Spectral zeta function analysis for separable partial differential equations

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Joint work with Thalia Jeffres, Tianshi Lu (Wichita State University) and Guglielmo Fucci (East Carolina University)

Outline

Motivations

What are spectral functions?

- Eigenvalue problem for a suitable differential operator P def. on $\mathcal{M}$:

$$P u_\ell(x) = \lambda_\ell u_\ell(x), \qquad \mathcal{B} u_\ell|_{x\in\partial\mathcal{M}} = 0$$

$$0 < \lambda_1 \leq \lambda_2 ..., \quad \lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty.$$

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- Casimir energy:

$$\text{" } E_P = \frac{1}{2} \sum_{\ell=1}^{\infty} \lambda_\ell^{1/2} \text{ "}$$

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- Relation between zeta function and heat kernel:

$$\text{Res}\ \zeta_P(s) = \frac{a_{D/2-z}(P,\mathcal{B})}{\Gamma(z)}, \qquad z = \frac{D}{2}, \frac{D-1}{2}, \dots, \frac{1}{2}, -\frac{2n+1}{2}, n \in \mathbb{N}_0,$$

$$\zeta_P(-q) = (-1)^q q! a_{D/2+q}(P,\mathcal{B}), \qquad q \in \mathbb{N}_0$$

Zeta function $\zeta_P(s) = \sum_{\ell=1}^{\infty} \lambda_\ell^{-s}$ as best organization of the spectrum

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More precisely:

$$\zeta_P \left(-\frac{1}{2} + \epsilon \right) = -\frac{1}{\epsilon} \frac{1}{\sqrt{4\pi}} a_{\frac{D+1}{2}} (P, \mathcal{B}) + \text{FP } \zeta_P \left(-\frac{1}{2} \right) + \mathcal{O}(\epsilon)$$

Examples for separation of variables

- Two dimensional ball: Laplacian in polar coordinates:

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\phi_{m,n}(r,\varphi)=\omega_{m,n}^2\phi_{m,n}(r,\varphi)$$

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- Impose boundary condition:

$$J_{|m|}\left(\omega_{m,n}\right)=0$$

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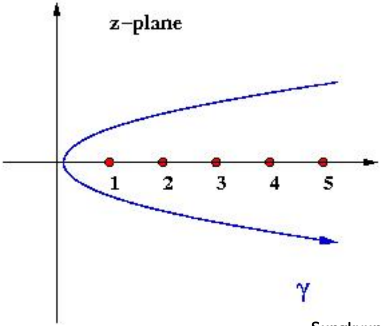

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- What is the suitable asymptotics to be subtracted?

$$I_m(mk) \sim \frac{1}{\sqrt{2\pi m}} \frac{e^{m\eta}}{(1+z^2)^{1/4}} \left(1 + \mathcal{O}\left(\frac{1}{m}\right)\right)$$

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$$\zeta'(0) = \frac{5}{12} + 2\zeta_R'(-1) + \frac{1}{2}\ln\pi + \frac{1}{6}\ln 2.$$

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M. Bordag, B. Geyer, KK and E. Elizalde, Commun. Math. Phys. 179 (1996) 215-234
M. Bordag, KK and J.S. Dowker, Commun. Math. Phys. 182 (1996) 371-394
KK, P. Loya and J. Park, J. Geom. Anal. 18 (2008) 835-888

Spherical suspension

- Let $\mathcal{M} = \mathcal{I} \times \mathcal{N}$, $I \subseteq [0, \theta_0]$, $\theta_0 \in (0, \pi)$, $\mathcal{N}$ a smooth, compact Riemannian d-dimensional manifold:

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- Separate variables:

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + \left[\left(-\frac{1}{4} - \omega^2\right) - \left(m^2 + \frac{(d-1)^2}{4}\right)\frac{1}{\sin^2\theta}\right]\psi = 0$$

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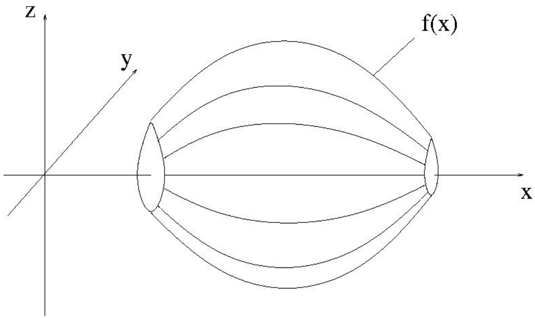
G. Fucci and KK, Commun. Math. Phys. 314 (2012) 483-507

Surfaces of revolution



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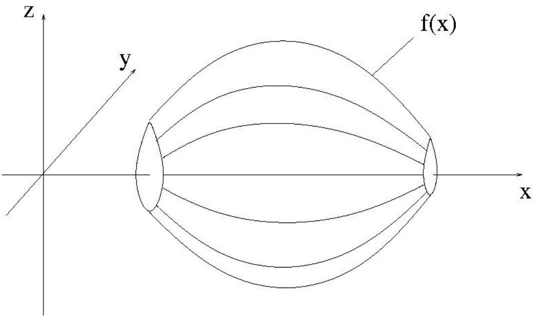

- Parameterization and metric of the surface

$$S = \begin{pmatrix} x \\ f(x)\cos\theta \\ f(x)\sin\theta \end{pmatrix},$$

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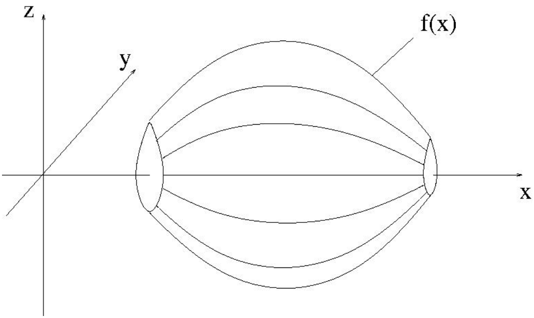

- Parameterization and metric of the surface

$$S = \begin{pmatrix} x \\ f(x)\cos\theta \\ f(x)\sin\theta \end{pmatrix}, \qquad g(x) = \begin{pmatrix} 1+(f'(x))^2 & 0 \\ 0 & f^2(x) \end{pmatrix}$$

Surfaces of revolution

- Laplacian on the surface

$$\begin{aligned} \Delta\varphi &= \frac{1}{\sqrt{\det g}}\partial_\mu\left(g^{\mu\nu}\sqrt{\det g}\,\partial_\nu\right)\varphi \\ &= \frac{1}{1+(f')^2}\left(\frac{\partial^2\varphi}{\partial x^2} - \frac{f'f''}{1+(f')^2}\,\frac{\partial\varphi}{\partial x} + \frac{f'}{f}\,\frac{\partial\varphi}{\partial x}\right) + \frac{1}{f^2}\,\frac{\partial^2\varphi}{\partial\theta^2} \end{aligned}$$

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$$\varphi_{m,n}(x,\theta)=\psi_{m,n}(x)\;e^{im\theta},\quad m\in\mathbb{Z}$$

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$$\varphi_{m,n}(x,\theta)=\psi_{m,n}(x)\; e^{im\theta}, \quad m\in\mathbb{Z}\Longrightarrow$$

$$\psi''_{m,n}(x)+\psi'_{m,n}(x)\left(\frac{f'}{f}-\frac{f'f''}{1+(f')^2}\right)+\left(\lambda_{m,n}-\frac{k^2}{f^2}\right)\left(1+(f')^2\right)\psi_{m,n}(x)=0$$

$$\psi_{m,n}(a)=\psi_{m,n}(b)=0$$

Surfaces of revolution

- Let

$$u = \frac{f'}{f} - \frac{f'f''}{1+(f')^2}, \qquad v = \left(\lambda - \frac{m^2}{f^2}\right)\left(1+(f')^2\right)$$

- Consider

$$\psi''_{m,\lambda} + u\psi'_{m,\lambda} + v\psi_{m,\lambda} = 0,$$

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$$\zeta(s) = \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty}\int_\gamma d\lambda\, \lambda^{-s}\frac{d}{d\lambda}\ln\psi_{m,\lambda}(b), \qquad \Re s > 1$$

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- Rewriting:

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$$= \frac{2\sin \pi s}{\pi} \sum_{m=1}^{\infty} \int_{0}^{\infty} dz \ \ z^{-s} m^{-2s} \frac{d}{dz} \ln \psi_{m,-m^2 z}(b) + \ldots$$

Relevant asymptotics from scratch

- Setup for surfaces of revolution for $\psi_{m,-m^2z}(b)$:

$$\left(\partial_x^2 + u\partial_x + v\right)\psi(x) = 0 \qquad \text{with}$$

$$u = \frac{f'}{f} - \frac{f'f''}{1+(f')^2}, \qquad v = -m^2\left(z + \frac{1}{f^2}\right)\left(1+(f')^2\right)$$

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- Ansatz for solution

$$\psi(x) = c\ \exp\left\{-\int dt\ \frac{u}{2}\right\}\ \exp\left\{\int dx\ S(x)\right\}$$

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- Assume

$$q = \sum_{i=-2}^{\infty} m^{-i} q_i \quad \Longrightarrow \quad S = \sum_{i=-1}^{\infty} m^{-i} S_i$$

- Explicitly

$$S_{-1} = \pm\sqrt{-q_{-2}}$$

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S_{i+1} &= -\frac{q_i + S_i' + \sum_{j=0}^{i} S_j S_{i-j}}{2S_{-1}}
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- For the surface of revolution

$$
\begin{aligned}
S_{-1}^{\pm} &= \pm\sqrt{\left(z + \frac{1}{f^2}\right)(1+(f')^2)} \\
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- Large-m expansion for $\psi_{m,-m^2 z}(b)$

$$\psi_{m,-m^2 z}(x) = \exp\left\{ -\int_a^x dt\ \frac{u}{2} \right\} \left\{ A \exp\left(\int_a^x dt\ S^+(t) \right) + B \exp\left(\int_a^x dt\ S^-(t) \right) \right\}$$

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$$\psi_{m,-m^2z}(a) = 0 \Longrightarrow A = -B$$

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$$\psi'_{m,-m^2z}(a) = 1 \Longrightarrow A = (S^+(a) - S^-(a))^{-1}$$

- Large-m expansion for $\psi_{m,-m^2z}(b)$

$$\psi_{m,-m^2z}(x) = \exp\left\{-\int_a^x dt\ \frac{u}{2}\right\}\left\{A\exp\left(\int_a^x dt\ S^+(t)\right) + B\exp\left(\int_a^x dt\ S^-(t)\right)\right\}$$

- Impose the initial conditions $\psi_{m,-m^2z}(a) = 0$, $\psi'_{m,-m^2z}(a) = 1$

$$\psi_{m,-m^2z}(a) = 0 \Longrightarrow A = -B$$
$$\psi'_{m,-m^2z}(a) = 1 \Longrightarrow A = (S^+(a) - S^-(a))^{-1}$$

- Relevant expansion

$$\partial_z \ln \psi_{m,-m^2z}(b) = -\partial_z \ln S^+_{-1}(a) + \partial_z \int_a^b dt\ \left(mS^+_{-1}(t) + S_0(t) + \frac{1}{m}S^+_1(t)\right) + \mathcal{O}\left(\frac{1}{m^2}\right)$$

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Very valuable check of the calculation!

- Notation:

$$\phi(q) = \prod_{k=1}^{\infty}(1 - q^k), \qquad A = \int_a^b \frac{\sqrt{1 + f'(x)^2}}{f(x)} dx$$

- Functional determinant:

$$\zeta'(0) = -2 \ln \phi\left(e^{-2\pi^2/A}\right) + \frac{\pi^2}{6A} + \frac{1}{6}\int_a^b \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}} dx$$
$$+ \frac{1}{2}\int_a^b \frac{f''(x)}{(1 + f'(x)^2)^{3/2}} dx$$

T.D. Jeffres, KK and Tianshi Lu, J. Phys. A: Math. Theor. 45 (2012) 345201 (16pp)

Warped product manifolds

- Let $\mathcal{M} = \mathcal{I} \times \mathcal{N}$, $I = [a, b]$, $\mathcal{N}$ a smooth, compact Riemannian d-dimensional manifold:

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G. Fucci and KK, Commun. Math. Phys. 317 (2013) 635-665

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- Relevant asymptotic behavior determined as before:

$$\psi_{m,-m^2 z}(x) = \exp\left[-\int_a^x dt\ \frac{u}{2}\right]\ \left\{A \exp\left[\int_a^x dt\ S^+(t)\right] + B \exp\left[\int_a^x dt\ S^-(t)\right]\right\}$$

J.M. Muñoz-Castañeda, KK and M. Bordag, *Heat kernel coefficients, spectral zeta functions and selfadjoint extensions*
G. Fucci and KK, *Spectral functions for regular Sturm-Liouville problems*

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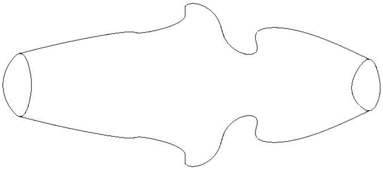

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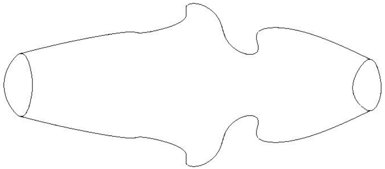

- Cusp-like singularities?