# Variations of almost Hermitian structures

Kouei Sekigawa

Niigata University, Japan

(Joint work with Yunhee Euh,

Jungchan Lee and Jeong Hyeong Park)

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In this talk, we discuss the following two topics:

I. Critical almost Hermitian structures of the functional $\mathcal{F}_{\lambda,\mu}$ (introduced by Koda),

II. Curvature identities on compact almost Hermitian surfaces.

# Background

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$M$ : compact orientable smooth manifold.

$\mathcal{M}(M)$ : set of all Riemannian metrics on $M$.

$\mathcal{M}_c(M) = \{g \in \mathcal{M}(M) | Vol(M,g) = c\}$, $c$ : positive constant

$\mathcal{M}_c(M) \subset \mathcal{M}(M)$

$\mathcal{F}$ : Einstein-Hilbert functional on $\mathcal{M}_c(M)$

$$\mathcal{F}(g) = \int_M \tau dv_g$$

$\tau$ : scalar curvature of $g$, $dv_g$ : volume element of $g$

$g \in \mathcal{M}_c(M)$ is a critical point of $\mathcal{F}$ on $\mathcal{M}_c(M)$ if and only if $g$ is an Einstein metric.

D. Hilbert, Die Grundlagen der Physik, Nachr. Ges. Wiss. Gött.

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$\mathcal{AH}(M)$ : set of all almost Hermitian structures on $M$

$\mathcal{AH}_c(M) = \mathcal{AH}(M) \cap \mathcal{M}_c(M) = \{g \in \mathcal{AH}(M) | Vol(M, g) = c\}$,

$c$ : positive constant (and hence, $\mathcal{AH}_c(M) \subset \mathcal{AH}(M)$)

$(\lambda, \mu) \in \mathbb{R}^2 - (0, 0)$

$$\mathcal{F}_{\lambda,\mu}(J, g) = \int_M (\lambda\tau + \mu\tau^*) dv_g$$

$\tau$ : scalar curvature of $(J, g)$, $\tau^*$ : $*$-scalar curvature of $(g, J)$

T. Koda, Critical almost Hermitian structures, Indian J. Pure Appl. Math. 26 (1995), 679–690.

We study the critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on the space $\mathcal{AH}(M)$ and the subspace $\mathcal{AH}_c(M)(c > 0)$.

If $(g, J)$ is a critical point of $\mathcal{F}_{\lambda,\mu}$, then $(g, J)$ is also the critical point of $\mathcal{F}_{a\lambda,a\mu}$ for any $a \neq 0$.

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$\mathcal{AK}(M)$ : set of all almost Kähler structures on $M$

$\mathcal{AK}(M,\omega_0) \subset \mathcal{AK}(M,[\omega_0]) \subset \mathcal{AK}_c(M) \subset \mathcal{AK}(M) \subset \mathcal{AH}(M)$

$(g,J) \in \mathcal{AK}(M)$ is critical point of $\mathcal{F}_{1,-1}$ on $\mathcal{AK}(M,\omega_o)$ if and only if the Ricci operator of $g$ commutes with the almost complex structure $J$.

D. E. Blair and S. Ianus, Critical associated metrics on symplectic manifolds, Contemporary Math. 51 (1986), 23–29.

$(g,J)$ is a critical point of $\mathcal{F}_{\lambda,\mu}(\lambda \neq \mu)$ on $\mathcal{AK}(M,[\omega_0])$ if and only if $\rho$ is $J$-invariant.

T. Oguro, K. Sekigawa and A. Yamada, proceedings of the 8th International Workshop on Complex Structures and Vector fields, Sofia, Bulgaria, August 21-26, 2006, World Scientific, 2007.

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$(g, J) \in \mathcal{AK}(M)$ is critical point of $\mathcal{F}_{-1,1}$ on $\mathcal{AK}(M)$ if and only if $(g, J)$ is a Kähler structure.

T. Oguro and K. Sekigawa, Some critical almost Kähler structures, Colloq. Math. 111 (2008), no. 2, 205–212.

# Preliminaries

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$(M, g, J)$ : $2n$-dimensional almost Hermitian manifold

( $g$ : Hermitian metric and $J$ : almost complex structure )

$(g, J)$ is called an almost Hermitian structure.

$\omega$ : the Kähler form of $(M, g, J)$ defined by

$$\omega(X, Y) = g(X, JY), \qquad X, Y \in \mathfrak{X}(M)$$

$\nabla$ : the Levi-Civita connection

$R$ : the curvature tensor of $(M, g, J)$ defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in \mathfrak{X}(M)$$

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Ricci tensor : $\rho(X,Y) = \mathrm{tr}(Z \longmapsto R(Z,X)Y)$

Ricci $*$-tensor : $\rho^*(X,Y) = \mathrm{tr}\ (Z \longmapsto R(X,JZ)JY)$

scalar curvature : $\tau = \mathrm{tr}\ Q$

$*$-scalar curvature : $\tau^* = \mathrm{tr}\ Q^*$

where $g(QX,Y) = \rho(X,Y)$ and $g(Q^*X,Y) = \rho^*(X,Y)$

# Special classes of almost Hermitian manifolds

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$$\mathcal{K} = \text{the class of Kähler manifolds}$$

$$\mathcal{AK} = \text{the class of almost Kähler manifolds}$$

$$\mathcal{NK} = \text{the class of nearly Kähler manifolds}$$

$$\mathcal{H} = \text{the class of Hermitian manifolds}$$

| **Class** | **Defining conditions** |
|---|---|
| $\mathcal{K}$ | $\nabla J = 0$ |
| $\mathcal{AK}$ | $d\omega = 0$ $(\underset{X,Y,Z}{\mathfrak{S}} g((\nabla_X J)Y, Z) = 0)$ |
| $\mathcal{NK}$ | $(\nabla_X J)Y + (\nabla_Y J)X = 0$ |
| $\mathcal{H}$ | $J$ : integrable $(N_J = 0)$ |

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The inclusion relations between the above classes

$$\mathcal{K} \begin{matrix} \subset \mathcal{AK} \\ \\ \subset \mathcal{NK} \end{matrix}, \qquad \mathcal{NK} \cap \mathcal{AK} = \mathcal{K}$$

A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. di Mat. Pura. Appl. 123 (1980), 35–58

Note:

(1) $\tau^* - \tau = \frac{1}{2}|\nabla J|^2$ holds on $(M, g, J) \in \mathcal{AK}$,

(2) $\tau - \tau^* = |\nabla J|^2$ holds on $(M, g, J) \in \mathcal{NK}$.

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We first give a necessary and sufficient condition for an almost Hermitian structure $(g, J)$ to be a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$ and $\mathcal{AH}_c(M)$. Some results obtained by applying the Theorems to some special classes of almost Hermitian structure.

J. C. Lee, J. H. Park, K. Sekigawa, Notes on critical almost Hermitan structures, Bull. Korean Math. Soc. 47 (2010), 167–178.

J. C. Lee, J. H. Park and K. Sekigawa, Some critical almost Hermitian structures, Results in Math., 63 (2013), No. 1, 31-45.

# Critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$

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$M$: 2n($\geq$4)-dimensional compact orientable smooth manifold admitting almost complex structure.

For $(g, J) \in \mathcal{AH}(M)$, we consider a smooth curve $(g(t), J(t)) \in \mathcal{AH}(M)$ such that $(g(0), J(0)) = (g, J)$.

We denote by $\omega(t)$ the Kähler form of $(g(t), J(t))$.

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$$\left.\frac{d}{dt}\right|_{t=0} g(t)_{ij} = h_{ij}, \qquad \left.\frac{d}{dt}\right|_{t=0} J(t)_j{}^i = K_j{}^i, \qquad \left.\frac{d}{dt}\right|_{t=0} \omega(t)_{ij} = A_{ij}.$$

$$\left.\frac{d}{dt}\right|_{t=0} g(t)^{ij} = -h^{ij}, \qquad \left.\frac{d}{dt}\right|_{t=0} dv_{g(t)} = \frac{1}{2}(g^{ij}h_{ij})dv_g.$$

$$\left.\frac{d}{dt}\right|_{t=0} \Gamma(t)_{ij}{}^k = \frac{1}{2}g^{ka}(\nabla_i h_{aj} + \nabla_j h_{ia} - \nabla_a h_{ij}).$$

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$$\left.\frac{d}{dt}\right|_{t=0} R(t)_{ijk}{}^{l} = \frac{1}{2}(-R_{ijk}{}^{a}h_a{}^{l} + R_{ija}{}^{l}h_k{}^{a} + \nabla_i\nabla_k h_j{}^{l} - \nabla_j\nabla_k h_i{}^{l} - \nabla_i\nabla^l h_{jk} + \nabla_j\nabla^l h_{ik}),$$

$$\left.\frac{d}{dt}\right|_{t=0} \rho(t)_{ij} = \frac{1}{2}(-R_{aij}{}^{b}h_b{}^{a} + \rho_{ia}h_j{}^{a} + \nabla_a\nabla_j h_i{}^{a} - \nabla_i\nabla_j h_a{}^{a} - \nabla^a\nabla_a h_{ij} + \nabla_i\nabla_a h_j{}^{a}),$$

$$\left.\frac{d}{dt}\right|_{t=0} \tau(t) = -\rho_{ij}h^{ij} + \nabla^i\nabla^j h_{ij} - \nabla^i\nabla_i h_a{}^{a},$$

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$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\rho^*(t)_{ij} =& \rho^*_{ia}h_j{}^a - \frac{1}{2}R_{iua}{}^b J_j{}^u J^{ac}h_{bc} \\ & - \frac{1}{2}J^{ab}J_j{}^c\nabla_i\nabla_a h_{bc} + \frac{1}{2}J^{ab}J_j{}^c\nabla_c\nabla_a h_{bi} \\ & + \frac{1}{2}(2J_j{}^q\rho^*{}_i{}^p - J_j{}^u J^{pa}J^{qb}R_{iuab})A_{pq}.\end{aligned}$$

$$\frac{d}{dt}\Big|_{t=0}\tau^*(t) = \rho^*_{ab}h^{ab} - J^{ia}J^{jb}\nabla_a\nabla_b h_{ij} - 2J^{ip}\rho^*_{iq}A_p{}^q.$$

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$$K_a{}^i J_j{}^a + J_a{}^i K_j{}^a = 0$$

$$h_{ij} = h_{ab} J_i{}^a J_j{}^b + K_{ia} J_j{}^a + J_{ia} K_j{}^a$$

$$K_j{}^i = -h_a{}^i J_j{}^a - A_j{}^i$$

$$h_{ij} = -h_{ab} J_i{}^a J_j{}^b + J_i{}^a A_{aj} + J_j{}^a A_{ai} \tag{1}$$

$$K_j{}^i = h_j{}^a J_a{}^i - A_b{}^a J_a{}^i J_j{}^b \tag{2}$$

Conversely, let $(h, A)$ be a pair of a symmetric $(0, 2)$-tensor $h = (h_{ij})$ and a 2-form $A = (A_{ij})$ satisfying (1) and define a $(1, 1)$-tensor $K$ by (2).

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$\Omega^2_{nd}(M)$: the space of all non-degenerate 2-forms on $M$

$\mathcal{AH}(M)$ can be regarded as a subspace of $\mathcal{M}(M) \times \Omega^2_{nd}(M)$ by the mapping $\iota : \mathcal{AH}(M) \to \mathcal{M}(M) \times \Omega^2_{nd}(M)$ defined by $\iota : (g, J) \longmapsto (g, \omega)$(where $\omega$ is the Kähler form of $(g, J)$). In the sequel, we shall identify $\mathcal{AH}(M)$ with the subspace $\iota(\mathcal{AH}(M))$ $(\equiv \mathcal{H})$ of $\mathcal{M}(M) \times \Omega^2_{nd}(M)$ alternatively. Gil-Medrano and Michor showed that, for any $(g, \omega) \in \mathcal{H}$, a pair $(h, A)$ of a symmetric $(0, 2)$-tensor $h$ and a 2-form $A$ on $M$ is tangent to the subspace $\mathcal{H}$ of $\mathcal{M}(M) \times \Omega^2_{nd}(M)$ at $(g, \omega)$ if and only if $(h, A)$ satisfies the equality (1). There exists a curve $(g(t), J(t)) \in \mathcal{AH}(M)$ through $(g, J)$ for sufficiently small $t$, where tangent vector at $t = 0$ is $(h, K)$.

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$$T_{(g,\omega)}(\mathcal{M}(M)\times\Omega^2_{nd}(M)) = \mathcal{N}^1_{(g,\omega)}\oplus\mathcal{N}^2_{(g,\omega)}\oplus\mathcal{N}^3_{(g,\omega)}\oplus\mathcal{N}^4_{(g,\omega)},$$
$$T_{(g,\omega)}\mathcal{H} = \mathcal{N}_{(g,\omega)}{}^1\oplus\mathcal{N}_{(g,\omega)}{}^2\oplus\mathcal{N}_{(g,\omega)}{}^3 \tag{3}$$

where

$$\begin{aligned}
\mathcal{N}^1_{(g,\omega)} =&\{(h,0)|h_{\bar{i}\bar{j}} = -h_{ij}\},\\
\mathcal{N}^2_{(g,\omega)} =&\{(0,A)|A_{\bar{i}j} = A_{i\bar{j}}\},\\
\mathcal{N}^3_{(g,\omega)} =&\{(h,A)|A_{ij} = -h_{\bar{i}j}, \quad h_{\bar{i}\bar{j}} = h_{ij}\}\\
\mathcal{N}^4_{(g,\omega)} =&\{(h,A)|A_{ij} = h_{\bar{i}j}, \quad h_{\bar{i}\bar{j}} = h_{ij}\}
\end{aligned}$$

O. Gil-Medrano and P. W. Michor, Geodesics on spaces of almost Hermitian structures, Israel Journal of Mathematics, 88 (1994), 319-332.

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$$\frac{d}{dt}\Big|_{t=0}\mathcal{F}_{\lambda,\mu}(g(t),J(t)) = \frac{d}{dt}\Big|_{t=0}\int_M(\lambda\tau(t)+\mu\tau^*(t))dv_{g(t)}$$

$$= \int_M\sum_{i,j}\Big\{\Big(-\lambda\rho_{ij}-\mu\rho_{\bar{i}\bar{j}}+\mu(\rho^*_{ij}+\rho^*_{ji})+\frac{1}{2}(\lambda\tau+\mu\tau^*)g_{ij}$$

$$-\mu\sum_{a,b}(J_{ia}\nabla_a\nabla_bJ_{jb}+J_{ja}\nabla_a\nabla_bJ_{ib}$$

$$+(\nabla_aJ_{ia})\nabla_bJ_{jb}+(\nabla_aJ_{jb})\nabla_bJ_{ia})\Big)h_{ij}$$

$$+2\mu\rho^*_{\bar{i}j}A_{ij}\Big\}dv_g$$

$$= \int_M\sum_{i,j}(T_{ij}h_{ij}+2\mu\rho^*_{\bar{i}j}A_{ij})dv_g.$$

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$$
\begin{aligned}
T_{ij} = -\lambda\rho_{ij} - \mu\rho_{\bar{i}\bar{j}} + \mu(\rho^*_{ij} + \rho^*_{ji}) + \frac{1}{2}(\lambda\tau + \mu\tau^*)g_{ij} \\
-\mu \sum_{a,b} \big( J_{ia}\nabla_a\nabla_b J_{jb} + J_{ja}\nabla_a\nabla_b J_{ib} \qquad\qquad (4) \\
+ (\nabla_a J_{ia})\nabla_b J_{jb} + (\nabla_a J_{jb})\nabla_b J_{ia} \big).
\end{aligned}
$$

## Lemma 1

$(g, J)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$ if and only if $(g, J)$ satisfies

$$
\int_M \sum_{i,j} (T_{ij}h_{ij} + 2\mu\rho^*_{ij}A_{ij})dv_g = 0, \qquad (5)
$$

for $(h, A)$ satisfying (1).

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We introduce two deformations.

(I) Blair-Ianus deformations :

The curve $(g(t), J(t))$ through $(g, J)$ which corresponds to $\omega(t) = \omega$ for any $t$, where $\omega(t)$(resp. $\omega$) is the Kähler form of $(g(t), J(t))$(resp. $(g, J)$).

The curve $(g(t), J(t))$ can be regarded as a curve in $\mathcal{AH}(M)$ through $(g, J)$ with an initial condition $(h, A)$ such that $A = 0$ and $J$-skew invariant $h$.

(II) The curve $(g(t), J(t))$ through $(g, J)$ with initial condition $(h, A)$ given by $h_{ij} = \frac{1}{2}(J_i{}^a A_{aj} + J_j{}^a A_{ai})$ for any 2-form $A = (A_{ij})$ on $M$.

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Remark. If $(g(t), J(t))((g(0), J(0)) = (g, J))$ is a Type (I)-deformation(resp. Type (II)-deformation) of $(g, J) \in \mathcal{AH}(M)$, then the tangent vector $(h, A)$ to the corresponding curve $(g(t), \omega(t)) = \iota(g(t), J(t))$ at $(g, \omega)$ belongs to the subspace ${\mathcal{N}_{(g,\omega)}}^1$(resp. ${\mathcal{N}_{(g,\omega)}}^2 \oplus {\mathcal{N}_{(g,\omega)}}^3$).

Gil-Medrano and Michor provided explicit examples of a Blair-Ianus deformation and Type (II) deformation in (3).

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## Lemma 2

Let $B = (B_{ij})$ be a symmetric $(0,2)$-tensor on $M$. Then

$$\int_M \sum_{i,j} B_{ij} D_{ij} dv_g = 0$$

for all symmetric $(0,2)$-tensor $D$ satisfying $D_{ij} + D_{\bar{i}\bar{j}} = 0$ if and only if $B$ is $J$-invariant.

D. E. Blair and S. Ianus, Critical associated metrics on symplectic manifolds, Contemp. Math. 51 (1986), 23–29.

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Considering Type (I)-deformation $(g(t), J(t))$ of $(g, J)$ and taking account of (5) in Lemma 1 and Lemma 2, we see that the tensor field $T$ is $J$-invariant. Further, considering a Type (II)-deformation $(g(t), J(t))$ of $(g, J)$, we have

$$
\begin{aligned}
0 &= \int_M \sum_{i,j} (T_{ij} h_{ij} + 2\mu \rho^*_{\bar{i}j} A_{ij}) dv_g \\
&= \int_M \sum_{i,j} (T_{i\bar{j}} + 2\mu \rho^*_{\bar{i}j}) A_{ij} dv_g .
\end{aligned}
$$

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### Theorem 3

$(g, J)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$ if and only if $(g, J)$ satisfies $T_{ij} = T_{\bar{i}\bar{j}}$ and $T_{i\bar{j}} + 2\mu\rho^*_{\bar{i}j} = 0$(and hence, especially $\rho^*$ is symmetric for a critical point $(g, J)$ of $\mathcal{F}_{\lambda,\mu}(\mu \neq 0)$ ).

Further, we have the following

### Theorem 4

The functional $\mathcal{F}_{\lambda,\mu}$ vanishes at its critical point.

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### Corollary 5

$(g, J) \in \mathcal{AK}(M)$ is a critical point of functional $\mathcal{F}_{-1,1}$ on $\mathcal{AH}(M)$ if and only if $(g, J)$ is a Kähler structure on $M$.

### Corollary 6

$(g, J) \in \mathcal{NK}(M)$ is a critical point of functional $\mathcal{F}_{-1,1}$ on $\mathcal{AH}(M)$ if and only if $(g, J)$ is a Kähler structure on $M$.

### Theorem 7

Let $M = (M, g, J)$ be a 4-dimensional Hermitian manifold. $(g, J) \in \mathcal{H}(M)$ is a critical point of $\mathcal{F}_{1,-1}$ on $\mathcal{AH}(M)$ if and only if $(g, J)$ is a Kähler structure on $M$.

# Critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$

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**Theorem 8**

$(g, J)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$ if and only if $\lambda\tau + \mu\tau^*$ is constant, and $T_{ij} = T_{\bar{i}\bar{j}}$ , $T_{ij} - 2\mu\rho^*_{ij} = Cg_{ij}$ hold on $M$ with respect to $(g, J)$, where $C = \frac{n-1}{2n}(\lambda\tau + \mu\tau^*)$.

Outline of the proof

Let $(g, J)$ be a critical point of $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$. First, we consider a Type (I)-deformation $(g(t), J(t))$ of $(g, J)$. Then, since $\omega(t) = \omega(0) = \omega(\omega$ is the Kähler form of $(g, J))$ for sufficiently small $|t|$, it follows that $(g(t), J(t)) \in \mathcal{AH}_c(M)$.

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Thus, applying the similar arguments as in the proof of Lemma 1 to the present case and taking account of Lemma 2, we see that the tensor $T$ defined by (4) is $J$-invariant.

Further, considering a Type (II)-deformation $(g(t), J(t))$ of $(g, J)$, we have

$$\begin{aligned} 0 &= \int_M \sum_{i,j} (T_{ij}h_{ij} + 2\mu\rho^*_{ij}A_{ij})dv_g \\ &= \int_M \sum_{i,j} (-T_{kj} + 2\mu\rho^*_{kj})J_{ik}A_{ij}dv_g, \end{aligned} \tag{6}$$

$$0 = \frac{d}{dt}\Big|_{t=0} \int_M dv_{g(t)} = \frac{1}{2}\int_M \sum_{i,j} J_{ij}A_{ij}dv_g. \tag{7}$$

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Thus, from (6) and (7), by applying the Lagrange's multiplier method, we see finally that

$$T_{ij} - 2\mu\rho^*_{ij} = Cg_{ij}$$

holds for some constant $C$.

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## Corollary 9

$(g, J)$ is a critical point of the functional $\mathcal{F}_{1,0}$ on $\mathcal{AH}_c(M)$ if and only if $(M, g, J)$ is Einstein.

Bérard and Bergery has constructed an Einstein non-Kähler Hermitian structure $(g, J)$ on the complex surface $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with positive scalar curvature and further, Koda has proved that $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, g, J)$ is a weakly $*$-Einstein manifold $(\rho^* = \frac{\tau^*}{4}g)$ with non-constant positive $*$-scalar curvature $\tau^*$.

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So, by Corollary 9, we see that the Einstein Hermitian structure $(g, J)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}(\lambda \neq 0, \mu = 0)$ on $\mathcal{AH}_c(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ for some positive constant $c$. However, from Theorem 4, we see also that $(g, J)$ can not be a critical point of functional on $\mathcal{AH}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ for all $(\lambda, \mu)(\neq (0, 0))$.

**Theorem 10**

$(g, J) \in \mathcal{NK}(M)$ is a critical point of $\mathcal{F}_{1,-1}$ on $\mathcal{AH}_c(M)$ for some positive constant $c$ if and only if

(1) $(g, J)$ is a Kähler structure on $M$, or

(2) $(g, J)$ is an Einstein and $*$-Einstein non-Kähler(or strictly) nearly Kähler structure on $M$ with $\tau = 5\tau^*$.

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### Theorem 11

$(g, J) \in \mathcal{AK}(M)$ is a critical point of $\mathcal{F}_{-1,1}$ on $\mathcal{AH}_c(M)$ for some positive constant $c$ if and only if $(g, J)$ satisfies the following conditions (i) and (ii):

(1) $\rho = \rho^*$ are both $J$-invariant,

(2) $8\pi\gamma = -4\hat{\rho} - \frac{1}{n}(\tau^* - \tau)\omega$,

where $\gamma$ denotes the first Chern form of $(M, g, J)$ and $\hat{\rho}$ is the 2-form defined by $\hat{\rho}(X, Y) = \rho(X, JY)$ for $X$, $Y \in \mathfrak{X}(M)$.

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**Corollary 12**

$(g, J) \in \mathcal{AK}(M)(dimM = 4)$ is a critical point of $\mathcal{F}_{-1,1}$ on $\mathcal{AH}_c(M)$ for some positive constant $c$ if and only if $(g, J) \in \mathcal{K}(M)$.

We may note that Corollary 12 is stronger than Corollary 5 (previous Corollary 4).

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**Theorem 13**

Let $M = (M, g, J)$ be a compact Hermitian surface. Then $(g, J) \in \mathcal{H}(M)$ is a critical point of $\mathcal{F}_{1,-1}$ on $\mathcal{AH}_c(M)$ for some positive constant $c$ if and only if $(g, J) \in \mathcal{K}(M)$.

The classification of compact irreducible strictly nearly Kähler Einstein manifolds has been given in the following.

Y, Euh and K. Sekigawa, Notes on strictly nearly Kähler Einstein manifolds, C. R. Acad. Bulgaria Sci. 64(2011), 791-798.

# Curvature identities on compact almost Hermitian surfaces

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We derive a curvature identity on a compact almost Hermitian surface from the integral formulas for the first Pontrjagin number and the first Chern number.

J. C. Lee, J. H. Park and K. Sekigawa, Curvature identities derived from an integral formula for the first Chern number, Bull. Korean Math. Soc., 50 (2013), No. 4, 1261-1275.

Y. Euh, J. H. Park and K. Sekigawa, Curvature idneities derived form the intefral furmula for the first Pontrjagin number, Differ. Geom. Appl., 31 (2013), 463–471.

## Theorem 14

Let $M = (M, J, g)$ be a compact almost Hermitian surface. Then the first Pontrjagin number $p_1(M)$ is given by the following integral formula:

$$p_1(M) = \frac{1}{16\pi^2}\int_M \{2\rho^{*ij}\rho*_{ij} + J^{ia}J^{jb}R_{abk}{}^{l}R_{ijl}{}^{k}\}dv_g.$$

## Theorem 15

Let $M = (M, g, J)$ be a compact almost Hermitian surface. Then the first Chern number $c_1(M)^2$ is given by the following integral formula:

$$c_1(M)^2 = \frac{1}{16\pi^2}\int_M \{(\tau^*)^2 - 2\sum \rho^*_{ij}\rho^*_{ji} - \sum \rho^*_{ij}(\nabla_j J_b{}^a)(\nabla_s J_{ta})J_i{}^s J_b{}^t \\ - \frac{\tau^*}{2}\sum(\nabla_c J_a{}^b)(\nabla_s J_t{}^a)J_b{}^t J^{cs}\}dv_g.$$

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Here, we recall the following theorem due to Wu.

**Theorem 16**

Let $M = (M, g, J)$ be a compact almost Hermitian surface.

Then the following equality holds:

$$c_1(M)^2 = 2\chi(M) + p_1(M)$$

where $\chi(M)$ is the Euler number of $M$.

W. T. Wu, Sur la structure presque complexe d'une variété différentiable réelle de dimension 4, C. R. Acad. Sci. Paris, 227 (1948), 1076–1078.

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Now, we set

$$\begin{aligned}\mathcal{F}_p(g,J) &= 16\pi^2 p_1(M)\\ \mathcal{F}_c(g,J) &= 16\pi^2 c_1(M).\end{aligned} \tag{8}$$

We regard $\mathcal{F}_p$ and $\mathcal{F}_c$ as the functionals on $\mathcal{AH}(M)$. First, we note that the first Pontrjagin number $p_1(M)$ in the deRham cohomology group is a topological invariant of $M$ due to the Novikovs theorem and hence, $\mathcal{F}_p$ is constant on $\mathcal{AH}(M)$.

S. P. Novikov, Topological invariance of rational classes of Pontrjagin, Dokl. Akad. Nauk SSSR, 163 (1965), 298–300.

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Since the Euler number $\chi(M)$ is a topological invariant of $M$, taking account of Theorem 16, we see further that the first Chern number $c_1(M)$ is a topological invariant of $M$, and hence $\mathcal{F}_c$ is also constant on $M$.

Therefore, we have finally the following.

**Theorem 17**

Let $M = (M, g, J)$ be a compact almost Hermitian surface. Then the functionals $\mathcal{F}_p$ and $\mathcal{F}_c$ are both constant on $\mathcal{AH}(M)$.

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Now, let $(g, J)$ be any point of $\mathcal{AH}(M)$ and $r(t) = (g(t), J(t))$ $(|t| < \varepsilon,\ \varepsilon > 0)$ by any curve in $\mathcal{AH}(M)$ through $(g(0), J(0)) = (g, J)$. Then from Theorem 17, we have

$$\frac{d^n}{dt^n}\Big|_{t=0} \mathcal{F}_p(g(t), J(t)) = 0 \tag{9}$$

and

$$\frac{d^n}{dt^n}\Big|_{t=0} \mathcal{F}_c(g(t), J(t)) = 0 \tag{10}$$

for $n = 1, 2, \cdots$.

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The following identity holds on any 4-dimensional Riemannian manifold:

$$R^{abc}{}_{i}R_{abcj}-2\rho_i{}^a\rho_{aj}-2\rho^{ab}R_{iabj}+\tau\rho_{ij}-\frac{1}{4}\left(|R|^2-4|\rho|^2+\tau^2\right)g_{ij}=0. \tag{11}$$

Y. Euh, J. H. Park and K. Sekigawa, A curvature identity on a 4-dimensional Riemannian manifold, Results Math., 63 (2013), 107–114.

The following identity holds on any almost Hermitian surface:

$$\rho^*_{ij}+\rho^*_{ji}-\rho_{ij}-J_i{}^aJ_j{}^b\rho_{ab}-\frac{\tau^*-\tau}{2}g_{ij}=0. \tag{12}$$

A. Gray, Curvature identities for Hermitian and almost Hermitian manifolds, Tohoku Math. J. 28 (1976), 601–612.

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From (9) and (10), we have the following.

## Theorem 18

Let $M = (M, g, J)$ be a compact almost Hermitian surface. Then in addition to (11) and (12), the following identities hold on $M$:

$$T_{ij} = T_{ab}J_i{}^a J_j{}^b S_{ij} = S_{ab}J_i{}^a J_j{}^b, \qquad T_{ib}J_j{}^b + S_{ij} = 0 \tag{13}$$

and

$$U_{ij} = U_{ab}J_i{}^a J_j{}^b, V_{ij} = V_{ab}J_i{}^a J_j{}^b, \qquad U_{ib}J_j{}^b + V_{ij} = 0 \tag{14}$$

where $T_{ij} = \frac{1}{2}(T'_{ij} + T'_{ji})$, $S_{ij} = \frac{1}{2}(S'_{ij} - S'_{ji})$

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$$
\begin{aligned}
T'_{ij} =& 2\big(\rho^{*a}{}_{j}\rho^{*}_{ai} - \rho^{*}_{ja}\rho^{*a}_{i} + \rho^{*ab}R_{auvi}J_b{}^{u}J_j{}^{v} \\
&\quad + 2\nabla_b\nabla_a(\rho^{*ac}J_i{}^{b}J_{cj}) + \frac{1}{2}\rho^{*ab}\rho^{*}_{ab}g_{ij}\big) \\
&- 4J_j{}^{a}J^{ub}R_{abk}{}^{l}R_{iul}{}^{k} + 4\nabla_b\nabla_a(J_u{}^{a}J_{vi}R^{uv}{}_{j}{}^{b}) \\
&+ \frac{1}{2}(J^{ua}J^{vb}R_{abk}{}^{l}R_{uvl}{}^{k})g_{ij} \\
S'_{ij} =& 2\rho^{*ab}(2J_{bj}\rho^{*}_{ai} - J_b{}^{c}J_i{}^{u}J_j{}^{v}R_{acuv}) - 2(J^{ua}R_{jak}{}^{l}R_{iul}{}^{k}),
\end{aligned}
$$

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$$
\begin{aligned}
U'_{ij} =& J_i{}^v \rho_v^{*w} (\nabla_w J_a{}^b)(\nabla_j J_u{}^a) J_b{}^u - J_i{}^w \rho_j^{*c} (\nabla_c J_a{}^b)(\nabla_w J_u{}^a) J_b{}^u \\
& + 4\nabla_u (J^{cv} \rho^*_{vi} (\nabla_c J_j{}^u)) - \frac{1}{2} J^{cv} R_{vwsi} J^{dw} J_j{}^s (\nabla_d J_a{}^b)(\nabla_c J_u{}^a) J_b{}^u \\
& - \nabla_s \nabla_c (J_j{}^d J_i{}^s J_k{}^c (\nabla^k J_a{}^b)(\nabla_d J_u{}^a) J_b{}^u) - 3 J^{cv} \rho_v^{*d} (\nabla_d J_{ai})(\nabla_c J_j{}^w) J_w{}^a \\
& - \frac{1}{2} J^{cv} \rho_v^{*d} (\nabla_d J_a{}^b)(\nabla_c J_u{}^a) J_b{}^u g_{ij} - \frac{1}{2} \rho^*_{ij} (\nabla_c J_a{}^b)(\nabla_u J_v{}^a) J_b{}^v J^{cu} \\
& + \frac{1}{2} \nabla_l \nabla_k (J_i{}^k J_j{}^l (\nabla_c J_a{}^b)(\nabla_u J_v{}^a) J_b{}^v J^{cu}) + 2\nabla_c (\tau^* (\nabla_u J_j{}^c) J_i{}^u) \\
& + \frac{3}{2} \tau^* (\nabla_c J_j{}^b)(\nabla_u J_{vi}) J_b{}^v J^{cu} + \tau^* (\nabla_j J_a{}^b)(\nabla_u J_v{}^a) J_b{}^v J_i{}^u \\
& - \frac{1}{4} \tau^* (\nabla_c J_a{}^b)(\nabla_u J_v{}^a) J_b{}^v J^{cu} g_{ij} - 2\rho^*_{lk} R^k{}_{abi} J^{la} J_j{}^b \\
& + 4\nabla_a \nabla^l (\rho^*_{kl} J_i{}^a J_j{}^k) - \rho^{*ab} \rho^*_{ba} g_{ij} + 2\tau^* \rho^*_{ij} \\
& - 2\nabla_b \nabla_a (\tau^* J_i{}^a J_j{}^b) + \frac{1}{2} (\tau^*)^2 g_{ij},
\end{aligned}
$$

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and

$$\begin{aligned}
V'_{ij} = &- J^{qv} J_j{}^p \rho^*_{vi} (\nabla_p J_a{}^b)(\nabla_q J_u{}^a) J_b{}^u - 2\nabla_p (J^{qv} \rho^{*p}_v (\nabla_q J_{wi}) J_j{}^w) \\
&+ \frac{1}{2} J^{qv} J^{pw} J_i{}^c J_j{}^d R_{vwcd} (\nabla_p J_a{}^b)(\nabla_q J_u{}^a) J_b{}^u \\
&+ \rho^{*p}_j (\nabla_p J_a{}^b)(\nabla_i J_u{}^a) J_b{}^u + J^{qv} \rho^{*p}_v (\nabla_p J_{ai})(\nabla_q J_j{}^a) \\
&- J_i{}^p \rho^*_{pj} (\nabla_c J_a{}^b)(\nabla_u J_v{}^a) J_b{}^v J^{cu} + \frac{1}{2} \tau^* (\nabla_c J_{ai})(\nabla_u J_j{}^a) J^{cu}) \\
&+ \frac{1}{2} \tau^* (\nabla_i J_a{}^b)(\nabla_j J_v{}^a) J_b{}^v - \nabla_u (\tau^* (\nabla_c J_j{}^b) J_{bi} J^{cu}) - 4 J_j{}^l \rho^*_{lk} \rho^{*k}_{\ i} \\
&+ 2\rho^*_{lk} J^{lc} J_i{}^a J_j{}^b R^k{}_{cab} + 4\tau^* J_i{}^a \rho^*_{aj}.
\end{aligned}$$

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## Remark A

The identities (13) and (14) hold on any almost Hermitian surfaces without compactness assumption.

Y. Euh, P. Gilkey, J. H. Park and K. Sekigawa, Transplanting geometrical strucures, Differ. Geom. Appl., 31 (2013), 374–387.

# An application

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It is well-known that the following curvature identity holds on any Kähler manifold $M = (M, g, J)$:

$$R_{ijkl} = R_{ijab} {J_k}^a {J_l}^b. \tag{15}$$

Now, let $M = (M, J, g)$ be a Kähler surface. From (11) and (13) in Theorem 18 together with Remark A, we have the following.

**Lemma 19**

In addition to the identities (11) and (15), the following identity holds on $M$:

$$8\rho_{ia}{\rho_j}^a - 2\rho^{ab}R_{iabj} - 2\tau\rho_{ij} - \frac{1}{2}(2|\rho|^2 - \tau^2)g_{ij} = 0.$$

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Further, from (11) and (14) in Theorem 18 together with Remark A, we have also the following.

**Lemma 20**

In addition to the identities (11) and (15), the identity holds on $M$

$$2\rho_i{}^a\rho_{ja} - \tau\rho_{ij} - \frac{1}{2}(|\rho|^2 - \frac{\tau^2}{2})g_{ij} = 0.$$

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Therefore, from Lemmas 19 and 20, we have the following.

## Theorem 21

Let $M = (M, g, J)$ be a Kähler surface. Then in addition to the identities (11) and (15), the following identity holds on $M$:

$$\rho^{ab} R_{iabj} - 2\rho_{ia}\rho_j{}^a = 0.$$

## Remark B

The above Theorem 21 was extended to the higher dimensional Kähler manifolds.

P. Gilkey, J. H. Park and K. Sekigawa, Universal curvature identities and Euler-Lagrange formulas for Kähler manifolds, J. Math. Soc. Japan, in press.

## Further Study

- Discuss second variation of the functional $\mathcal{F}_{\lambda,\mu}$
- Find other applications of Theorem 13

Thank you for your attention