Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory

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Imtak Jeon

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1. $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory

[1011.1324, 1105.6294, 1109.2035, 1112.0069, 1206.3478, 1210.5078]

Symmetries of SDFT

- **O(D,D) T-duality**

- **Gauge symmetries**

  1. **DFT-diffeomorphism (generalized Lie derivative)**
     - **Diffeomorphism**
     - ***B*-field gauge symmetry**

  2. **A pair of local Lorentz symmetries, Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$**

  3. **$\mathcal{N}=2$ Local SUSY**

- **Coordinate gauge symmetry: section condition $\partial_A\partial^A = 0$**

  [Lee, Park]

Field contents of $\mathcal{N} = 2$ $D = 10$ SDFT

- **Bosons**
  - **NS-NS sector** $\begin{cases} \textbf{DFT-dilaton:} & d \\ \textbf{DFT-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$

- **Fermions**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

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Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB}$
$p, q, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ vector	$\eta_{pq} = \mathbf{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ spinor	$C_{+\alpha\beta}\,, \quad (\gamma^p)^T = C_+\gamma^p C_+^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathbf{diag}(+--\cdots-)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}\,, \quad (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+\bar{\gamma}^{\bar{p}}\bar{C}_+^{-1}$

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All NS-NS fields, d, V_{Ap}, $\bar{V}_{A\bar{p}}$, will be equally treated as basic geometric objects.

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R-R potential is bi-fundamental spinor representation as a democratic description.

$cf.$ $\mathbf{O}(D,D)$ spinor representation Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach

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Field contents of $\mathcal{N}=2$ $D=10$ SDFT

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    - **DFT-vielbeins:** $V_{Ap}\,,\quad \bar{V}_{A\bar{p}}$
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***A priori*, O(D,D) rotates only the O(D,D) vector indices (capital Roman), and the R-R sector and all the fermions are O(D,D) T-duality singlet.**

IIA $\Leftrightarrow$ IIB exchange will follow only after parametrization and fixing a gauge.

Field contents of $\mathcal{N}=2$ $D=10$ SDFT

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- **Set the chiralities**

$$\gamma^{(D+1)}\mathcal{C}\bar{\gamma}^{(D+1)}=cc'\mathcal{C}\,.\qquad \begin{array}{ll}\gamma^{(D+1)}\psi_{\bar{p}}=c\psi_{\bar{p}}\,, & \gamma^{(D+1)}\rho=-c\rho\,,\\ \bar{\gamma}^{(D+1)}\psi'_{p}=c'\psi'_{p}\,, & \bar{\gamma}^{(D+1)}\rho'=-c'\rho'\,.\end{array}$$

c **and** c' **are sign factors, and equivalent up to a Pin**$(1,9)\times$ **Pin**$(9,1)$.
So we may fix $c=c'=+1$ **without loss of generality.**
However, the theory contains two 'types' of solutions, i.e. IIA and IIB.

Double-vielbein 1105.6294, 1109.2035

- The DFT-vielbeins satisfy the **four defining properties:**

$$V_{Ap}V^{A}{}_{q} = \eta_{pq}\,, \qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}\,, \qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}} = 0\,, \qquad V_{Ap}V_{B}{}^{p} + \bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}} = \mathcal{J}_{AB}\,.$$

- They generate **a pair of two-index projectors,**

$$P_{AB} := V_{A}{}^{p}V_{Bp}\,, \qquad \bar{P}_{AB} := \bar{V}_{A}{}^{\bar{p}}\bar{V}_{B\bar{p}}\,,$$

$P_{AB}, \bar{P}_{AB}$ are projection matrices('left and right'),

$$P_{A}{}^{B}P_{B}{}^{C} = P_{A}{}^{C}\,, \qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C} = \bar{P}_{A}{}^{C}\,, \qquad P_{A}{}^{B}\bar{P}_{B}{}^{C} = 0$$

which are related to $\mathcal{H}$ and $\mathcal{J}$,

$$P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}\,, \qquad P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}$$

- Projection will be the characteristic property of DFT geometry.

Semi-covariant derivatives

- We introduce **master 'semi-covariant' derivative**

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A \,.$$

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A \,, \qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A \,.$$

- **compatibility for the whole NS-NS sector**

$$\mathcal{D}_A d = 0 \,, \qquad \mathcal{D}_A V_{Bp} = 0 \,, \qquad \mathcal{D}_A \bar{V}_{B\bar{p}} = 0 \,. \qquad (cf.\ \mathcal{D}_\mu e_\nu{}^a = 0)$$

It follows that

$$\nabla_A d = 0 \,, \qquad \nabla_A P_{BC} = 0 \,, \qquad \nabla_A \bar{P}_{BC} = 0 \,, \qquad (cf. \nabla_\mu g_{\nu\lambda} = 0)$$

- Spin connections

$$\Phi_{Apq} = V^B{}_p \nabla_A V_{Bq} \,, \qquad \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}} \,,$$

- Torsion free conection ,

$$\Gamma^0_{[ABC]} = 0 \,,$$

is determined in terms of basic geometrical variables,

$$\Gamma^0_{CAB} = \; 2\,(P\partial_C P\bar{P})_{[AB]} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC}$$

$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \,,$$

- General torsionful conection ,

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{CAB} \,,$$

As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{BC}\psi_A \,, \quad \bar{\psi}_B\gamma_A\psi_C \,, \quad \bar{\rho}\gamma_{ABC}\rho \,, \quad \bar{\psi}_{\bar{p}}\gamma_{ABC}\psi^{\bar{p}} \,,$$

where we set $\psi_A = \bar{V}_A{}^{\bar{p}}\psi_{\bar{p}}$, $\gamma_A = V_A{}^p\gamma_p$.

Projection-aided covariant derivatives

"semi-covariant derivative" :

combined with the projections , we can generate various covariant quantities:

Examples:

- **For $\mathbf{O}(D,D)$ tensors:**

$$P_C{}^D \bar{P}_A{}^B \nabla_D T_B \,, \qquad \bar{P}_C{}^D P_A{}^B \nabla_D T_B \,,$$

$$P^{AB} \nabla_A T_B \,, \qquad \bar{P}^{AB} \nabla_A T_B \,, \qquad \text{Divergences} \,,$$

$$P^{AB} \bar{P}_C{}^D \nabla_A \nabla_B T_D \,, \qquad \bar{P}^{AB} P_C{}^D \nabla_A \nabla_B T_D \,. \qquad \text{Laplacians}$$

- Rule: need opposite chirality or contraction

Projection-aided covariant derivatives

- **For Spin**$(1, D-1)_L \times$ **Spin**$(D-1, 1)_R$ **tensors:**

$$\mathcal{D}_p T_{\bar{q}}\,, \qquad \mathcal{D}_{\bar{p}} T_q\,,$$

$$\mathcal{D}^p T_p\,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p}}\,,$$

$$\mathcal{D}_p \mathcal{D}^p T_{\bar{q}}\,, \qquad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_q\,,$$

where we set

$$\mathcal{D}_p := V^A{}_p \mathcal{D}_A\,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}} \mathcal{D}_A\,.$$

These are the pull-back of the previous results using the double-vielbeins.

Projection-aided covariant derivatives

- **Dirac operators for fermions,** ρ^{α}, $\psi_{\bar{p}}^{\alpha}$, $\rho'^{\bar{\alpha}}$, $\psi_{p}'^{\bar{\alpha}}$ **:**

$$\gamma^{p}\mathcal{D}_{p}\rho = \gamma^{A}\mathcal{D}_{A}\rho\,, \qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}} = \gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,, \qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}} = \mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho' = \bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,, \qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p} = \bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,, \qquad \mathcal{D}_{p}\psi'^{p} = \mathcal{D}_{A}\psi'^{A}\,,$$

Projection-aided covariant derivatives

- **For Spin$(1, D-1)_L \times$ Spin$(D-1, 1)_R$ bi-fundamental spinors,** $\mathcal{C}^{\alpha}{}_{\bar{\beta}}$ **:**

$$\gamma^A \mathcal{D}_A \mathcal{C} \,, \qquad\qquad \mathcal{D}_A \mathcal{C} \bar{\gamma}^A \,.$$

- Further define

$$\mathcal{D}_+ \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} + \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A \,,$$

$$\mathcal{D}_- \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} - \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A \,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent**

$$(\mathcal{D}^0_+)^2 \mathcal{C} \equiv 0 \,, \qquad\qquad (\mathcal{D}^0_-)^2 \mathcal{C} \equiv 0 \,,$$

- **The field strength of the R-R potential,** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$**, is then defined by**

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \,.$$

Projection-aided covariant derivatives

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Curvatures 1105.6294

- We define, as for a **key quantity** in our formalism, *cf*. [Siegel; Waldram; Hohm, Zwiebach]

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) \ .$$

This is Not covariant tensor, but contracting with projection operators, we can obtain covariant quatities.

  - Rank two-tensor:

$$P_I{}^A \bar{P}_J{}^B S_{AB}\,, \qquad \text{where } S_{AB} := S^C{}_{ACB}$$

  - Scalar curvature: defines the Lagrangian for NS-NS sector

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}$$

  - There is no covariant rank 4 tensor.

$\mathcal{N}=2$ $D=10$ SDFT

- **Lagrangian** (full order of fermions):

$$\mathcal{L}_{\text{Type II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^q$$
$$+ i\tfrac{1}{2}\bar{\rho}\gamma^p\mathcal{D}^\star_p\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^\star_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^q\mathcal{D}^\star_q\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^\star_{\bar{p}}\rho' + i\bar{\psi}'^p\mathcal{D}'^\star_p\rho' + i\tfrac{1}{2}\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\mathcal{D}'^\star_{\bar{q}}\psi'_p\Big]$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_+^{-1}\mathcal{F}^T C_+$.

- $\mathcal{D}_A$ in S_{ACBD}, $\mathcal{D}^\star_A$ and $\mathcal{D}'^\star_A$ are defined by their own torsionful connection ,
- The torsions are determined to satisfy usual 1.5 formalism ,

$$\delta\mathcal{L}_{\text{SDFT}} = \delta\Gamma_{ABC}\times 0\,.$$

- The Lagrangian is **pseudo : self-duality** of the R-R field strength needs to be imposed by hand, just like the 'democratic' type II SUGRA Bergshoeff, *et al.*

$$\Big(1-\gamma^{(D+1)}\Big)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^p\psi_{\bar{q}}\bar{\psi}'_p\bar{\gamma}^{\bar{q}}\right) \equiv 0\,.$$

$\mathcal{N} = 2$ $D = 10$ SDFT

- $\mathcal{N} = 2$ **Local SUSY** (full order of fermions):

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,$$

$$\delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,$$

$$\delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon \mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon \rho = -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,$$

$$\delta_\varepsilon \rho' = -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,$$

$$\delta_\varepsilon \psi_{\bar{p}} = \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,$$

$$\delta_\varepsilon \psi'_p = \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,.$$

$\hat{\mathcal{D}}$ is also defined by its own torsionful connection.

- **The action is invariant up to the self-duality.**

2. Unification of IIA and IIB SUGRAs

Parametrization: Reduction to Generalized Geometry

- We have used the DFT-variables. We may parametrize them in terms of Riemannian variables.
- Assuming that the upper half blocks are non-degenerate, the double-vielbein takes the most general form,

$$V_{Ap} = \frac{1}{\sqrt{2}}\begin{pmatrix} (e^{-1})_p{}^{\mu} \\ (B+e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}}\begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}.$$

Here $e_\mu{}^p$ and $\bar{e}_\nu{}^{\bar{p}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

$$e_\mu{}^p e_\nu{}^q \eta_{pq} = -\bar{e}_\mu{}^{\bar{p}} \bar{e}_\nu{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} = g_{\mu\nu}\,.$$

and $B_{\mu\nu}$ corresponds to the Kalb-Ramond two-form gauge field, with $B_{\mu p} = B_{\mu\nu}(e^{-1})_p{}^{\nu}$, $B_{\mu\bar{p}} = B_{\mu\nu}(\bar{e}^{-1})_{\bar{p}}{}^{\nu}$.

Parametrization: Reduction to Generalized Geometry

- Two parametrizations:

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^\mu \\ (B+e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^\mu \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}$$

versus

$$V_A{}^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta+\tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^p{}_\nu \end{pmatrix}, \qquad \bar{V}_A{}^{\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta+\bar{\tilde{e}})^{\mu p} \\ (\bar{\tilde{e}}^{-1})^p{}_\nu \end{pmatrix}.$$

- there is no longer preferred parametrization $\Longrightarrow$ Non-geometric flux
  c.f. "β-gravity" Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun *et al.*

Parametrization: Reduction to Generalized Geometry

- From now on, we take the former parametrization and impose $\frac{\partial}{\partial\tilde{x}_{\mu}} \equiv 0$.

- This reduces (S)DFT to generalized geometry

  Hitchin; Grana, Minasian, Petrini, Waldram

- For example, the $\mathbf{O}(D,D)$ covariant Dirac operators become

$$\sqrt{2}\gamma^{A}\mathcal{D}_{A}\rho \equiv \gamma^{m}\left(\partial_{m}\rho + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\rho + \tfrac{1}{24}H_{mnp}\gamma^{np}\rho - \partial_{m}\phi\rho\right),$$

$$\sqrt{2}\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}} \equiv \gamma^{m}\left(\partial_{m}\psi_{\bar{p}} + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\psi_{\bar{p}} + \bar{\omega}_{m\bar{p}\bar{q}}\psi^{\bar{q}} + \tfrac{1}{24}H_{mnp}\gamma^{np}\psi_{\bar{p}} + \tfrac{1}{2}H_{m\bar{p}\bar{q}}\psi^{\bar{q}} - \partial_{m}\phi\psi_{\bar{p}}\right)$$

$$\sqrt{2}\bar{V}^{A}{}_{\bar{p}}\mathcal{D}_{A}\rho \equiv \partial_{\bar{p}}\rho + \tfrac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\rho + \tfrac{1}{8}H_{\bar{p}qr}\gamma^{qr}\rho\,,$$

$$\sqrt{2}\mathcal{D}_{A}\psi^{A} \equiv \partial^{\bar{p}}\psi_{\bar{p}} + \tfrac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\psi^{\bar{p}} + \bar{\omega}^{\bar{p}}{}_{\bar{p}\bar{q}}\psi^{\bar{q}} + \tfrac{1}{8}H_{\bar{p}qr}\gamma^{qr}\psi^{\bar{p}} - 2\partial_{\bar{p}}\phi\psi^{\bar{p}}\,.$$

$\omega_{\mu} \pm \frac{1}{2}H_{\mu}$ and $\omega_{\mu} \pm \frac{1}{6}H_{\mu}$ naturally appear as spin connections. Liu, Minasian

Unification of type IIA and IIB SUGRAs

- In general, there can be different Riemaniann solution for each zehnbeins ${e_\mu}^p$ and ${\bar{e}_\mu}^{\bar{p}}$.
- Since the two zehnbeins correspond to the same spacetime metric, they must be related by a Lorentz rotation,

$$(e^{-1}\bar{e})_p{}^{\bar{p}}(e^{-1}\bar{e})_q{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq}\,.$$

- Further, there is a spinorial representation of this Lorentz rotation,

$$S_e\bar{\gamma}^{\bar{p}}S_e^{-1} = \gamma^{(11)}\gamma^p(e^{-1}\bar{e})_p{}^{\bar{p}}\,,$$

such that

$$S_e\bar{\gamma}^{(11)}S_e^{-1} = -\det(e^{-1}\bar{e})\gamma^{(11)}\,.$$

- To relate with the supergravity, we relate two zehnbeins equal to each other

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

This rotation may, or may not, flip the chirality

$$c' \to \det(e^{-1}\bar{e})c'$$

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$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

This rotation may, or may not, flip the chirality

$$c' \to \det(e^{-1}\bar{e})c'$$

Unification of type IIA and IIB SUGRAs

- In general, there can be different Riemaniann solution for each zehnbeins ${e_\mu}^p$ and ${\bar{e}_\mu}^{\bar{p}}$.
- Since the two zehnbeins correspond to the same spacetime metric, they must be related by a Lorentz rotation,

$$(e^{-1}\bar{e})_p{}^{\bar{p}}(e^{-1}\bar{e})_q{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq}\,.$$

- Further, there is a spinorial representation of this Lorentz rotation,

$$S_e\bar{\gamma}^{\bar{p}}S_e^{-1} = \gamma^{(11)}\gamma^p(e^{-1}\bar{e})_p{}^{\bar{p}}\,,$$

  such that

$$S_e\bar{\gamma}^{(11)}S_e^{-1} = -\det(e^{-1}\bar{e})\gamma^{(11)}\,.$$

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Unification of type IIA and IIB SUGRAs

- The $\mathcal{N}=2$ $D=10$ SDFT Riemannian solutions are then classified into two groups,

$$cc'\det(e^{-1}\bar{e})=+1 \quad : \quad \text{type IIA}\,,$$
$$cc'\det(e^{-1}\bar{e})=-1 \quad : \quad \text{type IIB}\,.$$

- We may safely put $\mathbf{c}\equiv\mathbf{c}'\equiv+1$ without loss of generality. However, the theory contains two 'types' of Riemannian solutions, as classified above.
- In conclusion, the single unique $\mathcal{N}=2$ $D=10$ SDFT unifies type IIA and IIB SUGRAs.

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Diagonal gauge fixing and Reduction to SUGRA

- Setting the diagonal gauge,

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

with $\eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}}$, $\bar{\gamma}^{\bar{p}} = \gamma^{(D+1)}\gamma^p$, $\bar{\gamma}^{(D+1)} = -\gamma^{(D+1)}$, breaks the local Lorentz symmetry,

$$\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R \quad \Longrightarrow \quad \mathbf{Spin}(1,D{-}1)_D\,.$$

- And it reduces SDFT to SUGRA:

$\mathcal{N}=2$ $D=10$ **SDFT** $\Longrightarrow$ $10D$ **Type II democratic SUGRA** Bergshoeff, *et al.*; Coimbra, Strickland-Constable, Waldram

$\mathcal{N}=1$ $D=10$ **SDFT** $\Longrightarrow$ $10D$ **minimal SUGRA** Chamseddine; Bergshoeff *et al.*

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$\mathcal{N} = 2$ $D = 10$ **SDFT** $\quad\Longrightarrow\quad$ $10D$ **Type II democratic SUGRA** Bergshoeff, *et al.*; Coimbra, Strickland-Constable, Waldram

$\mathcal{N} = 1$ $D = 10$ **SDFT** $\quad\Longrightarrow\quad$ $10D$ **minimal SUGRA** Chamseddine; Bergshoeff *et al.*

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \sum_p' \tfrac{1}{p!}\, \mathcal{C}_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p}$$

and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \sum_p' \tfrac{1}{(p+1)!}\, \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}}$$

where $\sum_p'$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1 a_2 \cdots a_p} = p\left(D_{[a_1}\mathcal{C}_{a_2 \cdots a_p]} - \partial_{[a_1}\phi\, \mathcal{C}_{a_2 \cdots a_p]}\right) + \tfrac{p!}{3!(p-3)!}\, H_{[a_1 a_2 a_3}\mathcal{C}_{a_4 \cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to an exterior derivative and its dual,

$$\begin{array}{ccc} \mathcal{D}^0_+ & \Longrightarrow & \mathrm{d} + (H - \mathrm{d}\phi)\wedge \\ \mathcal{D}^0_- & \Longrightarrow & *\left[\,\mathrm{d} + (H - \mathrm{d}\phi)\wedge\,\right]* \end{array}$$

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Diagonal gauge fixing and Reduction to SUGRA

- **In this way, ordinary SUGRA $\equiv$ gauge-fixed SDFT,**

$$\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R \quad\Longrightarrow\quad \mathbf{Spin}(1,D{-}1)_D\,.$$

Modified $\mathbf{O}(D,D)$ IIA $\leftrightarrow$ IIB

- In order to preserve the diagonal gauge, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, the $\mathbf{O}(D,D)$ transformation rule is modified.
- A compensating local Lorentz transformation, $\bar{L}_{\bar{q}}{}^{\bar{p}}$, $S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}} \in \mathbf{Pin}(D{-}1,1)_R$, must be accompanied:

$$\bar{V}_A{}^{\bar{p}} \quad \longrightarrow \quad M_A{}^B \bar{V}_B{}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}} \,, \qquad \bar{\gamma}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1} \bar{\gamma}^{\bar{p}} S_{\bar{L}} \,,$$

where

$$\bar{L} = \bar{e}^{-1} \left[\mathbf{a}^t - (g+B)\mathbf{b}^t\right] \left[\mathbf{a}^t + (g-B)\mathbf{b}^t\right]^{-1} \bar{e} \,,$$

in the parametrization of the generic $\mathbf{O}(D,D)$ group element,

$$M_A{}^B = \begin{pmatrix} \mathbf{a}^\mu{}_\nu & \mathbf{b}^{\mu\sigma} \\ \mathbf{c}_{\rho\nu} & \mathbf{d}_\rho{}^\sigma \end{pmatrix} .$$

Modified $\mathbf{O}(D,D)$ Transformation Rule After The Diagonal Gauge Fixing

$$
\begin{array}{ccc}
d & \longrightarrow & d \\
V_A{}^p & \longrightarrow & M_A{}^B\, V_B{}^p \\
\bar{V}_A{}^{\bar{p}} & \longrightarrow & M_A{}^B\, \bar{V}_B{}^{\bar{q}}\, \bar{L}_{\bar{q}}{}^{\bar{p}} \\
\mathcal{C}^\alpha{}_{\bar{\alpha}}\,,\quad \mathcal{F}^\alpha{}_{\bar{\alpha}} & \longrightarrow & \mathcal{C}^{\bar{\alpha}}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\bar{\alpha}}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}} \\
\rho^\alpha & \longrightarrow & \rho^\alpha \\
\rho'^{\bar{\alpha}} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\,\rho'^{\bar{\beta}} \\
\psi^\alpha_{\bar{p}} & \longrightarrow & (\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}}\,\psi^\alpha_{\bar{q}} \\
\psi'^{\bar{\alpha}}_p & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\,\psi'^{\bar{\beta}}_p
\end{array}
$$

- **All the barred indices are now to be rotated.** Consistent with Hassan

Modified $\mathbf{O}(D,D)$: IIA $\Leftrightarrow$ IIB

- **If and only if $\det(\bar{L}) = -1$, the modified $\mathbf{O}(D,D)$ rotation flips the chirality of the theory, since**

$$\bar{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L})\, S_{\bar{L}} \bar{\gamma}^{(D+1)} \, .$$

- Thus, the mechanism above naturally realizes the exchange of type IIA and IIB supergravities under $\mathbf{O}(D,D)$ T-duality.

Summary

- $\mathcal{N} = 2$ $D = 10$ **SDFT has been constructed to the full order in fermions. This unifies type II supergravities in very simple form in democratic fashion.**
- Parametrization independent variables in SDFT allows to have two types of solution from a single theory.
- Fixing ${e_\mu}^p = {\bar{e}_\mu}^{\bar{p}}$ reduces SDFT to type IIA or IIB SUGRA
- The gauge fixing modifies the $\mathbf{O}(10, 10)$ transformation rule and generate the exchange of IIA and IIB theory.

Thank you.

Appendix: Torsions

$$\Gamma_{ABC} = \Gamma^0_{ABC} + i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho - 2i\bar{\rho}\gamma_{BC}\psi_A - i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} + 4i\bar{\psi}_B\gamma_A\psi_C$$

$$+i\tfrac{1}{3}\bar{\rho}'\bar{\gamma}_{ABC}\rho' - 2i\bar{\rho}'\bar{\gamma}_{BC}\psi'_A - i\tfrac{1}{3}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_p + 4i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,.$$

and $\mathcal{D}^\star_A$ and $\mathcal{D}'^\star_A$ are defined by their own connection,

$$\Gamma^\star_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 2i\bar{\psi}_B\gamma_A\psi_C + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A\,,$$

$$\Gamma'^\star_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{4}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + i\tfrac{5}{24}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_p - 2i\bar{\psi}'_B\bar{\gamma}_A\psi'_C + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A\,.$$

Appendix: $10D\ \mathcal{N}=1$ SUGRA 1112.0069

- From $D=11$ SUGRA by Cremmer, Julia & Scherk with ansatz,

$$E_M{}^A = \begin{pmatrix} e^{-\frac{1}{3}\phi} e_\mu{}^a & 0 \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix},$$

$$A_{\mu\nu\lambda} = 0\,, \qquad A_{\mu\nu 11} = \tfrac{1}{2} B_{\mu\nu}\,.$$

$$\Psi_a = \tfrac{1}{6} 2^{\frac{1}{4}} e^{\frac{1}{6}\phi} \left(5\psi_a - \gamma_{ab}\psi^b - \gamma_a \rho\right)\,,$$
$$\Psi_z = -\tfrac{1}{3} 2^{\frac{1}{4}} e^{\frac{1}{6}\phi} \left(\rho + \gamma^a \psi_a\right)\,,$$

$$\gamma^{(10)}\psi_a = \psi_a\,, \qquad \gamma^{(10)}\rho = -\rho\,. \qquad \gamma^{(10)}\varepsilon = \varepsilon\,,$$

we can derive the $\mathcal{N}=1$ $10D$ SUGRA. *cf.* Chamseddine, Bergshoeff *et al.*
Consistent with Coimbra, Strickland-Constable & Waldram.

Appendix: $10D$ $\mathcal{N}=1$ SUGRA 1112.0069

- Action with full fermion order

$$\mathcal{L}_{10D} = e \times e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}$$

$$+i2\sqrt{2}\bar{\rho}\gamma^m[\partial_m\rho + \tfrac{1}{4}(\omega + \tfrac{1}{6}H)_{mnp}\gamma^{np}\rho] - i4\sqrt{2}\bar{\psi}^p[\partial_p\rho + \tfrac{1}{4}(\omega + \tfrac{1}{2}H)_{pqr}\gamma^{qr}\rho]$$

$$-i2\sqrt{2}\bar{\psi}^p\gamma^m[\partial_m\psi_p + \tfrac{1}{4}(\omega + \tfrac{1}{6}H)\gamma^{np}\psi_p + \omega_{mpq}\psi^q - \tfrac{1}{2}H_{mpq}\psi^q]$$

$$+\tfrac{1}{24}(\bar{\psi}^q\gamma_{mnp}\psi_q)(\bar{\psi}^r\gamma^{mnp}\psi_r) - \tfrac{1}{48}(\bar{\psi}^q\gamma_{mnp}\psi_q)(\bar{\rho}\gamma^{mnp}\rho)\Big].$$

- Supersymmetry

$$\delta_\varepsilon\phi = i\tfrac{1}{2}\bar{\varepsilon}(\rho + \gamma^a\psi_a)\,, \qquad \delta_\varepsilon e^a_\mu = i\bar{\varepsilon}\gamma^a\psi_\mu\,, \qquad \delta_\varepsilon B_{\mu\nu} = -2i\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}\,,$$

$$\delta_\varepsilon\rho = -\tfrac{1}{\sqrt{2}}\gamma^a[\partial_a\varepsilon + \tfrac{1}{4}(\omega + \tfrac{1}{6}H)_{abc}\gamma^{bc}\varepsilon - \partial_a\phi\varepsilon]$$

$$+i\tfrac{1}{48}(\bar{\psi}^d\gamma_{abc}\psi_d)\gamma^{abc}\varepsilon + i\tfrac{1}{192}(\bar{\rho}\gamma_{abc}\rho)\gamma^{abc}\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\gamma_{[a}\psi_{b]})\gamma^{ab}\rho\,,$$

$$\delta_\varepsilon\psi_a = \tfrac{1}{\sqrt{2}}[\partial_a\varepsilon + \tfrac{1}{4}(\omega + \tfrac{1}{2}H)_{abc}\gamma^{bc}\varepsilon]$$

$$-i\tfrac{1}{2}(\bar{\rho}\varepsilon)\psi_a - i\tfrac{1}{4}(\bar{\rho}\psi_a)\varepsilon + i\tfrac{1}{8}(\bar{\rho}\gamma_{bc}\psi_a)\gamma^{bc}\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\gamma_{[b}\psi_{c]})\gamma^{bc}\psi_a\,.$$