Isospectral Riemannian surfaces

Hyunsuk Kang
GIST

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$$\begin{aligned}\Delta &= -\frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x_i}(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_j}) \\ &= -g^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \text{lower order terms.}\end{aligned}$$

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- $4\pi^2|\gamma|^2, \gamma \in \Gamma^*$ for flat tori $\Gamma\backslash\mathbb{R}^n$.

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- Spectral geometry deals with the mutual influences between the spectrum of a Riemannian manifold and its geometry
  - D. Schueth.

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  $a_1 = \frac{1}{6} \int_M R$

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- Spectra on functions, one-forms, two-forms determine whether the manifold has a constant sectional curvature.
- For Einstein manifolds M_1, M_2 with same a_0, a_1, a_2, if M_1 has constant curvature K, so does M_2.

Counterexamples/ Developments

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[**Sunada** 1985] Seminal paper 'Riemannian coverings and isospectral manifolds'.

From this point on, the 'covering technique' is commonly used for construction of isospectral manifolds.

[**Buser** 1986, 1992, **Brooks-Tse** 1987] Isospectral Riemann surfaces (const. curv. -1) of genus ≥ 4.

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[**Gordon-Webb** 1994] Isospectral convex polygons in the hyperbolic plane (constant curvature=-1). (Dirichlet and Neumann isospectral)

[**Gordon et al, Schueth, Szabo** 1993-2005] Using torus action/ Riemannian submersion, produced isospectral manifolds with different local geometry such as isospectral left-invariant metrics on $S^m \times S^n$ and $S^m \times B^n$ and other compact Lie groups:

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[**Barden-K** 2012] Isospectral non-isometric Riemannian surfaces of genus $2, 3$ with variable curvature.

Sunada's theorem

Definition

Let G be a finite group and U, V be subgroups of G. Then U and V are said to be *almost conjugate* if $\forall g \in G$, $|[g] \cap U| = |[g] \cap V|$. If U, V are not conjugate, call (G, U, V) a Sunada triple.

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Example

$T = \Sigma_6$ permutation group on $\{1, 2, ..., 6\}$.

$$\begin{aligned} U_1 &= \{id, (12)(34), (13)(24), (14)(23)\} \\ U_2 &= \{id, (12)(34), (12)(56), (34)(56)\} \end{aligned}$$

Check both U_i meet the conjugacy class of permutations with cycle 2-2-1-1 in 3 elements. So almost conjugate.
U_1 has common fixed points $5, 6$ whereas U_2 has none. So non-conjugate.

Theorem

*(**Sunada** 1985) Let G act on a compact Riemannian manifold M by isometries. Suppose that U and V are almost conjugate subgroups of G and U and V act freely on M. Then $U\backslash M$ and $V\backslash M$ are isospectral.*

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Remark Giving a 'bumpy metric' (no two points have isometric nhds) on $M_0 = G \backslash M$, Sunada showed that isospectral manifolds can be non-isometric. When constructed isospectral manifolds turn out to be isometric, we give a bumpy metric in the common covered manifold to endow non-isometry between covering manifolds.

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- To check non-isometry, ad hoc method.

Isospectral surfaces

[**Perlis** 1977] For a Sunada triple, index[G:U] ≥ 7.

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- Both motivated came from number field theory;

Fields derived for U and V have the same zeta function.

$\Leftrightarrow$ G-sets $U\backslash G, V\backslash G$ are *linearly equivalent*, i.e. every $g \in G$ has the same number of fixed points in $U\backslash G, V\backslash G$.

$\Leftrightarrow$ U, V are almost conjugate in G.

[**Brooks-Tse** 1987] constructed genus 3 surfaces of variable curvature.

Use $\mathbf{SL}(3,\mathbb{Z}/2)$ of order 168 has two generators of order 7 with our choice of generators of order 7

$$E = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

with its product EF has order 7.

Theorem (Barden-K)

There are isospectral non-isometric Riemann surfaces of genus 3.

Isospectral surfaces of genus 2

Use the 2-generator group $G = \langle a, b \rangle$ of order 96 realised as a subgroup of A_{12}, group of even permutations on 12 symbols:

$$\begin{aligned} a &= (0\ 7\ 11)(1\ 5\ 6)(2\ 9\ 10)(3\ 4\ 8) \\ b &= (0\ 4\ 2)(1\ 5\ 9)(3\ 7\ 11)(6\ 10\ 8) \\ ab &= (0\ 11\ 4\ 6\ 5\ 10)(1\ 9\ 8\ 7\ 3\ 2) \end{aligned}$$

$\Rightarrow$ Obtain non-conjugate subgroups U, V of index 12 as direct products of cyclic groups C_2 and C_4.

$$\Rightarrow \chi(M_i) = 12\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{6} - 1\right) = -2$$

[**Barden-K** 2012] Isospectral non-isometric Riemannian surfaces of genus 2 with variable curvature.

- With singularities of order $3, 3, 6$, we do not have figure 8 geodesics as in genus 3 case. However with a continuous deformation of the fundamental domain, one claim:

[Claim] There exist isospectal but non-isometric Riemann surfaces of genus 2.

Future work

- What additional data other than spectrum is required in order to determine the geometry of Riemannian manifolds, in particular, for Riemann surfaces? In higher dimension, more data will be required.