$L^2$ harmonic 1-forms and first eigenvalue estimates of complete minimal submanifolds

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(\(M^n, \nabla\)) is said to be minimal in \((\overline{M}^{n+p}, \overline{\nabla})\) if the mean curvature vector \(\overrightarrow{H} = \sum h_{\alpha i} e_\alpha = \sum \langle \overrightarrow{\nabla} e_i, e_\alpha \rangle e_\alpha = \overrightarrow{0}\).

The length of the second fundamental form \(A\) is defined by \(|A|^2 := \sum (h_{ij}^\alpha)^2\).

An end of a manifold \(M\) is a connected component of \(M \setminus K\) where \(K\) is a sufficiently large compact subset of \(M\).
Question (Bernstein)

Is a minimal graph over $\mathbb{R}^n$ a hyperplane?

- Yes, if $n \leq 7$ (Fleming 1962, De Giorgi 1965, Almgren 1966, Simons 1968)
- No, if $n \geq 8$ (Bombieri-De Giorgi-Giusti 1969)

We now consider a complete minimal hypersurface which is not a graph.
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We now consider a complete minimal hypersurface which is not a graph.
Note that a minimal graph is stable.

- A minimal submanifold $M$ is said to be stable if the second variation of its volume is always nonnegative for any normal variation with compact support.

- In particular, when the codimension $= 1$, a minimal submanifold $M$ in a Euclidean space is stable iff

$$\frac{d^2 \text{Vol}(M_t)}{dt^2} = \int_M |\nabla \varphi|^2 - |A|^2 \varphi^2 dv \geq 0,$$

where $\varphi \in W_0^{1,2}(M)$. 
**Question**

Is a complete stable minimal hypersurface $M$ in $\mathbb{R}^{n+1}$ hyperplane for $n \leq 7$?

This question is still open but we have the following partial results.

- Yes, if $n = 2$ (Do Carmo and Peng 1979, Fischer-Colbrie and Schoen 1980)
- Yes, if $\int_M |A|^2 dv < \infty$ for any $n$ (Do Carmo and Peng 1980)
- Yes, if $\int_M |A|^n dv < \infty$ for any $n$ (The quantity $\int_M |A|^n dv$ is called a total scalar curvature. ) (Shen and Zhu 1998)
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Topological structure of minimal submanifolds

- If $M$ is a stable minimal hypersurface in a Euclidean space, then $M$ has only one end. (Cao-Shen-Zhu 1997)

- There is no nontrivial $L^2$ harmonic 1-form on a complete stable minimal hypersurface in a Euclidean space. (B. Palmer 1991)

Remark.
If a noncompact complete manifold $M$ has no nontrivial $L^2$ harmonic 1-form, then any codimension 1 cycle disconnects $M$. In particular, if $\dim(M) = 2$, then $M$ has no genus. (Dodziuk 1982)

Therefore
if there exists a codimension 1 cycle on a complete minimal hypersurface $M$ in Euclidean space which does not separate $M$, then $M$ is unstable.
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if there exists a codimension 1 cycle on a complete minimal hypersurface $M$ in Euclidean space which does not separate $M$, then $M$ is unstable.
Let $M^n$ be an $n$-dimensional complete immersed minimal submanifold in $\mathbb{R}^{n+p}$, $n \geq 3$. If

$$\left( \int_M |A|^n dv \right)^{\frac{1}{n}} < C_1 = \sqrt{\frac{n}{n-1}} C_s^{-1},$$

then $M$ has only one end. Here $C_s$ is a Sobolev constant. (L. Ni 2001)

The upper bound $C_1$ of total scalar curvature in the above theorem can be improved. (S. 2008)
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Denote by $\mathbb{H}^n$ the $n$-dimensional hyperbolic space of constant sectional curvature $-1$.

**Theorem (S. 2010)**

Let $M$ be an $n$-dimensional complete immersed minimal submanifold in $\mathbb{H}^{n+p}$, $n \geq 5$. If the total scalar curvature satisfies

$$\left(\int_M |A|^n dv\right)^{\frac{1}{n}} < \frac{1}{n-1} \sqrt{n(n-4)C_s^{-1}},$$

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Idea of proof

- Suppose not. Then there exists a nontrivial bounded harmonic function with finite total energy.
- Bochner’s formula for the harmonic function
- Ricci curvature estimate (Leung 1992)
- Multiply a nice test function $\varphi$ and integrate over $M$.
- Sobolev inequality (Hoffman and Spruck 1974)
- First eigenvalue estimate (Cheung and Leung 2001)

$$\frac{1}{4} (n - 1)^2 \leq \lambda_1(M) = \inf_{f} \frac{\int_{M} |\nabla f|^2}{\int_{M} f^2}.$$ 

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If we do not use the first eigenvalue estimate $\lambda_1(M) \geq \frac{(n-1)^2}{4}$, we have

**Theorem (S. 2010)**

Let $M$ be an $n$-dimensional complete immersed minimal submanifold in $\mathbb{H}^{n+p}$, $n \geq 3$. Assume that $\lambda_1(M) > \frac{(n-1)^2}{n}$ and the total scalar curvature satisfies

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\left( \int_M |A|^n \, dv \right)^{\frac{2}{n}} < \frac{n}{n-1} C_s^{-1} \left( \frac{n}{n-1} - \frac{n-1}{\lambda_1(M)} \right).
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Then $M$ must have only one end.
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Then $M$ must have only one end.
**Theorem (N.T. Dung and S. 2012)**

Let $N$ be an $(n + 1)$-dimensional Riemannian manifold with sectional curvature satisfying

$$K_1 \leq K_N \leq K_2,$$

where $K_1, K_2$ are constants and $K_1 \leq K_2 < 0$. Let $M$ be a complete minimal hypersurface in $N$. If

$$\left( \int_M |A|^n \right)^{\frac{1}{n}} \leq \frac{1}{n-1} \sqrt{\frac{n(nK_2 - 4K_1)}{K_2}} C_s^{-1}$$

for $n > 4 \frac{K_1}{K_2}$, then $M$ has only one end. (Here $C_s$ is the Sobolev constant.)
Higher codimensional case

Spruck (1975) proved that for an \( n \)-dimensional minimal submanifold \( M \) in \( \mathbb{R}^{n+m} \), a variational vector field \( E = \varphi \nu \), the second variation of its volume \( \text{Vol}(M_t) \) satisfies

\[
\frac{d^2 \text{Vol}(M_t)}{dt^2} \geq \int_M |\nabla \varphi|^2 - |A|^2 \varphi^2 dv,
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where \( \varphi \in W^{1,2}_0(M) \) and \( \nu \) is the unit normal vector field.

For an \( n \)-dimensional minimal submanifold \( M \) in \( \mathbb{H}^{n+m} \), a simple computation shows that for a variational vector field \( E = \varphi \nu \), the second variation of its volume \( \text{Vol}(M_t) \) satisfies

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\frac{d^2 \text{Vol}(M_t)}{dt^2} \geq \int_M |\nabla \varphi|^2 - (|A|^2 - n) \varphi^2 dv,
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- Spruck (1975) proved that for an $n$-dimensional minimal submanifold $M$ in $\mathbb{R}^{n+m}$, a variational vector field $E = \varphi \nu$, the second variation of its volume $\text{Vol}(M_t)$ satisfies

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where $\varphi \in W_0^{1,2}(M)$ and $\nu$ is the unit normal vector field.
Motivated by this, we will call a minimal submanifold $M$ in $\mathbb{H}^{n+m}$ 
**super stable** if for any $\varphi \in W^{1,2}_0(M)$

$$\int_M |\nabla \varphi|^2 - (|A|^2 - n)\varphi^2 \, dv \geq 0.$$ 

When $m = 1$, the concept of super stability is the same as the usual 
definition of stability.

Wang(2003) introduced the concept of super stability to prove that 
if $M^n(n \geq 3)$ is a complete super stable minimal submanifold with 
finite total scalar curvature in $\mathbb{R}^{n+p}$, then $M$ is an affine $n$-plane.
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**Theorem (S. 2010)**

Let $M$ be a complete super stable minimal submanifold in $\mathbb{H}^{n+m}$. Assume that the first eigenvalue of $M$ satisfies

$$(2n - 1)(n - 1) < \lambda_1(M).$$

Then there are no nontrivial $L^2$ harmonic 1-forms on $M$.

**Corollary**

Let $M$ be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ satisfying that

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Then there are no nontrivial $L^2$ harmonic 1-forms on $M$. 
Let $M$ be a complete noncompact Riemannian manifold and let $\Omega$ be a compact domain in $M$. Let $\lambda_1(\Omega) > 0$ denote the first eigenvalue of the Dirichlet boundary value problem

$$\begin{cases}
\Delta f + \lambda f = 0 & \text{in } \Omega \\
f = 0 & \text{on } \partial \Omega
\end{cases}$$

where $\Delta$ denotes the Laplace operator on $M$. Then the first eigenvalue $\lambda_1(M)$ is defined by

$$
\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega),
$$

where the infimum is taken over all compact domains in $M$. 
(Cheung and Leung)
For a complete minimal submanifold $M^n$ in $\mathbb{H}^m$,

$$\frac{1}{4} (n - 1)^2 \leq \lambda_1(M).$$

Here this inequality is sharp because equality holds when $M$ is totally geodesic by McKean's result.

(Candel)
Let $\Sigma$ be a complete simply connected stable minimal surface in the 3-dimensional hyperbolic space. Then the first eigenvalue of $\Sigma$ satisfies

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(Bérard-Castillon-Cavalcante 2011)
Let $\Sigma$ be a complete stable minimal surface in the 3-dimensional hyperbolic space. Then

$$\frac{1}{4} \leq \lambda_1(\Sigma) \leq \frac{4}{7}.$$  

(S. 2011)
Let $\Sigma$ be a simply connected stable minimal surface in a 3-dimensional simply connected Riemannian manifold $N^3$ with sectional curvature $K_N$ satisfying $-b^2 \leq K_N \leq -a^2 < 0$ for $0 < a \leq b$. Then the first eigenvalue of $\Sigma$ satisfies

$$\frac{1}{4} a^2 \leq \lambda_1(\Sigma) \leq \frac{4}{3} b^2.$$
Theorem (S. 2011)

Let $M$ be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^2 dv < \infty$. Then we have

$$\frac{(n - 1)^2}{4} \leq \lambda_1(M) \leq n^2.$$ 

It suffices to show that $\lambda_1(M) \leq n^2$ by the result of Cheung-Leung.
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Sketch of proof

- Step 1: Use $\lambda_1(B_R)$ for a ball $B_R$ of radius $R$ centered at $p \in M$.

$$\lambda_1(M) \leq \lambda_1(B_R) \leq \frac{\int_{B_R} |\nabla \phi|^2}{\int_{B_R} \phi^2}$$

for any compactly supported Lipschitz function $\phi$.

- Step 2: Substitute $|A|f$ for $\phi$ in the above inequality.


$$|A|\Delta |A| + |A|^4 + n|A|^2 \geq \frac{2}{n}|\nabla |A||^2.$$
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- **Step 2**: Substitute $|A|f$ for $\phi$ in the above inequality.

- **Step 3**: Use the following Simons-type inequality due to Chern, do Carmo, and Kobayashi (1970).

\[
|A|\Delta|A| + |A|^4 + n|A|^2 \geq \frac{2}{n}||\nabla|A||^2.
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$$|A|\Delta|A| + |A|^4 + n|A|^2 \geq \frac{2}{n} |\nabla|A||^2.$$
Step 4: Multiply both sides by a Lipschitz function $f^2$ with compact support in $B_R \subset M$ and integrate over $B_R$.

$$\int_{B_R} f^2 |A| \Delta |A| dv + \int_{B_R} f^2 |A|^4 dv + n \int_{B_R} f^2 |A|^2 dv \geq \frac{2}{n} \int_{B_R} f^2 |\nabla|A||^2 dv$$

Step 5: Use the stability inequality. (Substitute $|A|f$ for $f$)

$$\int_{B_R} |\nabla(|A|f)|^2 - (|A|^2 - n)|A|^2 f^2 dv \geq 0.$$
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Step 6: By the above steps and Schwarz inequality, we finally obtain

\[
\left\{1 + \frac{2n(1 + \alpha)}{\lambda_1(M)}\right\} \int_{B_R} |A|^2 |\nabla f|^2 dv \geq \left\{\frac{2}{n} - \frac{2n(1 + \frac{1}{\alpha})}{\lambda_1(M)}\right\} \int_{B_R} |\nabla|A||^2 f^2 dv.
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Step 7: Suppose that \(\lambda_1(M) > n^2\). Choosing \(\alpha > 0\) sufficiently large and letting \(R \to \infty\), we obtain \(\nabla|A| \equiv 0\), i.e., \(|A|\) is constant. However, since \(\int_M |A|^2 < \infty\) and the volume of \(M\) is infinite, it follows that \(|A| \equiv 0\) which means that \(M\) is a totally geodesic hyperplane. Since the first eigenvalue of totally geodesic hyperplane in \(\mathbb{H}^{n+1}\) is equal to \(\frac{(n - 1)^2}{4}\), this is a contradiction. Therefore we get \(\lambda_1(M) \leq n^2\).
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\[ \left\{ 1 + \frac{2n(1 + \alpha)}{\lambda_1(M)} \right\} \int_{B_R} |A|^2 |\nabla f|^2 \, dv \geq \left\{ \frac{2}{n} - \frac{2n(1 + \frac{1}{\alpha})}{\lambda_1(M)} \right\} \int_{B_R} |\nabla |A||^2 f^2 \, dv. \]

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Vanishing theorems for $L^p$ harmonic 1-forms

(S. 2014)

Let $N$ be an $(n + 1)$-dimensional complete Riemannian manifold with sectional curvature satisfying that $K \leq K_N$ where $K \leq 0$ is a constant. Let $M$ be a complete noncompact stable minimal hypersurface in $N$. Assume that, for $0 < p < \frac{n}{n - 1} + \sqrt{2n}$,

$$\lambda_1(M) > \frac{-2n(n - 1)^2 p^2 K}{2n - [(n - 1)p - n]^2}.$$

Then there is no nontrivial $L^{2p}$ harmonic 1-form on $M$. 


(Schoen and Yau 1976)
Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature. If $f$ is a harmonic function on $M$ with finite $L^2$ energy, then $f$ is constant.

Recall that a function $f$ on a Riemannian manifold $M$ has finite $L^p$ energy if $|\nabla f| \in L^p(M)$.

(S. 2014)
Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on $M$ with finite $L^p$ energy for $0 < p < \frac{n}{n-1} + \sqrt{2n}$. 
Application

(Schoen and Yau 1976)
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Recall that a function $f$ on a Riemannian manifold $M$ has finite $L^p$ energy if $|\nabla f| \in L^p(M)$.

(S. 2014)
Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on $M$ with finite $L^p$ energy for $0 < p < \frac{n}{n-1} + \sqrt{2n}$. 
(Schoen and Yau 1976)
Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature. If $f$ is a harmonic function on $M$ with finite $L^2$ energy, then $f$ is constant.

Recall that a function $f$ on a Riemannian manifold $M$ has finite $L^p$ energy if $|\nabla f| \in L^p(M)$.

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Let $M$ be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on $M$ with finite $L^p$ energy for $0 < p < \frac{n}{n-1} + \sqrt{2n}$. 
Thank you for your attention.