi) = 0 \). Then

\[
E_{f_k}(\phi) = E_{f_k}^+(\phi_0) - E_{f_k}^-(\phi_0) \\
= K_{f_k}(\phi_0) = K_{f_k}(\phi_t) \quad \text{(constant)} \\
\leq E_{f_k}(\phi_t).
\]
Generalized Jacobi fields

- **The generalized Jacobi field** $V$ is the kernel of $J^T_\phi$, where

  $$J^T_\phi(V) = \left(\nabla^\phi_{tr}\right)^* \left(\nabla^\phi_{tr}\right) V - \nabla^\phi_K V - \text{tr}_Q R^Q_\phi(V, dT\phi) dT\phi.$$

- (Jung-Jung, 2014 [5]) Let $M$ be a closed, oriented, connected Riemannian manifold and let $\phi : (M, g, \mathcal{F}, J) \to (M', g', \mathcal{F}', J')$ be a transversally holomorphic map. Assume that $\text{Ric}^Q \geq 0$ and $> 0$ at some point or $\mathcal{F}$ is minimal. Then $V \in \phi^{-1}Q'$ is a generalized Jacobi field along $\phi$ if and only if $V$ is a transversally holomorphic section.
Generalized Jacobi fields

- (Jung-Jung, 2014 [5]) Let \((M, g, \mathcal{F}, J)\) be a closed, oriented, connected Riemannian manifold with a Kähler foliation \(\mathcal{F}\). If \(\text{Ric}^Q \geq 0\) and \(> 0\) at some point or \(\mathcal{F}\) is minimal, then for any \(Y \in V(\mathcal{F})\), \(\pi(Y)\) is a transversally holomorphic field, that is, \(\theta(Y)J = 0\) if and only if \(\pi(Y)\) is a generalized Jacobi field of \(\mathcal{F}\), that is, \(J_{\text{id}}(\pi(Y)) = 0\).

- When \(\mathcal{F}\) is minimal, it is proved by S. Nishikawa and Ph. Tondeur [10].

- Let \((M, g, \mathcal{F}, J)\) be a compact Riemannian with a Kähler foliation. If \(\text{Ric}^Q \leq 0\) and \(< 0\) at some point, then any transversally holomorphic field \(\bar{Y}\) is trivial (Jung-Liu, 2012 [6]).
References


Thank You for your attention!