L^2 harmonic 1-forms and first eigenvalue estimates of complete minimal submanifolds

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Notations

- (M^n, ∇) is said to be minimal in $(\overline{M}^{n+p}, \overline{\nabla})$ if the mean curvature vector $\overrightarrow{H} = \sum h_{ii}^{\alpha} e_{\alpha} = \sum \langle \overline{\nabla}_{e_i} e_i, e_{\alpha} \rangle e_{\alpha} = \overrightarrow{0}$.
- The length of the second fundamental form A is defined by $|A|^2 := \sum (h_{ij}^{\alpha})^2$.
- An end of a manifold M is a connected component of $M \setminus K$ where K is a sufficiently large compact subset of M.

Question (Bernstein)

Is a minimal graph over $\mathbb{R}^n$ a hyperplane?

- Yes, if $n \leq 7$ (Fleming 1962, De Giorgi 1965, Almgren 1966, Simons 1968)
- No, if $n \geq 8$ (Bombieri-De Giorgi-Giusti 1969)

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Note that a minimal graph is stable.

- A minimal submanifold M is said to be stable if the second variation of its volume is always nonnegative for any normal variation with compact support.
- In particular, when the codimension $= 1$, a minimal submanifold M in a Euclidean space is stable iff

$$\frac{d^2\mathrm{Vol}(M_t)}{dt^2} = \int_M |\nabla\varphi|^2 - |A|^2\varphi^2 dv \geq 0,$$

where $\varphi \in W_0^{1,2}(M)$.

Question

Is a complete stable minimal hypersurface M in $\mathbb{R}^{n+1}$ hyperplane for $n \leq 7$?

This question is still open but we have the following partial results.

- Yes, if $n = 2$ (Do Carmo and Peng 1979, Fischer-Colbrie and Schoen 1980)
- Yes, if $\int_M |A|^2 dv < \infty$ for any n (Do Carmo and Peng 1980)
- Yes, if $\int_M |A|^n dv < \infty$ for any n (The quantity $\int_M |A|^n dv$ is called a total scalar curvature.) (Shen and Zhu 1998)

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Topological structure of minimal submanifolds

- If M is a stable minimal hypersurface in a Euclidean space, then M has only one end. (Cao-Shen-Zhu 1997)
- There is no nontrivial L^2 harmonic 1-form on a complete stable minimal hypersurface in a Euclidean space. (B. Palmer 1991)

Remark.
If a noncompact complete manifold M has no nontrivial L^2 harmonic 1-form, then any codimension 1 cycle disconnects M. In particular, if $\dim(M) = 2$, then M has no genus. (Dodziuk 1982)

Therefore
if there exists a codimension 1 cycle on a complete minimal hypersurface M in Euclidean space which does not separate M, then M is unstable.

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- Let M^n be an n-dimensional complete immersed minimal submanifold in $\mathbb{R}^{n+p}$, $n \geq 3$. If

$$\Big(\int_M |A|^n dv\Big)^{\frac{1}{n}} < C_1 = \sqrt{\frac{n}{n-1} C_s^{-1}},$$

  then M has only one end. Here C_s is a Sobolev constant. (L. Ni 2001)

- The upper bound C_1 of total scalar curvature in the above theorem can be improved. (S. 2008)

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Denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space of constant sectional curvature -1.

Theorem (S. 2010)

Let M be an n-dimensional complete immersed minimal submanifold in $\mathbb{H}^{n+p}$, $n \geq 5$. If the total scalar curvature satisfies

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Idea of proof

- Suppose not. Then there exists a nontrivial bounded harmonic function with finite total energy.
- Bochner's formula for the harmonic function
- Ricci curvature estimate(Leung 1992)
- Multiply a nice test function φ and integrate over M.
- Sobolev inequality (Hoffman and Spruck 1974)
- First eigenvalue estimate (Cheung and Leung 2001)

$$\frac{1}{4}(n-1)^2 \leq \lambda_1(M) = \inf_f \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

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If we do not use the first eigenvalue estimate $\lambda_1(M) \geq \frac{(n-1)^2}{4}$, we have

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Let M be an n-dimensional complete immersed minimal submanifold in $\mathbb{H}^{n+p}$, $n \geq 3$. Assume that $\lambda_1(M) > \frac{(n-1)^2}{n}$ and the total scalar curvature satisfies

$$\Big(\int_M |A|^n dv\Big)^{\frac{2}{n}} < \frac{n}{n-1} C_s{}^{-1}\Big(\frac{n}{n-1} - \frac{n-1}{\lambda_1(M)}\Big).$$

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Theorem (N.T. Dung and S. 2012)

Let N be an $(n+1)$-dimensional Riemannian manifold with sectional curvature satisfying

$$K_1 \leq K_N \leq K_2,$$

where K_1, K_2 are constants and $K_1 \leq K_2 < 0$. Let M be a complete minimal hypersurface in N. If

$$\left(\int_M |A|^n\right)^{\frac{1}{n}} \leq \frac{1}{n-1}\sqrt{\frac{n(nK_2 - 4K_1)}{K_2}C_s^{-1}}$$

for $n > 4\frac{K_1}{K_2}$, then M has only one end. (Here C_s is the Sobolev constant.)

Higher codimensional case

- Spruck(1975) proved that for an n-dimensional minimal submanifold M in $\mathbb{R}^{n+m}$, a variational vector field $E = \varphi\nu$, the second variation of its volume $\mathrm{Vol}(M_t)$ satisfies

$$\frac{d^2\mathrm{Vol}(M_t)}{dt^2} \geq \int_M |\nabla\varphi|^2 - |A|^2\varphi^2 dv,$$

where $\varphi \in W_0^{1,2}(M)$ and ν is the unit normal vector field.

- For an n-dimensional minimal submanifold M in $\mathbb{H}^{n+m}$, a simple computation shows that for a variational vector field $E = \varphi\nu$, the second variation of its volume $\mathrm{Vol}(M_t)$ satisfies

$$\frac{d^2\mathrm{Vol}(M_t)}{dt^2} \geq \int_M |\nabla\varphi|^2 - (|A|^2 - n)\varphi^2 dv,$$

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- Motivated by this, we will call a minimal submanifold M in $\mathbb{H}^{n+m}$ super stable if for any $\varphi \in W_0^{1,2}(M)$

$$\int_M |\nabla\varphi|^2 - (|A|^2 - n)\varphi^2 dv \geq 0.$$

- When $m = 1$, the concept of super stability is the same as the usual definition of stability.
- Wang(2003) introduced the concept of super stability to prove that if $M^n (n \geq 3)$ is a complete super stable minimal submanifold with finite total scalar curvature in $\mathbb{R}^{n+p}$, then M is an affine n-plane.

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Theorem (S. 2010)

Let M be a complete super stable minimal submanifold in $\mathbb{H}^{n+m}$. Assume that the first eigenvalue of M satisfies

$$(2n-1)(n-1) < \lambda_1(M).$$

Then there are no nontrivial L^2 harmonic 1-forms on M.

Corollary

Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ satisfying that

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First eigenvalue of complete noncompact manifolds

Let M be a complete noncompact Riemannian manifold and let Ω be a compact domain in M. Let $\lambda_1(\Omega) > 0$ denote the first eigenvalue of the Dirichlet boundary value problem

$$\begin{cases} \Delta f + \lambda f = 0 & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases}$$

where Δ denotes the Laplace operator on M. Then the first eigenvalue $\lambda_1(M)$ is defined by

$$\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega),$$

where the infimum is taken over all compact domains in M.

- (Cheung and Leung)
  For a complete minimal submanifold M^n in $\mathbb{H}^m$,
  $$\frac{1}{4}(n-1)^2 \le \lambda_1(M).$$
  Here this inequality is sharp because equality holds when M is totally geodesic by McKean's result.
- (Candel)
  Let Σ be a complete simply connected stable minimal surface in the 3-dimensional hyperbolic space. Then the first eigenvalue of Σ satisfies
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- (Bérard-Castillon-Cavalcante 2011)
  Let Σ be a complete stable minimal surface in the 3-dimensional hyperbolic space. Then

$$\frac{1}{4} \leq \lambda_1(\Sigma) \leq \frac{4}{7}.$$

- (S. 2011)
  Let Σ be a simply connected stable minimal surface in a 3-dimensional simply connected Riemannian manifold N^3 with sectional curvature K_N satisfying $-b^2 \leq K_N \leq -a^2 < 0$ for $0 < a \leq b$. Then the first eigenvalue of Σ satisfies

$$\frac{1}{4}a^2 \leq \lambda_1(\Sigma) \leq \frac{4}{3}b^2.$$

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Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^2 dv < \infty$. Then we have

$$\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq n^2.$$

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Sketch of proof

- Step1 : Use $\lambda_1(B_R)$ for a ball B_R of radius R centered at $p \in M$.

$$\lambda_1(M) \le \lambda_1(B_R) \le \frac{\int_{B_R} |\nabla \phi|^2}{\int_{B_R} \phi^2}$$

  for any compactly supported Lipschitz function ϕ.

- Step2 : Substitute $|A|f$ for ϕ in the above inequality.
- Step3 : Use the following Simons-type inequality due to Chern, do Carmo, and Kobayashi(1970).

$$|A|\Delta|A| + |A|^4 + n|A|^2 \ge \frac{2}{n}|\nabla|A||^2.$$

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- Step4 : Multiply both sides by a Lipschitz function f^2 with compact support in $B_R \subset M$ and integrate over B_R.

$$\int_{B_R} f^2|A|\Delta|A|dv + \int_{B_R} f^2|A|^4 dv + n\int_{B_R} f^2|A|^2 dv \geq \frac{2}{n}\int_{B_R} f^2|\nabla|A||^2 dv$$

- Step5 : Use the stability inequality. (Substitute $|A|f$ for f)

$$\int_{B_R} |\nabla(|A|f)|^2 - (|A|^2 - n)|A|^2 f^2 dv \geq 0.$$

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- Step6 : By the above steps and Schwarz inequality, we finally obtain

$$\left\{1+\frac{2n(1+\alpha)}{\lambda_1(M)}\right\}\int_{B_R}|A|^2|\nabla f|^2dv \geq \left\{\frac{2}{n}-\frac{2n(1+\frac{1}{\alpha})}{\lambda_1(M)}\right\}\int_{B_R}|\nabla|A||^2f^2dv.$$

- Step7 : Suppose that $\lambda_1(M) > n^2$. Choosing $\alpha > 0$ sufficiently large and letting $R \to \infty$, we obtain $\nabla|A| \equiv 0$, i.e., $|A|$ is constant. However, since $\int_M |A|^2 < \infty$ and the volume of M is infinite, it follows that $|A| \equiv 0$ which means that M is a totally geodesic hyperplane. Since the first eigenvalue of totally geodesic hyperplane in $\mathbb{H}^{n+1}$ is equal to $\dfrac{(n-1)^2}{4}$, this is a contradiction. Therefore we get $\lambda_1(M) \leq n^2$.

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Vanishing theorems for L^p harmonic 1-forms

- (S. 2014)
  Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying that $K \leq K_N$ where $K \leq 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that, for $0 < p < \frac{n}{n-1} + \sqrt{2n}$,

$$\lambda_1(M) > \frac{-2n(n-1)^2 p^2 K}{2n - [(n-1)p - n]^2}.$$

  Then there is no nontrivial L^{2p} harmonic 1-form on M.

Application

- (Schoen and Yau 1976)
  Let M be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature. If f is a harmonic function on M with finite L^2 energy, then f is constant.
- Recall that a function f on a Riemannian manifold M has finite L^p energy if $|\nabla f| \in L^p(M)$.
- (S. 2014)
  Let M be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on M with finite L^p energy for $0 < p < \frac{n}{n-1} + \sqrt{2n}$.

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Thank you for your attention.