Smooth scalar curvature decrease of any Riemannian metric of dimension≥ 3

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In 1974, Muto has studied the behavior of the total scalar curvature and shown that for any Riemannian metric g on a compact manifold, the total scalar curvature can decrease as one deforms g to $g + th$ for some symmetric (2,0)-tensor h. Moreover, Muto could choose h supported in any ball.

In 1995, Lohkamp has made the following conjecture in Riemannian geometry.

Conjecture 1 *Let (M^k, g_0), $k \geq 3$, be a manifold and $B \subset M$ an open ball. Then there is a C^∞-continuous path of Riemannian metrics g_t, $0 \leq t \leq \varepsilon$ on M with $g(0) = g$ and*
(i) *Ricci curvature of g_t is strictly decreasing in t on B.*
(ii) *$g_t \equiv g_0$ on $M \backslash B$.*

In this talk, such a g_t shall be called a (Ricci) melting of g on B in M.

A related scalar-curvature deformation result can be found in Lohkamp's paper in 1999, where it is shown that for any metric g in M and a ball $B \subset M$, and a smooth function f such that $f = s(g)$ outside B and $f < s(g)$ inside B, for any $\varepsilon > 0$ there exists a deformed Riemannain metric g_ε whose scalar curvature $s(g_\varepsilon)$ lies $f - \varepsilon \leq s(g_\varepsilon) \leq f$ on B_ε and $g_\varepsilon = g$ on $M \setminus B_\varepsilon$, where B_ε is a ε-neighborhood of B. From the construction, these metrics can be close to g in C^0 sense, but not in C^∞ topology.

Concerning this conjecture, not even the scalar-curvature case is thoroughly studied, although one may believe that for generic metrics such a scalar-curvature-decreasing path of metrics should exist.

Note that the conjecture is concerned with *local* deformation of metrics on a ball in a manifold; on a compact Riemannian manifold, Muto's integral decrease and conformal deformation argument (Yamabe solutions) would produce a scalar-curvature decrease on the whole manifold. More simply, homothetic changes on a metric can decrease the scalar curvature if it is not zero. Here we prove the following result.

Theorem 2 *Given a Riemannian metric g on a manifold M of dimension≥ 3 and a ball B in M, we obtain C^∞-continuous paths of Riemannian metrics g_t, $0 \leq t < \varepsilon$ on M with $g_0 = g$ such that the scalar curvatures $s(g_t)$ strictly decrease, i.e. $s(g_{t_1}) > s(g_{t_2})$ for $t_1 < t_2$ on B, $g_t = g$ on $M \setminus B$.*

1 Frame of Proof

Consider the scalar curvature functional $s : \mathcal{M} \to C^\infty(M)$ by $s : g \mapsto s(g)$ where $\mathcal{M}$ is the space of smooth Riemannian metrics on a manifold M. Its derivative is $Ds_g : T_g\mathcal{M} \to C^\infty(M)$; for a symmetric $(2,0)$ tensor field h,

$$Ds_g(h) = \Delta_g(\mathrm{tr_g h}) + \delta_\mathrm{g}(\delta_\mathrm{g}\mathrm{h}) - \mathrm{g}(\mathrm{ric_g}, \mathrm{h}). \tag{1}$$

The formal L^2 adjoint of Ds_g is

$$Ds^*_g(\psi) = \nabla_g d\psi + \frac{s}{n-1}(\Delta_g\psi)g - \psi r_g.$$

Finding its kernel is equivalent to solving for a function ψ;

$$\nabla_g d\psi + \frac{s}{n-1}\psi g - \psi r_g = 0. \tag{2}$$

For an open set U, let $L^2_{loc}(U)$ be the space of functions on U which is L^2 on each compact subsets. And $H^2_{loc}(U)$ is the space of functions f on U such that f, $|\nabla f|_g$ and $|\nabla\nabla f|_g$ are L^2 on each compact subsets of U.

Consider $Ds^*_g : H^2_{loc}(U) \to L^2_{loc}(U)$ and set

$$Ds^*_g(\psi) = \nabla_g d\psi + \frac{s}{n-1}\psi g - \psi r_g = 0, \quad \psi \in H^2_{loc}(U). \tag{3}$$

Lemma 1 *Suppose that there is a point in B that has a neighborhood where there is a nonzero solution of* (3)*. Then there exists a scalar curvature melting on a smaller ball in B.*

Lemma 2 *Suppose that there is a point p in B that does not have a neighborhood where there is a nonzero solution of* (3)*. Then there exists a scalar-curvature melting of g on a smaller ball.*

Lemma 3 *If there is a melting on a ball, then there is a melting on any larger ball.*

Above three lemmas prove the theorem.

2 Argument

We mainly discuss about Lemma 1, which needs a second order perturbation $g \mapsto g + th + t^2k$ of a given metric g as its main ingredient.

For the Euclidean metric g_0 on $\mathbb{R}^n$, there is nonzero C^∞ symmetric $(2,0)$ tensor field h_0 supported in a ball B in $\mathbb{R}^n$ such that $tr_{g_0}h_0 = 0$ and $\delta_{g_0}h_0 = 0$. Then $Ds_{g_0}(h_0) = 0$. It is known that $\frac{d^2s(g_0+th_0+t^2Q_0)}{dt^2}|_{t=0} \leq 0$ where Q_0 is a symmetric $(2,0)$ tensor field related to h_0.

Next,

Lemma 4 *If a Riemannian metric g satisfies $\| g - g_0 \|_{C^{k+1}(g_0)} < \varepsilon$ on B, then there exists a symmetric $(2,0)$ tensor h such that h is supported in B and $tr_g h = 0$ and $\delta_g h = 0$ and that $\| h - h_0 \|_{C^{l+1}(g_0)} < k(\varepsilon)$, where $k(\varepsilon)$ is a positive number depending on ε and $\lim_{\varepsilon\to 0} k(\varepsilon) = 0$.*

Proof. Set $h = h_0 + U$. We look for a tensor field U with small norm $\|U\|_{C^3(g_0)}$ supported in B such that

$$\delta_g(h_0 + U) = 0 \tag{4}$$

To find such a compactly supported smooth solution U, we apply the existence and regularity theory for elliptic differential equations defined for the weighted Sobolev and *Hölder* function spaces, developed recently by Corvino, Delay etc.. In fact, we set up the following equation for u;

$$\delta_g(h_0 + \psi^2 \delta_g^*(u)) = 0 \tag{5}$$

where ψ is some weight function, defined in terms of the distance function x to the boundary of B. ($\psi = x^\alpha e^{-\frac{1}{x}}$.) By Schauder estimate for interior and near-boundary, we get smooth h which has support in $\overline{B}$.

∎

With h obtained above, we should deform h just a little bit;

Lemma 5 *Suppose that h satisfies $\delta_g h = 0$, $tr_g h = 0$ as obtained in Lemma 4. Then there exists a smooth compactly supported $(2,0)$ tensor field $\tilde{h}$ so that $Ds_g(\tilde{h}) = 0$ and small norm $\|h - \tilde{h}\|$.*

We define g_t on M by

$$g_t := \begin{cases} g + t\tilde{h} + t^2 Q(\tilde{h}) & \text{on } B, \\ g & \text{on } M \setminus B. \end{cases} \tag{6}$$

This g_t almost gives a scalalr curvaure melting; one more (slight) geometric diffusion is needed to produce a melting $G(t)$ on a smaller ball in B. Lemma 1 is proved.

The above diffusion argument can prove other lemmas and so the theorem.

Remark 1

A. Extending our argument, we can prove that the melting can be done in a 'big' scale.

B. The space of Riemannian metrics of non-positive scalar curvature on a compact or noncompact manifold can be shown to be contractible. - Lohkamp proved that the space of metrics with negative scalar curvature is contractible.

C. Our work here may be solved by some Yamabe type problem?