# Picard, Lefschetz, Cartier, Grothendieck ...

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## Example: Quadrics again

$Q_t := (x^2 + y^2 + z^2 - tw^2 = 0)$
$t \neq 0$ intersecting lines $L_t, L'_t$
degenerate to same line $L_0$ on $Q_0$ ($2L_0$ is Cartier)
compute self-intersections:
special fiber: $\frac{1}{2}(a+b)^2$ general fiber: $2ab$
Note that $\frac{1}{2}(a+b)^2 - 2ab = \frac{1}{2}(a-b)^2 \geq 0$.
If $a+b$ is even:

- $aL_t + bL'_t$ Cartier on every fiber
- globally Cartier iff $\left(aL_t + bL'_t\right)^2 = \left(aL_0 + bL_0\right)^2$

## Numerical Cartier condition (weak form)

– $f : X \to C$ is a flat, proper, pure relative dimension $n$
– normal fibers
– $D$ divisor such that each $D_c$ is Cartier and ample.
Then

1. $c \mapsto (D_c^n)$ is upper semi-continuous and
2. $D$ is Cartier iff the above function is locally constant.

## Corollary: Numerical criterion of stability

– $f : X \to C$ a proper, flat, pure relative dimension $n$.
– fibers are (semi) log canonical with
– ample canonical class $K_{X_c}$.

Then

1. $c \mapsto \left(K_{X_c}^n\right)$ is upper semi-continuous
2. $f : X \to C$ is stable ($=\mathbb{Q}$-Gorenstein) iff the above function is locally constant.

## Numerical Cartier condition (strong form)

- $S$ reduced scheme over a field $k$
- $f : X \to S$ flat, proper, pure relative dimension $n$
- $S_2$ fibers
- $Z \subset X$ such that $Z \cap X_s$ has codim $\geq 2$
- $L^0$ line bundle on $X \setminus Z$
- $L^0|_{X_s \setminus Z}$ extends to an ample line bundle $L_s$ on $X_s$. Then

1. $s \mapsto (L_s^n)$ is an upper semi-continuous function on $S$ and
2. $L^0$ extends to a line bundle $L$ on $X$ iff the above function is locally constant.

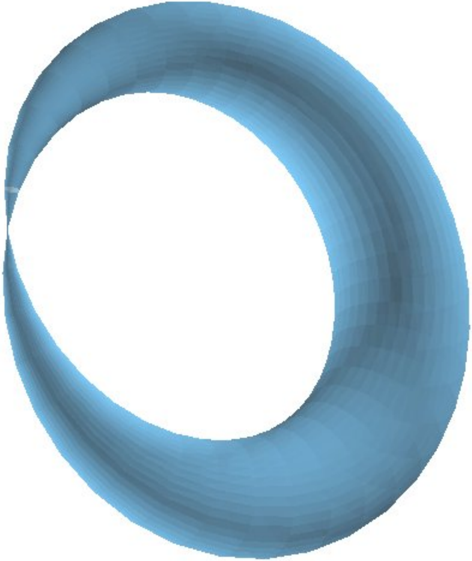
## Numerical Cartier condition (strong local form)

– $S$ reduced scheme over a field $k$
– $f : X \to S$ flat, projective, pure relative dimension $n$
– $S_2$ fibers
– $L^0$ line bundle on $X \setminus Z$
– $L^0|_{X_s \setminus Z}$ extends to an arbitrary line bundle $L_s$ on $X_s$.
– $H$ relatively ample on $X/S$

Then

1. $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is upper semi-continuous and
2. $L^0$ extends to a line bundle $L$ on $X$ iff the above function is locally constant.

# Unrelared picture



## Mostly flat families of $S_2$ sheaves

- $f : X \to S$ flat
- $U \subset X$ open, $Z = X \setminus U$
- $F$ coherent on $U$ with pure, $S_2$ fibers
- $Z \cap X_s$ has codimension $\geq 2$ in the support of $F_s$
- $S_2$-extension (or HULL) : $F^H := j_*(F)$
- fiberwise $S_2$ extension: $(F_s)^H := (j_s) * \left(F|_{U_s}\right)$

QUESTION: when is $\left(F^H\right)_s \cong (F_s)^H$?

Restriction map:

$$r_s^F : \left(F^H\right)_s \hookrightarrow (F_s)^H$$

## Weak criterion for flat families of $S_2$ sheaves

In addition: $f$ is projective, $\mathcal{O}_X(1)$ ample

Recall: $r_s^F : \left(F^H\right)_s \hookrightarrow (F_s)^H$

Equivalent:

1. $\left(F^H\right)_s \cong (F_s)^H$ for every $s$
2. Hilbert pol. $\chi\left(X_s, (F_s)^H(m)\right)$ is independent of $s$.

## Weak flatness criterion for line bundles

In addition: $f$ is projective, $\mathcal{O}_X(1)$ ample
Equivalent:

1. $L^0$ extends to a line bundle $L$ on $X$
2. Hilbert pol. $\chi\big(X_s, L_s(m)\big)$ is independent of $s$
3. all the $\big(L_s^i \cdot \mathcal{O}_{X_s}(1)^j \cdot Td_{n-i-j}(X_s)\big)$ are independent of $s$.

Main Claim:
$\big(L_s^2 \cdot \mathcal{O}_{X_s}(1)^{n-2}\big)$ independent of $s \Rightarrow$

all the $\big(L_s^i \cdot \mathcal{O}_{X_s}(1)^j \cdot Td_{n-i-j}(X_s)\big)$ are independent of $s$

## Upper semi-continuity

– $C$ spectrum of DVR, closed point $0$, generic point $g$.
By semicontinuity

$$h^0(X_0, L_0) \geq h^0(X_0, L|_{X_0}) \geq h^0(X_g, L|_{X_g}) = h^0(X_g, L_g).$$

If $L_0$ and $L_g$ are ample, then
applying it to $L^m$ and using Riemann–Roch:

$$(L_0)^n = \lim \frac{h^0(X_0, L_0^{\otimes m}) \cdot n!}{m^n} \geq \lim \frac{h^0(X_g, L_g^{\otimes m}) \cdot n!}{m^n} = (L_g^n).$$

## Proof in dimension 2

– $f : X \to T$ flat, proper, dimension $2$ with $S_2$ fibers
– $L$ (need not be ample)

$\chi\left(X_t, L_t^{\otimes m}\right) = a_t m^2 + b_t m + c_t$
cokernel of $r_0^m : L^{[\otimes m]}|_{X_0} \hookrightarrow L_0^{\otimes m}$ is Artinian, so
$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g$ for every $m$.
RR: $a_t = \frac{1}{2}\left(L_t \cdot L_t\right)$ and $c_t = \chi\left(X_t, \mathcal{O}_{X_t}\right)$.

Assume now that $(L_0 \cdot L_0) = (L_g \cdot L_g)$. Then $a_0 = a_g$ thus

$$b_0 m + c_0 \geq b_g m + c_g \quad \text{for every } m.$$

$m \gg 1$ gives $b_0 \geq b_g$ and $m \ll -1$ gives $-b_0 \geq -b_g$.
So $b_0 = b_g$ and $c_0 = c_g$ since $f$ is flat.
$\Rightarrow \chi(X_0, coker(r_0^1)) = 0$ so $coker(r_0^1) = 0$.
by Nakayama lemma, $L$ is locally free

## Local form; second look

– $f : X \to S$ flat, projective, pure dim $n$ with $S_2$ fibers
– $L^0$ line bundle on $X \setminus Z$
– every $L_s$ line bundle

1. $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is upper semi-continuous and
2. $L^0$ extends to a line bundle $L$ on $X$ iff the above function is locally constant.

Note: $(H_s^{n-2} \cdot$ takes general surface section
$\Rightarrow$ upper semi-continuity follows from 2-dim case

Claim: Codim $\geq 3$ singularities do not matter!

## Grothendieck–Lefschetz

– $(x \in X)$ local scheme, $x \in D \subset X$ Cartier divisor
– $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$
– $L$ be a coherent, rank 1, $S_2$ sheaf on $U$
– such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$
– Assume that depth$_x \mathcal{O}_D \geq 3$

$\Rightarrow L \cong \mathcal{O}_U$.

Proof. $0 \to L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \to 0$ gives

$$\begin{array}{ccccccc} H^0(U, L) & \xrightarrow{t} & H^0(U, L) & \xrightarrow{r} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \to \\ H^1(U, L) & \xrightarrow{t} & H^1(U, L) & \to & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). & \end{array}$$

$\operatorname{depth}_x \mathcal{O}_D \geq 3 \Rightarrow H^1(U_D, \mathcal{O}_{U_D}) = 0$ and so
$\Rightarrow t : H^1(U, L) \to H^1(U, L)$ is surjective.

$\dim U \geq 4$ implies $H^1(U, L)$ has finite length
$\Rightarrow t : H^1(U, L) \to H^1(U, L)$ isomorphism.

Thus $r : H^0(U, L) \to H^0(U_D, L_D)$ is surjective.

Lift back the constant 1 section to $L$. $\square$

## Stronger Grothendieck–Lefschetz

– $(x \in X)$ local scheme, $x \in D \subset X$ Cartier divisor
– $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$
– $L$ line bundle on $U$
– such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$
– remove assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 3$
– new assumption: $\operatorname{depth}_x \mathcal{O}_D \geq 2$ and $\dim D \geq 3$.

$\Rightarrow L \cong \mathcal{O}_U$.

– Conjectured around 2010
– Proved for semi-log-canonical 2012 (arXiv:1211.0317)
– Bhatt – de Jong: $X$ normal over field (arXiv:1302.3189)
– General case (over a field) (arXiv:1407.5108)

## Normal case in characteristic $p$

$\pi : X^+ \to X$ normalization in algebraic closure of $k(X)$

– Hochster–Huneke: $X^+$ is CM

– previous proof runs on $X^+$ (almost)

– $L$ becomes trivial on some finite degree cover

– use norm map to show that $L^m \cong \mathcal{O}_U$ for some $m > 0$

– work a little more ...