Explicit Resolution of Linear Systems in Dimension Three and Applications

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Birational Geometry

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   Try to classify varieties up to birational equivalence.
2. Minimal Model Program:
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Goal: study birational classification theory of threefolds explicitly by using MMP.

X: complex projective varieties with at worst $\mathbb{Q}$-factorial terminal singularities. $n = \dim(X)$, mostly, $n = 3$.

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Three building blocks:

- varieties with $\kappa = -\infty$;
- varieties with $\kappa = 0$;
- varieties with $\kappa = \dim$, which are varieties of general type

There exist $d(X)$ and $r(X)$ such that
φ_m is stabilized (birationally) for $m \geq r(X)$ and divisible by $d(X)$.

Question. Does there exist $d(n)$ and $r(n)$ such that
φ_m is stabilized (birationally) for $m \geq r(n)$ and divisible by $d(n)$
for any variety X of dimension n?

Iitaka fibration

[Hacon-McKernan '06, Takayama, Tsuji]
$\kappa(X) = \dim X$, then $r(n)$ exist (and $d(n) = 1$).
[Fujino-Mori '00]
$\kappa(X) = 1$, then $r(n)$ and $d(n)$ exist.
[Viehweg-Zhang '07]
$\kappa(X) = 2$, then $r(n)$ and $d(n)$ exist.

Do we have explicit bound for threefolds?

Iitaka fibration

[Kawamata, Morrison '86]
$\dim X = 3$, $\kappa(X) = 0$
Then $r(X) \leq 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
$= lcm\{m | \phi(m) \leq 20\}$.

[Ringler 07', D.Q. Zhang-Viehweg]
$\dim X = 3$, $\kappa(X) = 2$
Then $r(x) \leq 48$ and $d(X) = 12$.

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Question 1. For threefolds of general type, what is $r(3)$?
It is well-known that $r(3) \geq 27$.

Iitaka fibration

<u>Threefolds with $\kappa = 1$</u> [Fujino-Mori '00]
$\dim X = 3$, $\kappa(X) = 1$
Then exists computable $r(X)$ and $d(X)$.
If general fiber is abelian surface or bielliptic surafce, then $r(X) \leq n$ such that

$$-2Nn + \sum \lfloor n \frac{m_i - b_i}{m_i} \rfloor > 0$$

for all $b_i < \min\{m_i, N+1\}$ with $-2N + \sum \frac{m_i - b_i}{m_i} > 0$ and $N = 2520$.
If general fiber is K3 surface or Enriques, then one needs to compute the bound with $N = lcm\{m | \phi(m) \leq 22\}$.

Threefolds with $\kappa = 1$

$$\chi(m + m_0\mu K_X) - \chi(mK_X) = -2m_0\mu\chi(\mathcal{O}_X) + m_0\mu\frac{1}{12}\sum\frac{r_i^2-1}{r_i}.$$

On the other hand, if $q(F) = 0$ then

$$\chi(m + m_0\mu K_X) = \chi(f_*(m + m_0\mu)K_X) + \chi(R^2 f_*(m + m_0\mu)K_X)$$

$$= \chi(f_*(mK_X) + \chi(R^2 f_*(mK_X) + d\mu\chi(\mathcal{O}_F)$$

$$= \chi(mK_X) + d\mu\chi(\mathcal{O}_F).$$

Hence we have

$$\delta := \frac{d}{m_0} = \frac{1}{12}\sum\frac{r_i^2-1}{r_i} - 2\chi(\mathcal{O}_X).$$

Threefolds with $\kappa = 1$

F: Enriques surface. $\chi(\mathcal{O}_F) = 1$, $\chi(\mathcal{O}_X) = 1$.

$$\delta \geq \{c | c = \frac{1}{12} \sum \frac{r_i^2 - 1}{r_i} - 2\}.$$

F: K3 surface. $\chi(\mathcal{O}_F) = 2$, $\chi(\mathcal{O}_X) = 0, 1, 2$.

$$\delta \geq \{c | c = \frac{1}{24} \sum \frac{r_i^2 - 1}{r_i} - 2\}.$$

$\delta \geq 1/120960$.

Threefolds with $\kappa = 1$

[Chen-Hacon '07] If general fiber is K3 surface or Enriques surafce, then $r(X) = 362880$ and $d(X) = 120960$.

Question 2. Is there threefolds with $\kappa = 1$ and $r(X) = 362880$? The singularities has index $2, 3, 4, 7, 9, 16$

Question 3. What is the distribution of birational invariants? Are there any non-trivial relation among them?

Canonical Volume

Asymptotic Riemann-Roch formula:

$$P_m(X) = \frac{c}{n!} m^n + l.o.t$$

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$Vol(X) := c$.

If X is minimal, i.e. K_X is nef, then $Vol(X) = K_X^n$.

Canonical Volume

- $\dim X = 1,\ Vol(X) = \deg(K_X) = 2p_g(X) - 2.$

Canonical Volume

- $\dim X = 1$, $Vol(X) = \deg(K_X) = 2p_g(X) - 2$.
- $\dim X = 2$. [Bogomolov-Miyaoka-Yau Inequality]
  $3c_2 - c_1^2 \geq 0$.

  $$Vol(X) \leq 9\chi(\mathcal{O}_X) \leq 9p_g + 9.$$

  [Noether Inequality]
  $Vol(X) \geq 2p_g - 4$.

Difficulties in higher dimensions

- Minimal models exists but contain singularities in dimension 3 or higher.
- $Vol(X) > 0$ is an integer if $\dim X \leq 2$.
  $Vol(X) > 0$ is a rational number if $\dim X \geq 3$.
- $\chi(\mathcal{O}_X)$ could be positive or negative in higher dimensions.

Canonical Volume

X is a Gorenstein minimal 3-fold of general type. Miyaoka-Yau Inequality yields:

$$Vol(X) \leq 72\chi(\omega_X) \tag{1}$$

[Meng Chen-Hacon] Suppose furthermore that $p_g(X) > 0$, then

$$Vol(X) \leq 144 p_g(X). \tag{2}$$

If X is non-Gorenstein, by Reid,

$$Vol(X) \leq 72\chi(\omega_X) + 3\sum_{i=1}^{s}(r_i - \frac{1}{r_i}), \tag{3}$$

where i runs through Reid's basket $B(X) = \{\frac{1}{r_i}(1, -1, b_i) | i = 1, \cdots, s\}$.

Noether Inequality

What is the three dimensional analogue of Noether Inequality?

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Kobayashi constructed examples of threefolds with $p_g(X) = 3k + 4$ and $Vol(X) = 4k + 2$ for $k \geq 1$.
Hence the inequality $Vol(X) \geq 2p_g(X) - 6$ fails in dimension 3.

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Hence the inequality $Vol(X) \geq 2p_g(X) - 6$ fails in dimension 3.
One can only expect that $Vol(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}$.

Theorem (-, Meng Chen)

The inequality

$$Vol(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}$$

holds for all projective Gorenstein minimal 3-folds X of general type.

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holds for all projective Gorenstein minimal 3-folds X of general type.

We can almost prove the same inequality for minimal 3-folds. The remaining difficulty is when $|K_X|$ induced a fibration to $\mathbb{P}^1$ fibered by surfaces of $(1, 2)$-type.

[Corti] Any two birational MFS of dimension three can be connected by a finite sequence of Sarkisov links.
It is measured by Sarkisov degree (c, e, μ), where c is the canonical threshold, c is the number of crepant divisors, $\mu > 0 \in \frac{1}{2}\mathbb{Z} + \frac{1}{3}\mathbb{Z}$.

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It is measured by Sarkisov degree (c, e, μ), where c is the canonical threshold, c is the number of crepant divisors, $\mu > 0 \in \frac{1}{2}\mathbb{Z} + \frac{1}{3}\mathbb{Z}$.
Question 4. Can we have an effective Sarkisov program in

dimension three?

Questions

It is natural to consider linear system (or pairs) explicitly in various settings:

- $(X, \mathcal{H})$ in Sarkisov program. Study canonical threshold of $(X, \mathcal{H})$.
- $\kappa(X) = 3$ with $Vol = 1/1680$, $\mathcal{H}$ the system obtained from φ_{14}. We need to study the pair $(X, \mathcal{H})$. Where the minimal model of $\mathcal{H}$ is a surface of general type of type $(1, 2)$.
- $\kappa(X) = 1$ with F general fiber of Iitaka fibration. We consider (X, F).

Terminal Singularities in Dimension Three

$P \in X$ is terminal, then exists a canonical cover

$$(\tilde{P} \in \tilde{X}) \to (P \in X)$$

so that

$$(P \in X) \cong (\tilde{P} \in \tilde{X})/\mu_r$$

for some cyclic group μ_r of order r.

$r = 1$,

$P \in X$ is an isolated cDV point and Gorenstein.

1. cA: $(xy + z^{n+1} + ug(x, y, z, u) = 0) \in \mathbb{C}^4$.
2. cD: $(x^2 + y^2 z + z^{n-1} + ug(x, y, z, u) = 0) \in \mathbb{C}^4$.
3. cE_6: $(x^2 + y^3 + z^4 + ug(x, y, z, u) = 0) \in \mathbb{C}^4$.
4. cE_7: $(x^2 + y^3 + yz^3 + ug(x, y, z, u) = 0) \in \mathbb{C}^4$.
5. cE_8: $(x^2 + y^3 + z^5 + ug(x, y, z, u) = 0) \in \mathbb{C}^4$.

Terminal Singularities in Dimension Three

$r > 1$,

$P \in X$ is a quotient of a smooth or an isolated cDV point.

1. $\mathbb{C}^3/\frac{1}{r}(a, r-a, 1) \cong \mathbb{C}^3/\frac{1}{r}(1, -1, b)$, $(r, a) = (r, b) = 1$.
2. cA/r: $(xy + f(z, u) = 0) \in \mathbb{C}^4/\frac{1}{r}(a, r-a, 1, r)$.
3. $cAx/2$: $(x^2 + y^2 + f(z, u) = 0) \in \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 0)$.
4. $cAx/4$: $(x^2 + y^2 + f(z, u) = 0) \in \mathbb{C}^4/\frac{1}{4}(1, 3, 1, 2)$.
5. $cD/2$: $P \in X$ is given by $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 1, 0, 1)$ with φ being one the following:
$$\begin{cases} x^2 + yzu + y^{2a} + u^{2b} + z^c, & a \geq b \geq 2, c \geq 3 \\ x^2 + y^2 z + \lambda y u^{2l+1} + f(z, u^2). \end{cases}$$
6. $cD/3$: $P \in X$ is given as $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{3}(0, 2, 1, 1)$ with φ being one of the following:
$$\begin{cases} x^2 + y^3 + zu(z+u); \\ x^2 + y^3 + zu^2 + yg(z, u) + h(z, u); & g \in \mathfrak{m}^4, h \in \mathfrak{m}^6; \\ x^2 + y^3 + z^3 + yg(z, u) + h(z, u); & g \in \mathfrak{m}^4, h \in \mathfrak{m}^6. \end{cases}$$
7. $cE/2$: $(x^2 + y^3 + yg(z, u) + h(z, u) = 0) \in \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1)$.

Theorem

Let X be an algebraic 3-fold with at worst terminal singularities. For any terminal singularity $P \in X$, there exists a sequence of birational morphisms:

$$\tau_P : Y = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X,$$

such that Y is smooth on $\tau_P^{-1}(P)$ and, for all i, the morphism $\pi_i : X_{i+1} \to X_i$ is a divisorial contraction to a singular point $P_i \in X_i$ of index $r_i \geq 1$ with discrepancy $1/r_i$.

Explicit Elimination of Indeterminacies

Suppose that $|M|$ is a moving linear system (i.e. without fixed part) on the given projective terminal 3-fold X with $\mathrm{Bs}|M| \neq \emptyset$.

(i) If there is a point $P \in \mathrm{Bs}|M| \cap \mathrm{Sing}(X)$, we take a partial resolution $Z_1 \to X \ni P$ and consider the linear system $|M_1|$, where M_1 is the proper transform of M on Z_1.

(ii) Inductively, we will end up with a chain of partial resolutions $Z_n \to \ldots \to Z_1 \to X$ so that $|M_n|$ is free out of singularities of Z_n.

(iii) If $|M_n|$ is base point free on Z_n, then we stop.

(iv) If $|M_n|$ has base points, then $\mathrm{Bs}|M_n|$ consists of smooth points of Z_n. Usual resolution of indeterminancies over $\mathrm{Bs}|M_n|$: $Z_k \to \ldots \to Z_n$, which is a sequence of blow-ups.

(v) Thus we end up with a 3-fold Z_k so that $|M_k|$ is base point free.

$$\mu\colon Z_k \xrightarrow{\tau_k} \ldots \xrightarrow{\tau_{n+1}} Z_n \xrightarrow{\tau_n} \ldots \xrightarrow{\tau_1} X \tag{5}$$

a explicit elimination of indeterminancies of $|M|$.

Corollary

Let X and X' be birational threefolds with at worst terminal singularities and X'. There exists Z and birational morphisms $p : Z \to X$, $q : Z \to X'$ such that

$$q : Z \to X'$$

is a explicit elimination of indeterminancies of $|\mathcal{H}_X|$, where $\mathcal{H}_X$ is very ample on X.

Corollary

Let $|M|$ *be a moving linear system on a terminal 3-fold* X *and* $D \in |M|$ *be a general member. Let* $\mu : Z_k \to X$ *be the explicit elimination of indeterminancies. Then* $2D_{Z_k/X} \geq K_{Z_k/X}$.

Sketch of the Proof of Noether Inequlity

Let X be a projective Gorenstein minimal 3-fold of general type.
Set $d := \dim \varphi_1(X)$.
The following inequalities are already known:

- If $d \neq 2$, then

$$K_X^3 \geq \min\{2p_g(X) - 6,\ \frac{7}{5}p_g(X) - 2\}$$

  by [Meng Chen] and [Catanese–Chen–Zhang].
- If $d = 2$ and X is canonically fibred by curves C of genus $g(C) \geq 3$, then $K_X^3 \geq 2p_g(X) - 4$ by [Meng Chen].
- It remains to consider that X is fibred by curve C of genus 2.

Write $|K_X| = |M| + F$, where $|M|$ is the moving part and F is the fixed part. Let

$$\mu : X' = Z_k \to \ldots \to Z_1 \to X$$

be the Gorenstein resolution of indeterminacies. Let D be a general member of $|M|$ and $S := D_{X'}$.

$$\mu^* K_X = \mu^* M + \mu^* F = S + D_{X'/X} + \mu^* F.$$

Set $E' := D_{X'/X} + \mu^* F$.
On the surface S, set $L := \mu^*(K_X)|_S$. We also have $S|_S \equiv aC$ where $a \geq p_g(X) - 2$ and C is a general fiber.

$$(\mu^* K_X^2 \cdot S) \geq (\mu^* K_X \cdot_S S) \geq a(L \cdot C) \geq (L \cdot C)(p_g(X) - 2).$$

If $(L \cdot C) \geq 2$, then we have already

$$K_X^3 \geq (\mu^* K_X^2 \cdot S) \geq 2p_g(X) - 4.$$

It remains to consider the case $(L \cdot C) = 1$.
Denote $E'|_S := E'_V + E'_H$, where E'_V is the vertical part and E'_H is the horizontal part. Since $(E'_H \cdot C) = (L \cdot C) = 1$, E'_H is an irreducible curve.
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By Key Lemma, $E'_H = E_H$ and $2E'_V \geq E_V$.

Let $G := E_H = E'_H$.

$$\begin{array}{rl} & (2\mu^* K_X|_S + E'_V) \cdot G \\ = & (\mu^* K_X|_S + S|_S + 2E'_V + E'_H) \cdot G \\ \geq & (\mu^* K_X|_S + S|_S + E_V + E_H) \cdot G \\ = & (K_S \cdot G) \geq -2 - G^2 \end{array}$$

We also have

$$\begin{aligned} (\mu^* K_X|_S - E'_V) \cdot G \quad &= (S|_S \cdot G) + (E'_H \cdot G) \\ &= a(C \cdot G) + G^2 \\ &\geq p_g(X) - 2 + G^2. \end{aligned}$$

Combining these, we get $3(\mu^*(K_X)|_S \cdot G) \geq p_g(X) - 4$.

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Combining these, we get $3(\mu^*(K_X)|_S \cdot G) \geq p_g(X) - 4$.

$$\begin{aligned} K_X^3 \quad & = \mu^*(K_X)^3 \geq (\mu^*(K_X)^2 \cdot S) \\ & = (\mu^*(K_X)|_S \cdot S|_S) + (\mu^*(K_X)|_S \cdot E'|_S) \\ & \geq (p_g(X) - 2) + \frac{1}{3}(p_g(X) - 4) = \frac{2}{3}(2p_g(X) - 5). \end{aligned}$$

Canonical threshold and Sarkisov degree

Suppose that $P \in D \subset X$ is a singular point, we can define weighted multiplicity as following: Let $\pi : Y \to X$ be a divisorial contraction to P. We can write $K_Y = \pi^* K_X + \frac{a}{r} E$ and $D_Y = \pi^* D - \frac{b}{r} E$.

$$wm_P(X, D) := \max\{\frac{b}{a} | \pi : Y \to X \ni P\}.$$

If P is of type $cA, CA/r, cAx/2, cAx/4, cE/r$, then $1/wm_p(X, D)$ is the canonical threshold (over P), and all divisorial contractions are weighted blowups.
What P is of type cD/r?

Thank you for your attention!