Homogeneity of Lorentzian three-manifolds with reccurent curvature

E. García-Río, P. Gilkey, S. Nikčević

Korea, August 2014.

Abstract

k-Curvature homogeneous three-dimensional Walker metrics are described for $k \leq 2$. This allows a complete description of locally homogeneous three-dimensional Walker metrics, showing that there exists exactly three isometry classes of such manifolds. As an application one obtains a complete description of all locally homogeneous Lorentzian manifolds with recurrent curvature. Moreover, potential functions are constructed in all the locally homogeneous manifolds resulting in steady gradient Ricci and Cotton solitons.

Walker Lorentzian 3 dimensional manifolds

Let $\mathcal{M} = (M, g_M)$ be a 3-dimensional Lorentzian manifold which admits a parallel null vector field, i.e. $\mathcal{M}$ is a 3-dimensional Walker manifold. Such a manifold admits local adapted coordinates $(x, y, \tilde{x})$ so that the (possibly) non-zero components of the metric are given by

$$g(\partial_x, \partial_x) = -2f(x, y), \quad g(\partial_x, \partial_{\tilde{x}}) = g(\partial_y, \partial_y) = 1\,.$$

define a signature $(2, 1)$ metric on $\mathbb{R}^3$. We examine curvature homogeneity in this setting; this extends previous work where we assumed $f = f(y)$ was a function of only one variable.
We shall denote this manifold by $\mathcal{M}_f$.

Introduction

The study of Lorentzian three-manifolds admitting a parallel null vector field is central both in geometry and physics. Physically they represent the simplest non-trivial pp-waves and, from a geometrical point of view, they are the underlying structure of many Lorentzian situations without Riemannian counterpart. A Lorentzian manifold is said to be *irreducible* if the holonomy group does not leave invariant any non-trivial subspace. Moreover, the action of the holonomy is said to be *indecomposable* it leaves invariant only non-trivial subspaces for which the restriction of the metric degenerates. Then the de Rham-Wu's Theorem shows that any complete and simply connected Lorentzian manifold is a product of indecomposable ones. Thus Walker three-manifolds constitute the basic material to build many Lorentzian metrics.

When trying to analyze the curvature of a given manifold, one must deal not only with the curvature tensor itself, but also with its covariant derivatives. In the locally symmetric case, the curvature tensor is parallel and hence the study can be reduced to the purely algebraic level. Generalizing this condition, one naturally considers the case when the curvature tensor is *recurrent* (i.e., $\nabla R = \omega \otimes R$ for some 1-form ω) but not parallel ($\nabla R \neq 0$). The class of recurrent Lorentzian manifolds reduces to the study of plane waves and the three-dimensional Walker manifolds (A.S.Galaev-Lorentzian manifolds with recurrent curvature tensor, arXiv; A.G.Walker 1950 -On Ruse's spaces of recurrent curvature). Homogeneous plane waves were discussed in (M.Blau and M.O'Loughlin 2003) and hence one of the purposes of this paper is to give a complete description of all locally homogeneous Walker three-manifolds.

Our main result shows that there exists exactly three isometry classes of such manifolds , which allows a complete description of all locally homogeneous Lorentzian manifolds with recurrent curvature. Recall that, in any locally homogeneous pseudo-Riemannian manifold, the curvature tensor and all its covariant derivatives are the same at each point . Generalizing this condition, a manifold (M, g) is said to be k-*curvature homogeneous* if for any two points there exists a linear isometry between the corresponding tangent spaces which preserves the curvature tensor and its covariant derivatives up to order k. Clearly any locally homogeneous space is curvature homogeneous of any order; conversely, given m, there is $k = k(m)$ so that any k-curvature homogeneous manifold is in fact homogeneous (I.M.Singer-1960, F.Podesta and A.Spiro-1996, B.Opozda-1997).

Moreover it is shown (Bueken,Djoric-2000) that a three-dimensional Lorentzian manifold which is curvature homogeneous up order two is also locally homogeneous. Our approach is based on a careful analysis of the curvature homogeneity of a Walker three-manifold.
A purpose of this work to give a complete description of all Walker three-manifolds . It is our second purpose in this paper to analyze their geometry, thus showing that any 1-curvature homogeneous Walker three-manifold is either a gradient Ricci soliton or a gradient Cotton soliton.

Recall that, generalizing the Einstein condition, a Lorentzian manifold (M, g) is said to be a *Ricci soliton* if and only if it is a self-similar solution of the Ricci flow, i.e., the one-parameter family of metrics $g(t) = \sigma(t)\psi_t^* g$ is a solution of the Ricci flow $\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}$, for some smooth function $\sigma(t)$ and some one-parameter family of diffeomorphisms ψ_t of M, where $Ric_{g(t)}$ denotes the Ricci tensor of $(M, g(t))$. From a physical viewpoint, Ricci solitons can be interpreted as special solutions of the Einstein field equations, where the stress-energy tensor essentially corresponds to the Lie derivative of the metric. We analyze in detail the structure of gradient Ricci solitons on Walker three-manifolds , showing that any 1-curvature homogeneous Lorentzian three-manifold with recurrent curvature is indeed a steady gradient Ricci soliton. While one of the possible 1-curvature homogeneous Walker three-manifold is indeed a plane wave and thus a expanding, steady and shrinking Ricci soliton, it is shown that the non-homogeneous family does not support any non-steady Ricci soliton.

The Cotton tensor, C, measures the failure of the Schouten tensor to be Codazzi. The existence of self-similar solutions to the Cotton flow $\frac{\partial}{\partial t}g(t) = -\lambda C_{g(t)}$ also provide a family of three-dimensional metrics which generalize the locally conformally flat manifolds. Locally homogeneous Walker three-manifolds split essentially into two families, one of which is locally conformally flat. The existence of gradient Cotton solitons is also studied, showing that any locally homogeneous Walker three-manifold is a steady gradient Cotton soliton. Moreover, a locally homogeneous Walker three-manifold admits a non-steady Cotton soliton if and only if it is locally conformally flat.

Preliminaries

A Lorentzian manifold admitting a parallel null vector field will be refereed in what follows as a *Walker manifold*. Let $(x, y, \tilde{x})$ be coordinates on $\mathbb{R}^3$, let $\mathcal{O}$ be a connected open subset of $\mathbb{R}^2$, let $M := \mathcal{O} \times \mathbb{R}$, let $f \in C^\infty(\mathcal{O})$, and let $\mathcal{M}_f := (M, g_f)$ where $g = g_f$ is the Lorentz metric on M given by:

$$g(\partial_x, \partial_x) = -2f(x, y), \quad g(\partial_x, \partial_{\tilde{x}}) = g(\partial_y, \partial_y) = 1\,.$$

We shall assume that $f_{yy} > 0$ and thus the curvature does not vanish identically; the case $f_{yy} < 0$ is similar. This implies that $\mathcal{M}_f$ is 0-curvature homogeneity.

We note for future reference that the non-zero covariant derivatives are given by:

$$\nabla_{\partial_x}\partial_x = -f_x\partial_{\tilde{x}} + f_y\partial_y \quad \text{and} \quad \nabla_{\partial_x}\partial_y = \nabla_{\partial y}\partial_x = -f_y\partial_{\tilde{x}}.$$

Since $\nabla_x\partial_x = -f_x\partial_{\tilde{x}} + f_y\partial_y$ involves a ∂_y component which can depend on y, these manifolds are not generalized plane wave manifolds and in fact can be geodesically incomplete; they can exhibit Ricci blowup.

The only (potentially) non-zero components of the $(1,3)$ curvature tensor $\mathcal{R}$ are given by

$$\mathcal{R}(\partial_x, \partial_y)\partial_y = f_{yy}\partial_{\tilde{x}} \quad \text{and} \quad \mathcal{R}(\partial_x, \partial_y)\partial_x = -f_{yy}\partial_y.$$

Thus the $(0,4)$ curvature tensor is determined by $R(\partial_x, \partial_y, \partial_y, \partial_x) = f_{yy}$. Similarly, when considering ∇R, the only possible contributions from the Christoffel symbols arise when plugging ∂_x or ∂_y in the last entry $\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \cdot\,)$.

Consequently the (possibly) non-zero entries in ∇R are given by:

$$\nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x) = f_{xyy} \quad \text{and} \quad \nabla R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y) = f_{yyy}.$$

It follows from previous equations that $\mathcal{M}_f$ is locally symmetric if and only if f_{yy} is a constant. Whenever $f_{yy} = \text{const.} \neq 0$, the resulting manifold is a Cahen-Wallach symmetric space (CLPTV-1990). In dimension 3, all Walker metrics have recurrent curvature in a neighborhood of any point of non-zero curvature, i.e., $\nabla R = \omega \otimes R$, for a 1-form

$$\omega = (\ln f_{yy})_x \, dx + (\ln f_{yy})_y \, dy.$$

We refer to (V,R,G,N,L-2009) and the references therein for more information on Walker three-manifolds.

We compute similarly that the (possibly) non-zero entries in $\nabla^2 R$ are:

$$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x, \partial_x) = f_{xxyy} - f_y f_{yyy},$$

$$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_x, \partial_y) = f_{xyyy},$$

$$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \partial_x) = f_{xyyy},$$

$$\nabla^2 R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \partial_y) = f_{yyyy}\,.$$

Definition

Let $\mathcal{M}_f$ be a three-dimensional Walker manifold given by Walker-metric.

1. Let $\mathcal{N}_b$ be defined by taking $f(x,y) = b^{-2}e^{by}$ for $b \neq 0$.
2. Let $\mathcal{P}_c$ be defined by taking $f(x,y) = \frac{1}{2}y^2\alpha(x)$ where $\alpha_x = c\alpha^{3/2}$ and $\alpha > 0$.
3. Let $\mathcal{CW}$ be the three-dimensional Cahen-Wallach symmetric space defined by taking $f(x,y) = \varepsilon y^2$.

Theorem

There is a geodesic $\gamma(t)$ in $\mathcal{P}_c$ which defined for $t \in [0,1)$ and there exists a parallel vector field $Y(t)$ along $\gamma(t)$ with

$$g_f(\dot\gamma,\dot\gamma) = 1, \quad g_f(\dot\gamma, Y) = 0, \quad g_f(Y,Y) = 1, \quad \lim_{t\to 1^-} R(\dot\gamma, Y, Y, \dot\gamma) = \infty\,.$$

Consequently, $\mathcal{P}_c$ is geodesically incomplete and can not be embedded in a geodesically complete manifold.

Theorem

1. *The manifolds $\mathcal{CW}_\varepsilon$ are locally symmetric.*
2. *The manifolds $\mathcal{N}_b$ and $\mathcal{P}_c$ are locally homogeneous.*
3. *The manifolds $\{\mathcal{CW}_\varepsilon, \mathcal{N}_b, \mathcal{P}_c\}$ have non-isomorphic 1-curvature models and represent different local isometry types.*

The manifolds $\mathcal{P}_c$ are pp-waves one supposes since the function $f(x, y)$ is quadratic in y; they are not generalized plane. The manifolds $\mathcal{N}_b$ are not pp-waves since f_{yy} is not quadratic in y. Note that the Cahen-Wallach symmetric spaces $\mathcal{CW}_\varepsilon$ are geodesically complete. The geodesic equations corresponding to the manifolds $\mathcal{N}_b$ can be integrated explicitly.

Geometric Solitons

The objective of the different geometric evolution equations is to improve a given initial metric by considering a flow associated to the geometric object under consideration. The Ricci, Yamabe, and mean curvature flows are examples extensively studied in the literature. Under suitable conditions, the Ricci flow evolves an initial metric to an Einstein metric while the Yamabe flow evolves an initial metric to a new one with constant scalar curvature within the same conformal class. There are however certain metrics which, instead of evolving by the flow, remain invariant up to scaling and diffeomorphisms, i.e., they are self-similar solutions of the flow. For any solution of the form $g(t) = \sigma(t)\psi_t^* g(0)$, where $\sigma(t)$ is a smooth function and $\{\psi_t\}$ a one-parameter family of diffeomorphisms of M, there exists a vector field X (the soliton vector field) which relates the Lie derivative of the metric $\mathcal{L}_X g$ with the geometric object defining the flow under consideration.

Ricci solitons

A *Ricci soliton* is a pseudo-Riemannian manifold (M, g) which admits a smooth vector field X (which is called a soliton vector field) on M such that

$$\mathcal{L}_X g + Ric = \lambda g, \tag{1}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, Ric is the Ricci tensor, and λ is a real number ($\lambda = \frac{1}{n}(2\mathrm{div}X + Sc)$, where $n = \dim\, M$ and Sc denotes the scalar curvature of (M, g)). A Ricci soliton is said to be *shrinking, steady* or *expanding*, if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Moreover we say that a Ricci soliton (M, g) is a *gradient Ricci soliton* if the vector field X satisfies $X = \mathrm{grad}\, h$, for some potential function h. In such a case Equation (1) can be written in terms of h as $2\mathrm{Hes}_h + Ric = \lambda g$.

Three-dimensional Walker metrics admitting a non-trivial (i.e., not Einstein) gradient Ricci soliton were completely described in BVGRGF. where it is shown that one of the following two possibilities must occur:
[(R.1)] There exist coordinates $(x, y, \tilde{x})$ for some function f satisfying $f_{yy}^{-1} f_{yyy} = b$. Hence

$$f(x, y) = \frac{1}{\kappa^2} e^{\kappa y} \alpha(x) + y\beta(x) + \gamma(x)$$

for some arbitrary functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$. The potential function of the soliton is given by

$$h(x, y, \tilde{x}) = \frac{\kappa}{2} y + \hat{h}(x), \text{ where } \hat{h}_{xx} = \frac{\kappa}{2} \beta(x),$$

and the soliton vector field is spacelike and given by $\operatorname{grad} h = \frac{\kappa}{2} \partial_y + \hat{h}_x(x) \partial_{\tilde{x}}$.

[(R.2)] There exist coordinates $(x, y, \tilde{x})$ for some function f satisfying $f_{yyy} = 0$. Hence

$$f(x, y) = y^2\alpha(x) + y\beta(x) + \gamma(x)$$

for some arbitrary functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$.
The potential function of the soliton is given by

$$h(x, y, \tilde{x}) = \hat{h}(x), \text{ where } \hat{h}_{xx} = -\alpha(y),$$

and the soliton vector field is lightlike and given by
$\operatorname{grad} h = \hat{h}_x(x)\partial_{\tilde{x}}$.
Moreover, in both cases the Ricci soliton is steady. As we shall see all gradient Ricci solitons above are 1-curvature homogeneous, provided that f_{yy} has constant sign.

Further, note that all metrics corresponding to the second case above are plane waves (since the function f is quadratic on y), and hence they admit non-gradient vector fields X resulting in expanding and shrinking Ricci solitons BBGG-2011. However metrics corresponding to the first case above only admit steady Ricci solitons

Theorem

A gradient Ricci soliton $\mathcal{M}_f$ admits a vector field X resulting in a non-steady Ricci soliton if and only if it is a locally conformally flat metric.

Recall here that two Ricci soliton vector fields differ in a homothetic vector field . Hence, a gradient Ricci soliton Walker metric admits a vector field resulting in a non-steady Ricci soliton if and only if it admits a non-Killing homothetic vector field. Locally conformally flat Walker metrics are plane waves CGRVA-2005, and thus they admit homothetic vector fields resulting in expanding, steady and shrinking Ricci solitons ....

Cotton solitons

The *Schouten tensor* of any pseudo-Riemannian manifold is given by $S_{ij} = Ric_{ij} - \frac{Sc}{4} g_{ij}$. Then the *Cotton tensor*, $C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik}$ measures the failure of the Schouten tensor to be a Codazzi tensor. The Cotton tensor is the unique conformal invariant in dimension three and it vanishes if and only if the manifold is locally conformally flat. Using the Hodge $\star$-operator, the $(0,2)$-Cotton tensor is given by $C_{ij} = \frac{1}{2\sqrt{g}} C_{nmi} \epsilon^{nm\ell} g_{\ell j}$, where $\epsilon^{123} = 1$. Moreover, the $(0,2)$-Cotton tensor is trace-free and divergence-free York-1971 and it appears naturally in many physical contexts (see, for example Chow-Pope-Sezgin 2010, Garcia-Hehl-Heinicke 2004 and the references therein).

The only non-zero component of the $(0,2)$-Cotton tensor of a Walker manifold $\mathcal{M}_f$ is given by $C(\partial_x, \partial_x) = -\frac{1}{2} f_{yyy}$ (and hence the manifold is locally conformally flat if and only if $f_{yyy} = 0$).
A geometric flow associated to the Cotton tensor was introduced in Kisisel, Sarioglu,Tekin-2008 as

$$\frac{\partial}{\partial_t} g(t) = -\lambda \, C_{g(t)},$$

where $C_{g(t)}$ is the $(0,2)$-Cotton tensor corresponding to $(M, g(t))$.
Then one naturally considers soliton solutions of the Cotton flow.
Following K,S,T -2008, a *Cotton soliton* is a triple (M, g, X) of a three-dimensional manifold and a vector field X satisfying

$$\mathcal{L}_X g + C = \lambda \, g, \tag{2}$$

where λ is a real number. The Cotton soliton is said to be *shrinking*, *steady* or *expanding* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

The necessary and sufficient conditions for a Walker manifold to be a gradient Cotton soliton were discussed in (CLGRVL-2012), where it is shown that $\mathcal{M}_f$ is a non-trivial (i.e., not locally conformally flat) gradient Cotton soliton if and only if it is steady and $f_{yyyy} = \kappa f_{yy}$ for some non-zero constant κ. Now one of the following three possibilities must occur:

(C.1)There exist coordinates $(x, y, \tilde{x})$ for some function f satisfying $f_{yyyy} = \kappa^2 f_{yy}$.

$$f(x, y) = \frac{1}{\kappa^2}(e^{\kappa y}\alpha_1(x) + e^{-\kappa y}\alpha_2(x)) + y\beta(x) + \gamma(x)$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\beta(x)$ and $\gamma(x)$ are arbitrary functions. Moreover, the potential function of the soliton is given by $h(x, y, \tilde{x}) = \frac{\kappa}{2}y + \hat{h}(x)$, where

$$\hat{h}_{xx}(x) = \frac{\kappa}{2}(e^{\kappa y}\alpha_1(x) - e^{-\kappa y}\alpha_2(x) + 2\kappa\beta(x)).$$

The soliton vector field is spacelike and given by $\operatorname{grad} h = \kappa^2\partial_y + \hat{h}_x(x)\partial_{\tilde{x}}$.

[(C.2)] There exist coordinates $(x, y, \tilde{x})$ for some function f satisfying $f_{yyyy} = -\kappa^2 f_{yy}$. Hence

$$f(x,y) = -\frac{1}{\kappa^2}(\cos(\kappa y)\alpha_1(x) + \sin(\kappa y)\alpha_2(x)) + y\beta(x) + \gamma(x))$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\beta(x)$ and $\gamma(x)$ are arbitrary functions. Moreover, the potential function of the soliton is given by $h(x, y, \tilde{x}) = -\frac{\kappa}{2}y + \hat{h}(x)$, where

$$\hat{h}_{xx}(x) = \frac{\kappa}{2}(\cos(\kappa y) - \sin(\kappa y) - 2\kappa\beta(y)).$$

The soliton vector field is spacelike and given by $\operatorname{grad} h = -\kappa^2\partial_y + \hat{h}_x(x)\partial_{\tilde{x}}$.

[(C.3)] There exist coordinates $(x, y, \tilde{x})$ for some function f satisfying $f_{yyyy} = 0$. Hence

$$f(x, y) = y^3\alpha_1(x) + y^2\alpha_2(x) + y\beta(x) + \gamma(x)$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\beta(x)$ and $\gamma(x)$ are arbitrary functions. Moreover, the potential function of the soliton is given by $h(x, y, \tilde{x}) = \hat{h}(x)$, where

$$\hat{h}_{xx}(x) = -3\alpha_1(x).$$

The soliton vector field is spacelike and given by $\operatorname{grad} h = -\kappa^2\partial_y + \hat{h}_x(x)\partial_{\tilde{x}}$.

Moreover, in all cases above the gradient Cotton soliton is steady. Note that two Cotton soliton vector fields $(\mathcal{L}_{X_i} g + C = \lambda_i g,\ i = 1, 2)$ differ by a homothetic vector field since

$$\mathcal{L}_{X_1 - X_2} g = (\lambda_1 - \lambda_2) g.$$

Hence, as well as for Ricci solitons, no Walker metric corresponding to (C.1) and (C.2) supports any non-trivial Cotton soliton of non-steady type as a consequence of the following.

Theorem

Let $\mathcal{M}_f$ be a non-flat Walker manifold satisfying $f_{yyyy} = b f_{yy}$ $(b \neq 0)$. Then any homothetic vector field on $\mathcal{M}_f$ is necessarily a Killing vector field.

Remark

Let $\mathcal{M}_f$ be given by $f(x,y) = y^3 e^{-\lambda x} + y^2 + y e^{\lambda x} + \gamma(x)$. Then the vector field $X = \frac{1}{2}\partial_x + \frac{\lambda}{2} y \partial_y + \{\lambda \tilde{x} + \theta(x)\}\, \partial_{\tilde{x}}$ is a Cotton soliton for any function $\theta(x)$ which satisfies the identity $\theta_x = 3e^{-\lambda x} + (\frac{1}{2} - \lambda)\gamma(x)$. Moreover the Cotton soliton is expanding or shrinking depending on the sign of λ. Hence there are gradient Cotton solitons (C.3) which also admit non-Killing homothetic vector fields.

Curvature homogeneity

A pseudo-Riemannian manifold (M, g) is said to be *k-curvature homogeneous* if for each pair of points $p, q \in M$ there is a linear isometry $\Phi_{pq} : T_pM \to T_qM$ such that

$$\Phi_{pq}^* R(q) = R(p),\ \Phi_{pq}^* \nabla R(q) = \nabla R(p),\ \dots,\ \Phi_{pq}^* \nabla^k R(q) = \nabla^k R(p)$$

where R, ∇R, $\dots$, $\nabla^k R$ stands for the curvature tensor and its covariant derivatives up to order k.

Clearly any locally homogeneous manifold is curvature homogeneous and the converse holds true if k is sufficiently large. An open question in the study of curvature homogeneity is to decide the minimum level of curvature homogeneity needed to show that a space is locally homogeneous. A general estimate of the form $k_M + 1 \leq n(n-1)/2$ (where n is the dimension of the manifold) was obtained by Singer.

However, there are sharper bounds in low dimensions. A Riemannian manifold which is 1-curvature homogeneous is locally homogeneous in dimension ≤ 4. However one needs 2-curvature homogenenity to ensure local homogeneity in the three-dimensional Lorentzian setting (see PG-2007) for more information and references).

A 0-curvature homogeneous manifold is said to be modeled on a symmetric space if its curvature tensor at each point is that of a symmetric space. A complete and simply connected indecomposable Lorentzian symmetric space is either irreducible, and hence of constant sectional curvature, or otherwise it is a Cahen-Wallach symmetric space (CLPTV-1990). Now, an immediate application of Schur's lemma shows that a curvature homogeneous Lorentz manifold modeled on an irreducible symmetric space has constant sectional curvature.

On the other hand, curvature homogeneous Lorentzian manifolds modeled on indecomposable symmetric spaces need not to be symmetric, but they are Walker manifolds (CLPTV-1990).
The purpose of this work is to analyze the class of three-dimensional manifolds with recurrent curvature under different curvature homogeneity assumptions. The question of 1-curvature homogeneity is dealt with by the following result :

Theorem

Assume that $f_{yy} > 0$ and that f_{yy} is non-constant. Then $\mathcal{M}_f$ is 1-curvature homogeneous if and only if exactly one of the following two possibilities holds:

1. $f_{yy}(x, y) = \alpha(x)e^{by}$ *where* $0 \neq b \in \mathbb{R}$ *and where* $\alpha(x)$ *is arbitrary. This manifold is* 1-*curvature modeled on* $\mathcal{N}_b$.
2. $f_{yy}(x, y) = \alpha(x)$ *where* $\alpha = c \cdot \alpha_x^{1.5}$ *for some* $0 \neq c \in \mathbb{R}$. *This manifold is locally homogeneous and is locally isometric to* $\mathcal{P}_c$.

Theorem

Any 1-curvature homogeneous Lorentzian three-manifold with recurrent curvature is a steady gradient Ricci soliton.

Theorem

The manifold $\mathcal{M}_f$ *is* 2*-curvature homogeneous if and only if it falls into one of the three families:*

1. $f = b^{-2}\alpha(x)e^{by} + \beta(x)y + \gamma(x)$ *for* $\beta(x) = b^{-1}\alpha^{-1}\{\alpha_{xx} - \alpha_x^2\alpha^{-1}\}$ *where* $b \neq 0$ *and* $\alpha > 0$*. The manifold* $\mathcal{M}_f$ *is locally isometric to the manifold* $\mathcal{N}_b$ *and consequently is locally homogeneous.*
2. $f_{yy} = \alpha(x) > 0$ *where* $\alpha_x = c\alpha^{3/2}$ *for* $c > 0$*; thus* $\alpha = \tilde{c}(x - x_0)^{-2}$ *for some* $(\tilde{c}, x_0)$*. The manifold* $\mathcal{M}_f$ *is locally isometric to the manifold* $\mathcal{P}_c$ *and consequently locally homogeneous.*
3. $f = \varepsilon y^2 + \beta(x)y + \gamma(x)$ *where* $0 < \varepsilon \in \mathbb{R}$*. The manifold* $\mathcal{M}_f$ *is locally isometric to the manifold* $\mathcal{CW}_\varepsilon$ *and consequently is locally homogeneous.*

Theorem

Any 2*-curvature homogeneous Lorentzian three-manifold with recurrent curvature is a steady gradient Cotton soliton.*

Finally note that the model manifolds $\mathcal{P}_c$ and $\mathcal{CW}_\varepsilon$ admit Ricci and Cotton solitons of any kind (expanding, steady and shrinking), but $\mathcal{N}_b$ only admits steady Ricci and Cotton solitons.

There is a slightly different version of curvature homogeneity that is due to Kowalski and Vanžurová 2011. A manifold (M, g) is said to be *k-curvature homogeneous of type* $(1, 3)$(homothety curvature homogeneity) if for any two points there exists a linear homothety between the corresponding tangent spaces which preserves the $(1, 3)$-curvature tensor and its covariant derivatives up to order k. This concept lies between the notion of affine k-curvature homogeneity and k-curvature homogeneity since the conformal group lies between the orthogonal group and the general linear group. This is genuinely a different concept.

Theorem

Suppose f_{yy} is never zero and non-constant.

1. *If f_{yyy} never vanishes, then $\mathcal{M}_f$ is homothety 1-curvature homogeneous.*
2. *If $f_{yy} = \alpha(x)$ with α_x never zero, then $\mathcal{M}_f$ is homothety 1-curvature homogeneous if and only if $f = a(x - x_0)^{-2}y^2 + \beta(x)y + \gamma(x)$ where $0 \neq a \in \mathbb{R}$. This manifold is locally homogeneous.*
3. *Assume that $\mathcal{M}_f$ is homothety 2-curvature homogeneous, and that f_{yy} and f_{yyy} never vanish. Then $\mathcal{M}_f$ is locally isometric to one of the examples:*
   - 3.1 $\mathcal{M}_{\pm e^{ay}}$ *for some $a \neq 0$ and $M = \mathbb{R}^3$. This manifold is homogeneous.*
   - 3.2 $\mathcal{M}_{\pm \ln(y)}$ *for some $a \neq 0$ and $M = \mathbb{R} \times (0, \infty) \times \mathbb{R}$. This manifold is homothety homogeneous but not locally homogeneous.*
   - 3.3 $\mathcal{M}_{\pm y^{\varepsilon}}$ *for $\varepsilon \neq 0, 1, 2$ and $M = \mathbb{R} \times (0, \infty) \times \mathbb{R}$. This manifold is*

It will follow from our analysis that the manifolds $\mathcal{M}_{\pm\ln(y)}$ and $\mathcal{M}_{\pm y^c}$ are homothety homogeneous VSI manifolds which are cohomogeneity one, thereby exhibiting non-trivial examples in the VSI setting. We also refer to recent work of Dunn and McDonald (2013) for related work on homothetical curvature homogeneous manifolds.

References

A. Derdzinski, "Einstein metrics in dimension four", *Handbook of Differential Geometry. Vol. I* Edited by Dillen and Verstraelen, North-Holland, Amsterdam, (2000), 419–707.

C. Dunn, P. Gilkey, and S. Nikčević, "Curvature homogeneous signature $(2, 2)$ manifolds", *Differential Geometry and its Applications, Proc. Conf. Prague, August 30-September 3, 2004 Charles University, Prague (Czech Republic)*, Edited by Bures, Kowalski, Krupka, and Slovak, MATFYZPRESS (2005), 29–44.

P. Gilkey, "The geometry of curvature homogeneous pseudo-Riemannian manifolds", ICP Advanced texts in mathematics **2** (2007).

P. Gilkey and S. Nikčević, "Affine curvature homogeneous 3-dimensional Lorentz Manifolds", *Int. J. Geom. Methods Mod. Phys.* **2** (2005), 737–749.

W. Batat, M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández, "Ricci Solitons on Lorentzian Manifolds with Large Isometry Groups", *Bull. London Math. Soc.* **43** (2011), 1219–1227.

W. Batat, G. Calvaruso and B. de Leo, Homogeneous Lorentzian 3-manifolds with a parallel null vector field, *Balkan J. Geom. Appl.* **14** (2009), 11–20.

M. Blau and M. O'Loughlin, Homogeneous plane waves, *Nuclear Phys. B* **654** (2003), 135–176.

M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández, Locally conformally flat Lorentzian gradient Ricci solitons, *J. Geom. Anal.*, to appear (doi: 10.1007/s12220-011-9283-z).

M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević and R. Vázquez-Lorenzo, *The geometry of Walker manifolds*, Synthesis Lectures on Mathematics and Statistics **5**, Morgan & Claypool Publ., Williston, VT, 2009.

P. Bueken and M. Djorić, Three-dimensional Lorentz metrics and curvature homogeneity of order one, *Ann. Glob. Anal. Geom.* **18** (2000), 85–103.

M. Cahen, J. Leroy, M. Parker, F. Tricerri and L. Vanhecke, Lorentz manifolds modelled on a Lorentz symmetric space, *J. Geom. Phys.* **7** (1990), 571–581.

G. Calvaruso, Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds, *Geom. Dedicata* **127** (2007), 99–119.

G. Calvaruso, Einstein-like Lorentz metrics and three-dimensional curvature homogeneity of order one, *Canad. Math. Bull.* **53** (2010), 412–424.

E. Calviño-Louzao, E. García-Río and R. Vázquez-Lorenzo, A note on compact Cotton solitons, *Class. Quantum Grav.* **29** (2012), 205014.

E. Calviño-Louzao, E. García-Río, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Curvature operators and generalizations of symmetric spaces in Lorentzian geometry, *Adv. Geom.* **12** (2012), 83–100.

M. Chaichi, E. García-Río and M. E. Vázquez-Abal, Three-dimensional Lorentz manifolds admitting a parallel null vector field, *J. Phys. A* **38** (2005), 841–850.

D. D. K. Chow, C. N. Pope and E. Sezgin, Classification of solutions in topologically massive gravity, *Class. Quantum Grav.* **27** (2010), 105001.

A. S. Galaev, Lorentzian manifolds with recurrent curvature tensor, `arXiv:1011.6541v1`.

A. A. García, F. W. Hehl, C. Heinicke and A. Macías, The Cotton tensor in Riemannian spacetimes, *Class. Quantum Grav.* **21** (2004), 1099–1118.

E. García-Río, P. Gilkey and S. Nikčević, Kowalski-Vanžurová curvature homogeneity, to appear.

P. Gilkey, *The geometry of curvature homogeneous pseudo-Riemannian manifolds*, ICP Advanced Texts in Mathematics, **2**, Imperial College Press, London, 2007.

P. Gilkey and S. Nikčević, Affine curvature homogeneous 3-dimensional Lorentz Manifolds, *Int. J. Geom. Methods Mod. Phys.* **2** (2005), 737–749.

K. Y. Ha and J. B. Lee, Left invariant metrics and curvatures on simply connected three-dimensional Lie groups, *Math. Nachr.* **282** (2009), 868–898.

K. Honda and K. Tsukada, Three-dimensional conformally flat homogeneous Lorentzian manifolds, *J. Phys. A: Math. Theor.* **40** (2007), 831–851.

A. U. O. Kisisel, O. Sarioglu and B. Tekin, Cotton flow, *Class. Quantum Grav.* **25** (2008), 165019.

O. Kowalski and A. Vanžurová, On curvature-homogeneous spaces of type $(1,3)$, *Math. Nachr.* **284** (2011), 2127–2132.

O. Kowalski and A. Vanžurová, On a generalization of curvature homogeneous spaces, *Results. Math.*, to appear (doi: 10.1007/s00025-011-0177-y).

S. Nikčević and U. Simon, Gradient Ricci Solitons and Curvature, *Proceeding PADGE* (2012), (239-253).

B. Opozda, On curvaature homogeneous and locally homogeneous affine connections, *Proc. Amer. Math. Soc.***124** (1996), 1889-1893.

B. Opozda, Afine versions of Singer's theorem on locally homogeneous spaces, *Ann. Global Anal. Geom.***15** (1997), 187-199.

I. M. Singer, Infinitesimally homogeneous spaces, *Comm. Pure Appl. Math.* **13** (1960), 685–697.

A. G. Walker, On Ruse's spaces of recurrent curvature, *Proc. London Math. Soc. (2)*, **52** (1950), 36–64.

J. W. York, Jr., Gravitational degrees of freedom and the initial-value problem, *Phys. Rev. Lett.* **26** (1971), 1656–1658.