Interstellar

Makers of the movie "Interstellar" boasted how much they tried to be faithful to physics of gravity. One interesting part of the plot is centered on the Miller planet entirely covered by ocean, orbiting just outside of a giant black hole called "Gargantua." Those who visited its surface, in a small landing ship, finds upon their return to the mother ship that 23 earth-years has passed during their few hour of absence due to the time-dilating effect of gravity. Some journalists criticized the movie for not making much sense, referring to "extremely strong gravitational force, necessary for the extreme time dilation." Give your own scientific opinions, physics-wise, on this part of movie that apparently caught attention of a lot of movie-goers. To do this, consider the following:

1) Motion of an object of mass m around a non-rotating black hole of mass M (Gargantua is supposed to be a fast-rotating black hole but, here, for simplicity we will assume otherwise) follows Lagrange equation of motion with the action

$$-mc^2\int ds\sqrt{\left(1-\frac{2GM}{c^2r}\right)\dot{t}^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}^2/c^2-r^2\left(\dot{\theta}^2+(\sin\theta)^2\dot{\phi}^2\right)/c^2}$$

c is the speed of light and G is the Newton's gravitational constant. s is an arbitrary parametrization of the motion, $t(s), r(s), \ldots$, with respect to which the "time" derivative as in $\dot{t}, \dot{r}, \ldots$ is taken. You may wish to confine your attention to $r > r_H \equiv 2GM/c^2$, below which things become really weird. The name "Gargantua" refers to the movie fact that the black hole is so heavy that r_H is extremely large, about a billion kilometers.

2) Equivalently, one may choose to fix the "time" parameter s to be of special type, upon which one is allowed to solve for Lagrange equation of

$$-\frac{1}{2}\int ds\left[\left(1-\frac{r_H}{r}\right)\dot{t}^2-\left(1-\frac{r_H}{r}\right)^{-1}\dot{r}^2/c^2-r^2\left(\dot{\theta}^2+(\sin\theta)^2\dot{\phi}^2\right)/c^2\right]$$

instead. This should be a little easier to handle.

Note that there are two sets of symmetries: the time translation $t \to t + const$ and the spatial rotation. These two implies their conjugate momenta are conserved, which are, respectively the relativistic versions of the energy and the angular momentum. These two symmetries reduces the dynamics to a radial one, much as in the Newtonian Kepler problem. Note that the Hamiltonian of this second Lagrangian is yet another conserved quantity, which you set equal to any number. Changing this number is equivalent to multiplying s by a number, and does not affect shape of orbits at all. This way, you end up with a first order differential equation, again as in the Newtonian

Kepler problem. With these, classify possible orbits of the planet, which should be similar to Kepler ones when $r \gg r_H$ but qualitatively different when $1 < r/r_H < 10$ or so. What is the smallest possible circular orbit, for example?

3) The time lapse felt by different observers are different, depending on position and on velocity. When a far away observer feels time lapse of δt, the person moving near black hole feels, instead, a smaller time lapse of

$$\delta\tau = \delta t \times \sqrt{\left(1 - \frac{r_H}{r}\right) - \left(1 - \frac{r_H}{r}\right)^{-1} \dot{r}^2/(c^2\dot{t}^2) - r^2\left(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\right)/(c^2\dot{t}^2)}$$

Now, you are in position to address physics of the Miller planet and the plot around it. What is the maximum possible time-dilation can you imagine for a reasonable, in the context of the movie, orbit of the planet around Gargantua?

4) More questions: Was the criticism by journalists mentioned above valid? Why did the producers choose a very large black hole, do you think? What other serious flaws can you find with this part of movie, physics-wise?