Non-strongly isospectral lens spaces that are Hodge-isospectral

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Joint work with Emilio Lauret and Juan Pablo Rossetti

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Notation

- M a compact Riemannian manifold without boundary,
- $\Lambda^p T^*M$ the exterior vector bundle,
- $\Delta_p := dd^* + d^*d$ the Hodge-Laplace operator on p-forms,
- If M is compact, the spectrum of Δ_p is a sequence

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$$

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Two manifolds are *p-isospectral* if the spectrum of the operators Δ_p is the same for both manifolds.

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There are many examples of non-isometric isospectral manifolds (Milnor, Vigneras80, Ikeda80, Gordon86, Gordon-Wilson, Schueth).

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Question

Hodge-isospectral implies strongly isospectral? (Wolf88, Gordon, Webb and others)

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Our goal in this lecture will be to describe these pairs together with their spectral properties.

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Our goal in this lecture will be to describe these pairs together with their spectral properties.

Lens spaces are spherical space forms with cyclic fundamental groups. For $q \in \mathbb{N}$, and $s_1, \dots, s_m \in \mathbb{Z}$ coprime to q, let

$$L(q; s_1, \dots, s_m) = \langle\gamma\rangle \backslash S^{2m-1},$$

where

$$\gamma = \operatorname{diag}\left(\begin{bmatrix} \cos(2\pi s_1/q) & \sin(2\pi s_1/q) \\ -\sin(2\pi s_1/q) & \cos(2\pi s_1/q) \end{bmatrix}, \dots, \begin{bmatrix} \cos(2\pi s_m/q) & \sin(2\pi s_m/q) \\ -\sin(2\pi s_m/q) & \cos(2\pi s_m/q) \end{bmatrix} \right).$$

Proposition

Let $L = L(q; s_1, \ldots, s_m)$ and $L' = L(q; s'_1, \ldots, s'_m)$ be lens spaces. The following assertions are equivalent.

1. L is isometric to L'.
2. L is diffeomorphic to L'.
3. L is homeomorphic to L'.
4. There exist $t \in \mathbb{Z}$ coprime to q and $\epsilon \in \{\pm 1\}^m$ such that $(s_1, \ldots, s_m)$ is a permutation of $(\epsilon_1 t s'_1, \ldots, \epsilon_m t s'_m) \pmod{q}$.

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We now obtain a consequence from the previous proposition. To the lens space $L(q; s_1, \ldots, s_m)$ we associate the congruence lattice $\mathcal{L}(q, s) = \mathcal{L}(q; s_1, \ldots, s_m)$ of all $(a_1, \ldots, a_m) \in \mathbb{Z}^m$ such that

$$a_1 s_1 + \cdots + a_m s_m \equiv 0 \pmod{q},$$

Proposition

Let $L(q;s)$, $L(q;s')$ be lens spaces, $\mathcal{L}(q,s)$ and $\mathcal{L}(q,s')$ the associated lattices. Then, $L(q;s)$ and $L(q;s')$ are isometric if and only if $\mathcal{L}(q,s)$ and $\mathcal{L}(q,s')$ are $\|\cdot\|_1$-isometric.

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Proof

L and L' are isometric if and only if there exist t coprime to q and φ, a composition of permutations and changes of signs, such that $\varphi(ts) = \varphi(ts_1, \ldots, ts_m) = (s'_1, \ldots, s'_m) = s'$.

The congruence lattices associated to the parameters $s_1, \ldots, s_m$ and $ts_1, \ldots, ts_m$ are the same.

On the other hand, $\mathcal{L}(q, \varphi(s)) = \varphi(\mathcal{L}(q,s))$ and φ is an isometry of $\mathbb{R}^n$ with respect to $\|\cdot\|_1$.

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The converse follows from the fact that every $\|\cdot\|_1$-linear isometry of $\mathbb{R}^n$ is a composition of permutations and changes of signs.

We recall some facts on Lie group representations. Let G be a compact Lie group K a closed subgroup. (We may suppose $G = SO(2m)$, $K = SO(2m-1)$.)
For a finite dimensional unitary representation (τ, W_τ) of K, consider the *homogeneous vector bundle*

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We form the vector bundle $\Gamma\backslash E_\tau$ over $\Gamma\backslash G/K$ and denote by $L^2(\Gamma\backslash E_\tau)$ the closure of $C^\infty(\Gamma\backslash G/K;\tau)$ with respect to the inner product $(f_1, f_2) = \int_{\Gamma\backslash X} \langle f_1(x), f_2(x)\rangle \, \mathrm{d}x$.

The Casimir element C lies in the center of $U(\mathfrak{g})$ and defines second order elliptic *differential operators* Δ_τ on $C^\infty(G/K;\tau)$ and $\Delta_{\tau,\Gamma}$ on $\Gamma\backslash E_\tau$; C acts on an irreducible representation V_π of G by a scalar $\lambda(C,\pi)$.

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The left regular representation of G on $L^2(E_\tau) \simeq L^2(G/K;\tau)$ decomposes into a sum of irreducible subrepresentations with finite multiplicity. The multiplicity of each $\pi \in \widehat{G}$ is given by Frobenius reciprocity: $[\tau : \pi_{|K}] := \dim Hom_K(V_\tau, V_\pi)$.

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By taking Γ-invariants, we get

$$L^2(\Gamma\backslash G/K : \tau) = \sum_{\pi\in\widehat{G}} [\tau : \pi_{|K}]\, V_\pi^\Gamma, \tag{0.1}$$

This implies

Proposition

Let G be a compact Lie group and K a closed subgroup. Let Γ be a discrete cocompact subgroup of G that acts freely on $X = G/K$. Let $\Delta_{\tau,\Gamma}$ be the Laplace operator acting on sections of $\Gamma\backslash E_\tau$.

If $\lambda \in \mathbb{R}$, the multiplicity $d_\lambda(\tau,\Gamma)$ of the eigenvalue λ of $\Delta_{\tau,\Gamma}$ is given by

$$d_\lambda(\tau,\Gamma) = \sum_{\pi\in\widehat{G}:\, \lambda(C,\pi)=\lambda} d_\pi^\Gamma \, \dim\left(\mathrm{Hom}_K(W_\tau^*, V_\pi)\right). \qquad (0.2)$$

Lemma

Let $L = L(q; s_1, \ldots, s_m)$ *be a lens space and let* $\mathcal{L} = \mathcal{L}(q; s_1, \ldots, s_m)$*. If* (π, V_π) *is a finite dimensional representation of* $SO(2m)$*, then*

$$\dim V_\pi^\Gamma = \sum_{\mu \in \mathcal{L}} m_\pi(\mu), \tag{0.3}$$

where $m_\pi(\mu)$ *denotes the multiplicity of the weight* μ *in the representation* π*.*

Proof.
One has that $V_\pi = \oplus_{\mu \in P(G)} V_\pi(\mu)$, *where* $V_\pi(\mu)$ *is the* μ-*weight space, i.e. the space of vectors* v *such that* $\pi(h)v = h^\mu v$ *for every* $h \in T$. *Hence, if* $\Gamma \subset T$ *is generated by* γ, *then* $V_\pi^\Gamma = \oplus_{\mu \in P(G)} V_\pi(\mu)^\Gamma$.
Now, $v \in V_\pi(\mu)$, $v \neq 0$, *is* Γ-*invariant if and only if* $\gamma^\mu = 1$, *hence*

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We let

$$H_\gamma = \operatorname{diag}\left(\begin{pmatrix} 0 & 2\pi s_1/q \\ -2\pi s_1/q & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 2\pi s_m/q \\ -2\pi s_m/q & 0 \end{pmatrix} \right),$$

thus $\exp(H_\gamma) = \gamma$. *If* $\mu = \sum_{j=1}^m a_j \varepsilon_j \in P(SO(2m))$ *then*

$$\gamma^\mu = e^{\mu(H_\gamma)} = e^{-2\pi i \left(\frac{a_1 s_1 + \dots + a_m s_m}{q} \right)} = 1,$$

if and only if $a_1 s_1 + \dots + a_m s_m \equiv 0 \pmod{q}$, *that is,* $\mu \in \mathcal{L}$.

Let $\mathcal{L}$ be an arbitrary sublattice of $\mathbb{Z}^m$. For $\mu \in \mathbb{Z}^m$ we set $Z(\mu) = \#\{j : 1 \leq j \leq m,\ a_j = 0\}$. Denote, for any $0 \leq \ell \leq m$ and any $k \in \mathbb{N}_0$,

$$N_{\mathcal{L}}(k) = \#\left\{\mu \in \mathcal{L} : \|\mu\|_1 = k\right\}, \tag{0.4}$$

$$N_{\mathcal{L}}(k, \ell) = \#\left\{\mu \in \mathcal{L} : \|\mu\|_1 = k,\ Z(\mu) = \ell\right\}. \tag{0.5}$$

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Definition

Let $\mathcal{L}$ and $\mathcal{L}'$ be sublattices of $\mathbb{Z}^m$.

(i) $\mathcal{L}$ and $\mathcal{L}'$ are said to be $\|\cdot\|_1$-*isospectral* if $N_{\mathcal{L}}(k) = N_{\mathcal{L}'}(k)$ for every $k \in \mathbb{N}$.

(ii) $\mathcal{L}$ and $\mathcal{L}'$ are said to be $\|\cdot\|_1^*$-*isospectral* if $N_{\mathcal{L}}(k, \ell) = N_{\mathcal{L}'}(k, \ell)$ for every $k \in \mathbb{N}$ and every $0 \leq \ell \leq m$.

To investigate the p-spectrum of lens spaces we will need two useful lemmas on weight multiplicities.

Lemma

Let $k \in \mathbb{N}$ and $0 \leq p \leq m$. If $\mu = \sum_{j=1}^{m} a_j \varepsilon_j \in \mathbb{Z}^m$ we have

$$m_{\pi_{k\varepsilon_1}}(\mu) = \begin{cases} \binom{r+m-2}{m-2} & \text{if } \|\mu\|_1 = k - 2r \quad \text{with } r \in \mathbb{N}_0, \\ 0 & \text{otherwise,} \end{cases} \tag{0.6}$$

$$m_{\pi_{\Lambda_p}}(\mu) = \begin{cases} \binom{m-p+2r}{r} & \text{if } \|\mu\|_1 = p - 2r \quad \text{with } r \in \mathbb{N}_0, \text{ and } |a_j| \leq 1 \; \forall j, \\ 0 & \text{otherwise.} \end{cases} \tag{0.7}$$

Here $\Lambda_p = \sum_{j=1}^{p} \varepsilon_j$ and $\pi_{0,p} = \pi_{\Lambda_p}$ is the exterior representation of $SO(2m)$ on $\bigwedge^p \mathbb{C}^{2m}$.

The second lemma is crucial in the proof of the main theorem.

Lemma

Let $\mu, \mu' \in P(SO(2m)) \simeq \mathbb{Z}^m$. If $\|\mu\|_1 = \|\mu'\|_1$ and $Z(\mu) = Z(\mu')$ then $m_{\pi_{k,p}}(\mu) = m_{\pi_{k,p}}(\mu')$ for every $k \in \mathbb{N}$ and every $1 \leq p \leq m$. Here $\pi_{k,p}$ is the irreducible representation of $SO(2m)$ with highest weight $k\varepsilon_1 + \Lambda_p$ if $p < m$ and the sum of the irreducible representations with highest weights $k\varepsilon_1 + \Lambda_p$ and $k\varepsilon_1 + \overline{\Lambda}_p$, when $p = m$.

The next theorem gives an explicit formula for $\dim V^{\Gamma}_{\pi_{k,p}}$ in terms of weight multiplicities $m_{\pi_{k,p}}(\mu)$ and number of lattice points $N_{\mathcal{L}}(k,\ell)$, where $\mathcal{L}$ is the congruence lattice of L.

Theorem

Let $L = \Gamma\backslash S^{2m-1}$ be a lens space with associated lattice $\mathcal{L}$ and let $k \in \mathbb{N}$ and $0 \le p \le m$. Then

$$\dim V^{\Gamma}_{\pi_{k,p}} = \sum_{r=0}^{[(k+p)/2]} \sum_{\ell=0}^{m} m_{\pi_{k,p}}(\mu_{r,\ell})\, N_{\mathcal{L}}(k+p-2r,\ell), \qquad (0.8)$$

where $\mu_{r,\ell}$ is any weight such that $Z(\mu_{r,\ell}) = \ell$ and $\|\mu_{r,\ell}\|_1 = k+p-2r$.
In the particular case when $p = 0$ we get that

$$\dim V^{\Gamma}_{\pi_{k\varepsilon_1}} = \sum_{r=0}^{[k/2]} \binom{r+m-2}{m-2} N_{\mathcal{L}}(k-2r). \qquad (0.9)$$

We can now state

Theorem

Let $L = \Gamma\backslash S^{2m-1}$ and $L' = \Gamma'\backslash S^{2m-1}$ be lens spaces with associated congruence lattices $\mathcal{L}$ and $\mathcal{L}'$ respectively. Then

(i) *L and L' are 0-isospectral if and only if $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1$-isospectral.*

(ii) *L and L' are p-isospectral for all p if and only if $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1^*$-isospectral.*

Remark

A pair of non-isometric lens spaces as given by Theorem 0.4 cannot be strongly isospectral, since two strongly isospectral lens spaces are necessarily isometric.

Remark

Ikeda (80) gave many pairs of non-isometric lens spaces that are 0-isospectral. The simplest is $L(11;1,2,3)$ and $L(11;1,2,4)$ in dimension 5. It follows that the associated congruence lattices $\mathcal{L} = \mathcal{L}(11;1,2,3)$ and $\mathcal{L}' = \mathcal{L}(11;1,2,4)$ must be $\|\cdot\|_1$-isospectral.

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However, $\mathcal{L}$ and $\mathcal{L}'$ are not $\|\cdot\|_1^*$-isospectral.

In fact, $\pm(2,-1,0)$ and $\pm(1,1-1)$ are the only vectors in $\mathcal{L}$ with 1-norm equal to 3, while $\pm(2,-1,0)$ and $\pm(0,2,-1)$ are those in $\mathcal{L}'$ with the same 1-norm. Thus,

$$N_{\mathcal{L}}(3,0) = 2 \neq N_{\mathcal{L}'}(3,0) = 0 \text{ and } N_{\mathcal{L}}(3,1) = 2 \neq N_{\mathcal{L}'}(3,1) = 4.$$

Remark

We shall give later infinitely many pairs of congruence lattices that are $\|\cdot\|_1^*$-isospectral in dimension $m = 3$. Such examples cannot exist in dimension $m = 2$; Ikeda and Yamamoto showed that two 0-isospectral lens spaces are necessarily isometric for $n = 3$.

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Remark

In an important paper, Ikeda88 constructed, for each given p_0, pairs of lens spaces that are p-isospectral for every $0 \leq p \leq p_0$, but are not $p_0 + 1$ isospectral. No pair in his set of examples is p-isospectral for all p.

Finiteness conditions

If $q \in \mathbb{N}$, we define an equivalence relation in $\mathbb{Z}^m$: if $\mu = \sum_j a_j \varepsilon_j$, $\mu' = \sum_j a'_j \varepsilon_j \in \mathbb{Z}^m$ then $\mu \sim \mu'$ if and only if $\mu - \mu' \in (q\mathbb{Z})^m$ and $a_j a'_j \geq 0$ for every j such that $a_j \not\equiv 0 \pmod{q}$.

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If $C(q) = \left\{ \sum_j a_j \varepsilon_j \in \mathbb{Z}^m : |a_j| < q \ , \forall j \right\}$, then for any congruence lattice $\mathcal{L}$, $C(q)$ and $C(q) \cap \mathcal{L}$ give a complete set of representatives of $\sim$ on $\mathbb{Z}^m$ and on $\mathcal{L}$.

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Definition

Let $\mathcal{L}$ be a q-congruence lattice. For any $k \in \mathbb{N}_0$ and $0 \leq \ell \leq m$, set

$$N_{\mathcal{L}}^{\mathrm{red}}(k,\ell) = \#\{\mu \in C(q) \cap \mathcal{L} : \|\mu\|_1 = k,\ Z(\mu) = \ell\}.$$

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$$N_{\mathcal{L}}^{\mathrm{red}}(k, \ell) = \#\{\mu \in C(q) \cap \mathcal{L} : \|\mu\|_1 = k, \ Z(\mu) = \ell\}.$$

$N_{\mathcal{L}}(k, \ell) = N_{\mathcal{L}}^{\mathrm{red}}(k, \ell)$ for every $k < q$. Also, for each of the $m - \ell$ nonzero coordinates a_i of a q-reduced element one has $|a_i| \leq q - 1$, thus $N_{\mathcal{L}}^{\mathrm{red}}(k, \ell) = 0$ for every $k > (m - \ell)(q - 1)$. Hence, the totality of numbers $N_{\mathcal{L}}^{\mathrm{red}}(k, \ell)$ is at most $\binom{m+1}{2} q$.

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Theorem

Let $\mathcal{L}$ and $\mathcal{L}'$ be two q-congruence lattices.

(i) *If $k = \alpha q + r \in \mathbb{N}$ with $0 \leq r < q$, then*

$$N_{\mathcal{L}}(k,\ell) = \sum_{s=0}^{m-\ell} 2^s \binom{\ell+s}{s} \sum_{t=s}^{\alpha} \binom{t-s+m-\ell-1}{m-\ell-1} N_{\mathcal{L}}^{\mathrm{red}}(k-tq, \ell+s) \tag{0.10}$$

(ii) $N_{\mathcal{L}}(k,\ell) = N_{\mathcal{L}'}(k,\ell)$ *for every k and ℓ if and only if* $N_{\mathcal{L}}^{\mathrm{red}}(k,\ell) = N_{\mathcal{L}'}^{\mathrm{red}}(k,\ell)$ *for every k and ℓ.*

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Hence, if $N_{\mathcal{L}}^{\mathrm{red}}(k,\ell) = N_{\mathcal{L}'}^{\mathrm{red}}(k,\ell)$ for every k and ℓ, then $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1^*$-isospectral.

Computations and questions

We can use the finiteness theorem to produce, with the help of a computer, many examples of pairs of non-isometric congruence lattices that are $\|\cdot\|_1^*$-isospectral. Each such pair gives rise to a pair of non-isometric lens spaces that are p-isospectral for all p.

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We explain the computational procedure. For each m and q, one finds first a complete list of non-isometric q-congruence lattices in $\mathbb{Z}^m$. Then, for each lattice $\mathcal{L}$ in the list, one computes the (finitely many) numbers $N_{\mathcal{L}}^{\mathrm{red}}(k,\ell)$ for $0 \leq \ell \leq m$ and $0 \leq k \leq (m-\ell)(q-1)$. The program puts together the lattices for which these numbers coincide. By a previous theorem, such lattices are mutually $\|\cdot\|_1^*$-isospectral.

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By this procedure, using Sage, we found all $\|\cdot\|_1^*$-isospectral m-dimensional q-congruence lattices for $m = 3$, $q \leq 300$ and $m = 4$, $q \leq 150$. We point out that all such lattices come in pairs for these values of q and m.

q	$[s_1, s_2, s_3]$	$[s'_1, s'_2, s'_3]$	
49	[1, 6, 15]	[1, 6, 20]	*
64	[1, 7, 17]	[1, 7, 23]	*
98	[1, 13, 29]	[1, 13, 41]	*
100	[1, 9, 21]	[1, 9, 29]	*
100	[1, 9, 31]	[1, 9, 39]	
121	[1, 10, 23]	[1, 10, 32]	*
121	[1, 10, 34]	[1, 10, 43]	
121	[1, 10, 45]	[1, 10, 54]	
121	[1, 21, 34]	[1, 21, 54]	
121	[1, 21, 45]	[1, 21, 56]	
128	[1, 15, 33]	[1, 15, 47]	*
147	[1, 20, 43]	[1, 20, 62]	*
169	[1, 12, 27]	[1, 12, 38]	*
169	[1, 12, 53]	[1, 12, 64]	
169	[1, 12, 66]	[1, 12, 77]	
169	[1, 25, 40]	[1, 25, 64]	
169	[1, 25, 53]	[1, 25, 77]	
169	[1, 38, 53]	[1, 38, 79]	
169	[1, 12, 40]	[1, 12, 51]	

q	$[s_1, s_2, s_3, s_4]$	$[s'_1, s'_2, s'_3, s'_4]$
49	[1, 6, 8, 20]	[1, 6, 8, 22]
81	[1, 8, 10, 26]	[1, 8, 10, 28]
81	[1, 8, 10, 35]	[1, 8, 10, 37]
81	[1, 8, 19, 37]	[1, 8, 26, 37]
98	[1, 13, 15, 41]	[1, 13, 15, 43]
100	[1, 9, 11, 29]	[1, 9, 11, 31]
100	[1, 9, 21, 39]	[1, 9, 29, 31]
121	[1, 10, 12, 32]	[1, 10, 12, 34]
121	[1, 10, 12, 54]	[1, 10, 12, 56]
121	[1, 10, 23, 56]	[1, 10, 32, 56]

q	$[s_1, s_2, s_3, s_4]$	$[s'_1, s'_2, s'_3, s'_4]$
121	[1, 10, 34, 54]	[1, 10, 43, 45]
121	[1, 21, 23, 54]	[1, 21, 23, 56]
121	[1, 10, 12, 43]	[1, 10, 12, 45]
121	[1, 10, 23, 43]	[1, 10, 32, 34]
121	[1, 10, 23, 45]	[1, 10, 32, 54]
121	[1, 10, 23, 54]	[1, 10, 32, 45]
121	[1, 10, 34, 56]	[1, 10, 43, 56]

Next we attempt to explain in a unified manner the examples in the tables. Let r and t be positive integers and set $q = r^2 t$, $r > 1$. We let $\theta = 1 + rt$, as an element of $(\mathbb{Z}/q\mathbb{Z})^{\times}$, the group of units of $\mathbb{Z}/q\mathbb{Z}$. Then, the inverse of θ modulo q is $\theta^{-1} := 1 - rt$.
Clearly, for every $k \in \mathbb{Z}$,

$$\theta^k \equiv 1 + krt \pmod{q}.$$

In particular, θ has order r in $(\mathbb{Z}/q\mathbb{Z})^{\times}$. For example,

$$\mathcal{L} = \mathcal{L}(q; \theta^0, \theta^1, \theta^3) \qquad \text{and} \qquad \mathcal{L}' = \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-3}). \quad (0.11)$$

All pairs in the tables have a description in terms of suitable powers of θ for choices of r and t such that $q = r^2 t$. For instance, the fifth example in the table for m=3, if we take $r = 10$ and $t = 1$ can be written as

$$\begin{aligned} \mathcal{L}(100; 1, 9, 31) &= \mathcal{L}(q; \theta^0, -\theta^{-1}, \theta^3) \cong_1 \mathcal{L}(q; \theta^0, \theta^1, \theta^4), \\ \mathcal{L}(100; 1, 9, 39) &= \mathcal{L}(q; \theta^0, -\theta^{-1}, -\theta^{-4}) \cong_1 \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-4}), \end{aligned} \tag{0.12}$$

where $\cong_1$ denotes isometric in $\|\cdot\|_1$. Furthermore, the first pair in the table for $m = 4$, if $r = 7$ and $t = 1$ becomes

$$\begin{aligned} \mathcal{L}(49; 1, 6, 8, 20) &= \mathcal{L}(q; \theta^0, -\theta^{-1}, \theta^1, -\theta^{-3}) \cong_1 \mathcal{L}(q; \theta^0, \theta^1, \theta^3, \theta^4), \\ \mathcal{L}(49; 1, 6, 8, 22) &= \mathcal{L}(q; \theta^0, -\theta^{-1}, \theta^1, \theta^3) \cong_1 \mathcal{L}(q; \theta^0, \theta^{-1}, \theta^{-3}, \theta^{-4}). \end{aligned} \tag{0.13}$$

All examples shown in the tables respond to the following description:

$$\mathcal{L}(q;\theta^{d_0},\theta^{d_1},\ldots,\theta^{d_{m-1}}) \quad \text{and} \quad \mathcal{L}(q;\theta^{-d_0},\theta^{-d_1},\ldots,\theta^{-d_{m-1}}), \tag{0.14}$$

where $q = r^2 t$, $r > 1$, $\theta = 1 + rt$ and
$0 = d_0 < d_1 < \cdots < d_{m-1} < r$.

However, for some choices of m, r and t, there are sequences $0 = d_0 < d_1 < \cdots < d_{m-1} < r$ such that the lattices defined as above are not $\|\cdot\|_1^*$-isospectral.
For example, this happens when $m = 3$, $r = 8$, $t = 1$ and $[d_0, d_1, d_2] = [0, 1, 4]$.

The following questions come up naturally.

Question

Give conditions on the sequence $0 = d_0 < d_1 < \cdots < d_{m-1} < r$ for lattices as in (0.14) to be $\|\cdot\|_1^*$-isospectral.

Question

Are there examples of $\|\cdot\|_1^*$-isospectral lattices that are not of the type in (0.14) for some choice of θ?

Question

Are there families of $\|\cdot\|_1^*$-isospectral lattices having more than two elements?

Construction of $\|\cdot\|_1^*$-isospectral lattices

We now give an infinite two-parameter family of pairs of $\|\cdot\|_1^*$-isospectral lattices in $\mathbb{Z}^m$ for $m = 3$. Note :it was shown by Ikeda-Yamamoto that such lattices cannot exist in dimension $m = 2$.

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We fix $r, t \in \mathbb{N}$, $r > 1$, we set $q = r^2 t$ and consider the lattices

$$\mathcal{L} = \mathcal{L}(q; 1, rt-1, 2rt+1), \quad \mathcal{L}' = \mathcal{L}(q; 1, rt-1, 3rt-1). \tag{0.15}$$

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That is, $\mathcal{L}$ and $\mathcal{L}'$ are defined by the following equations

$$\begin{aligned} \mathcal{L}: &\quad a + (rt-1)b + (2rt+1)c \equiv 0 \pmod{r^2 t}, \\ \mathcal{L}': &\quad (rt-1)a' + b' + (3rt-1)c' \equiv 0 \pmod{r^2 t}. \end{aligned} \tag{0.16}$$

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The simplest case is the pair $\mathcal{L}(49; 1, 6, 15)$, $\mathcal{L}(49; 1, 6, 20)$ obtained by letting $t = 1$, $r = 7$ and $q = 49$.

Lemma

For any $\ell = 1, 2, 3$ *and any* $k \in \mathbb{N}$*, one has that* $N_{\mathcal{L}}(k, \ell) = N_{\mathcal{L}'}(k, \ell)$.

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It remains to prove that $N_{\mathcal{L}}(k, 0) = N_{\mathcal{L}'}(k, 0)$ for every k. By Lemma 5, this is equivalent to showing that $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1$-isospectral, since $N_{\mathcal{L}}(k) = \sum_{\ell=0}^{3} N_{\mathcal{L}}(k, \ell)$.
The proof of this fact requires quite some work, we omit it. Thus:

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The proof of this fact requires quite some work, we omit it. Thus:

Theorem

For any r*,* t *odd positive integers,* $r > 1$*,* $r \not\equiv 0 \pmod 3$*, the lattices* $\mathcal{L}(q; 1, rt - 1, 2rt + 1)$ *and* $\mathcal{L}(q; 1, rt - 1, 3rt - 1)$ *are* $\|\cdot\|_1^*$*-isospectral.*

Proposition

If L and L' are strongly isospectral lens spaces, then they are isometric.

Indeed, if $\Gamma\backslash S^{2m-1}$ and $\Gamma'\backslash S^{2m-1}$ are strongly isospectral spherical spaces forms, by Pesce the subgroups Γ and Γ' are representation equivalent, i.e. $L^2(\Gamma\backslash \mathrm{O}(2m))$ and $L^2(\Gamma'\backslash \mathrm{O}(2m))$ are equivalent representations of $\mathrm{O}(2m)$. Hence, Γ and Γ' are almost conjugate in $\mathrm{O}(2m)$ (Wolf). Since almost conjugate cyclic subgroups are necessarily conjugate, then L and L' are isometric.

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We observe that the examples in the theorem allow to obtain pairs of Riemannian manifolds in every dimension $n \geq 5$ that are p-isospectral for all p, but are not strongly isospectral.
Indeed, we may just take $M = L \times S^k$ and $M' = L' \times S^k$, for any $k \in \mathbb{N}_0$, where L, L' is any pair of non-isometric lens spaces in dimension 5 satisfying p-isospectrality for every p.

Relative to lens spaces of higher dimensions we have the following result.

Theorem

For any $n_0 \geq 5$*, there are pairs of non-isometric lens spaces of dimension* n*, with* $n > n_0$*, that are* p*-isospectral for all* p*.*

The proof uses Ikeda's duality and our previous result. We furthermore note that:

Lemma

The lens spaces $L(r^2t; 1, 1 + rt, 1 + 3rt)$, $L(r^2t; 1, 1 - rt, 1 - 3rt)$, $r \not\equiv 0 \pmod{3}$*, associated to the congruence lattices in the theorem are homotopically equivalent to each other.*

We end with complementary information on τ-isospectrality. Any representation τ of K induces a strongly elliptic natural operator $\Delta_{\tau,\Gamma}$ on the smooth sections of a natural vector bundle over $\Gamma\backslash S^{2m-1}$.

We will show many choices of representations τ of K such that $L(49; 1, 6, 15)$ and $L(49; 1, 6, 20)$ are not τ-isospectral. Denote by Γ and Γ' the finite cyclic subgroups of order $q = 49$ such that $L = \Gamma\backslash S^5$ and $L' = \Gamma'\backslash S^5$.

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Set $\Lambda_0 = 4\varepsilon_1 + 3\varepsilon_2$ with irreducible representation π_{Λ_0} of G. One can check that $\dim V^{\Gamma}_{\pi_{\Lambda_0}} = 8$, $\dim V^{\Gamma'}_{\pi_{\Lambda_0}} = 6$ and the only irreducible representation π such that $\lambda(C, \pi) = \lambda(C, \pi_{\Lambda_0}) = 47$ is $\pi = \pi_{\Lambda_0}$. Recall that $\lambda(C, \pi) = \langle\Lambda + \rho, \Lambda + \rho\rangle - \langle\rho, \rho\rangle$ is the scalar for which the Casimir operator C acts on V^{Γ}_{π}.

Since $\tau = \tau^*$, the multiplicity of the eigenvalue $\lambda_0 = 47$ in $\Delta_{\tau,\Gamma}$ and $\Delta_{\tau,\Gamma'}$ are given by

$$d_{\lambda_0}(\tau,\Gamma) = 8\,[\tau : \pi_{\Lambda_0}] \qquad \text{and} \qquad d_{\lambda_0}(\tau,\Gamma') = 6\,[\tau : \pi_{\Lambda_0}].$$

Hence, L and L' cannot be τ-isospectral for every τ such that $[\tau : \pi_{\Lambda_0}] > 0$. By applying the branching law from SO(6) to SO(5), one has that there are several such representations: those having highest weight $b_1\varepsilon_1 + b_2\varepsilon_2$ with $4 \geq b_1 \geq 3 \geq b_2 \geq 0$.

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For each π such that $\dim V_\pi^\Gamma \neq \dim_{V_\pi^{\Gamma'}}$, as above one obtains many irreducible representations τ of SO(5) such that L and L' are not τ-isospectral, so, we get that L and L' are 'very far' from being strongly isospectral.

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Question. Show there are only finitely many irreducible representations τ of SO(5) such that $L(49; 1, 6, 15)$ and $L(49; 1, 6, 20)$ are τ-isospectral. Which ones?