What is the Temperature of a Pure State?

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based on work with A. Liam Fitzpatrick, Daliang Li, Matt Walters, & Junpu Wang

Motivation

Understand the **universal features** of AdS quantum gravity directly from CFT.

- AdS Locality, Fock space, and Interactions
- (Large) Black Hole Thermodynamics?
- What CFT Data Controls Thermodynamics?
- How hard is it to UV complete quantum gravity?

Outline

I. Ensemble vs Microstate Thermodynamics

II. Probing thermodynamics using CFT correlators

III. Thermality from Virasoro Conformal Blocks

IV. New Constraints on Holographic CFT Data?

Ensemble vs Microstate Thermodynamic

Ensemble Thermodynamics

Most often we discuss thermodynamics in the canonical ensemble:

$$Z[T] = \sum_E e^{S(E) - \frac{E}{T}}$$

In thermo limit, we obtain the temperature:

$$\frac{dS}{d\langle E\rangle} \approx \frac{1}{T}$$

In an ensemble averaged view of the theory, we count states as function of energy and derive e.g. the Cardy formula.

Eigenstate Thermalization and Microstates

Microstate perspective — ask if

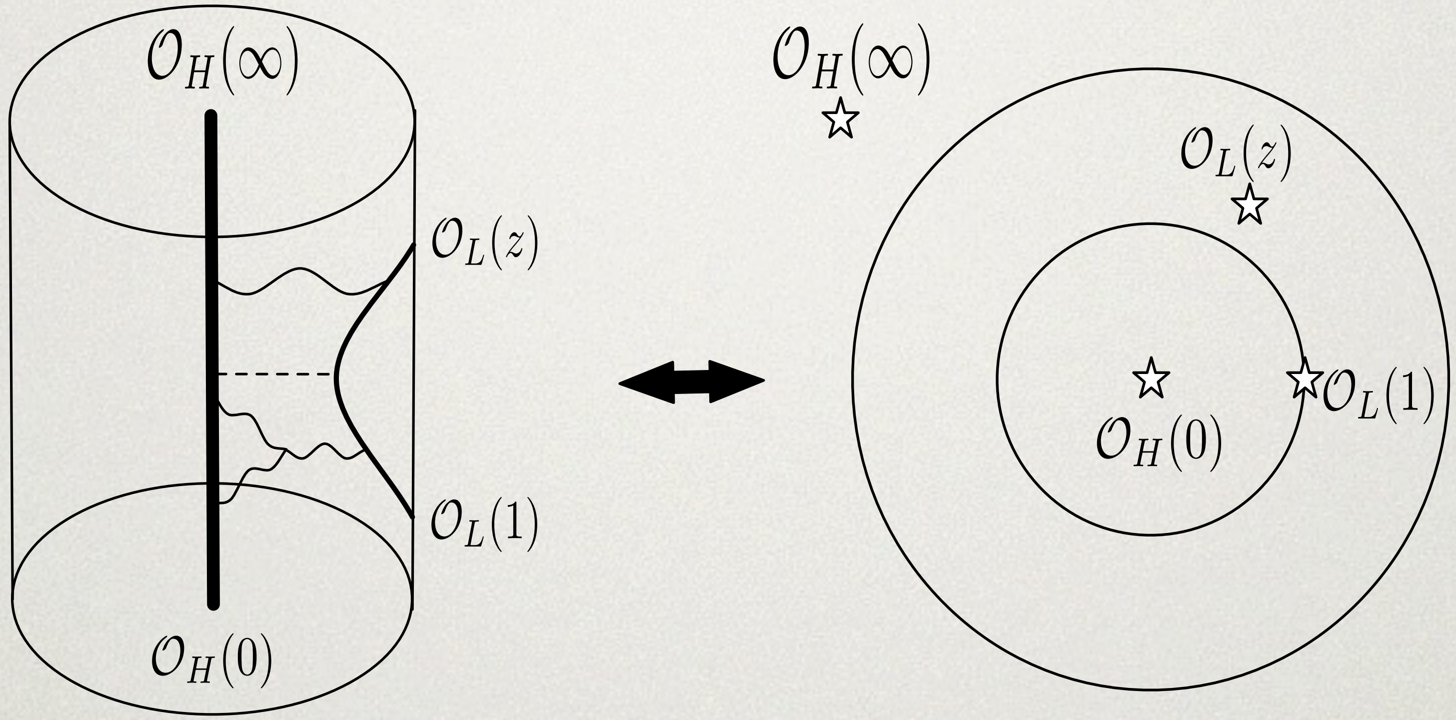
$$\langle\psi_E|\mathcal{O}_1\cdots\mathcal{O}_k|\psi_E\rangle$$

for a single pure microstate looks like
a correlator at a temperature set by the state:

$$\langle\mathcal{O}_1\cdots\mathcal{O}_k\rangle_{T(E)}$$

Related to `Eigenstate Thermalization Hypothesis'.
Thermo limit for us will be large central charge.

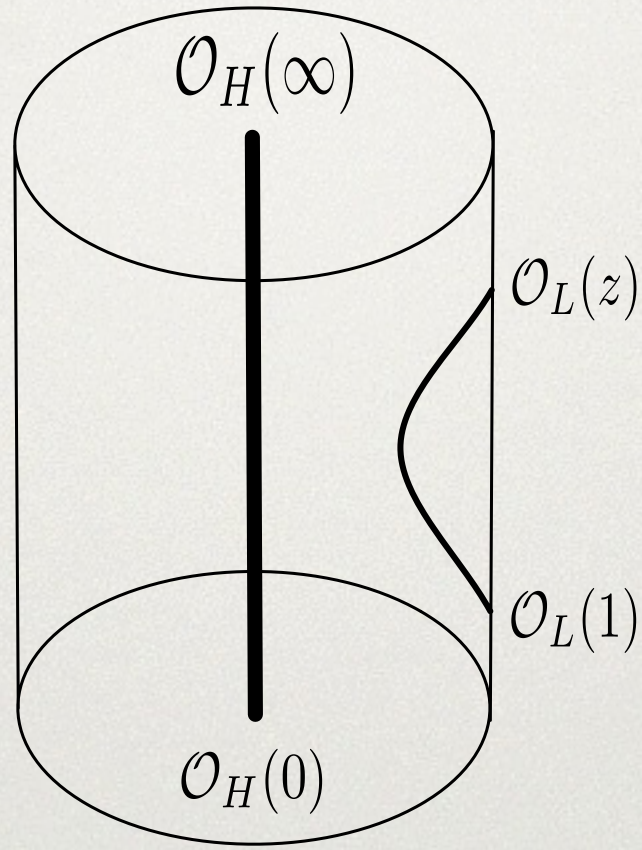
What Observable (in AdS/CFT)?



$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle$$

Some basic facts we can we learn from this…

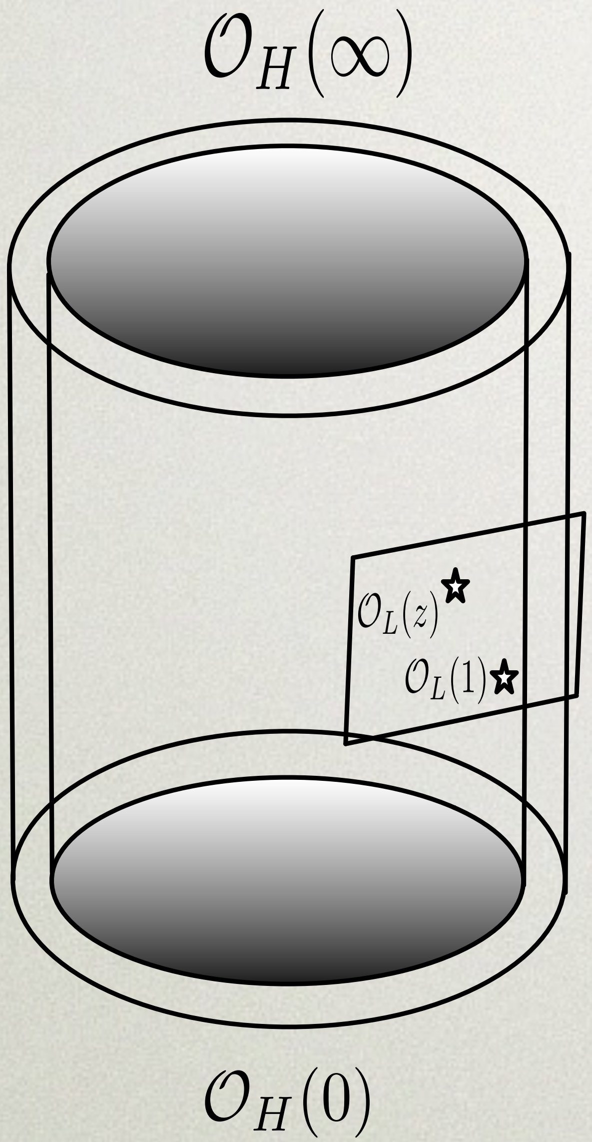
Can Measure Geodesic Lengths in AdS



$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle \approx e^{-\Delta_L L_{\text{geod}}(z,1)}$$

Follows at large mass from the geometric optics approximation in AdS.

Probing the Hawking Temperature



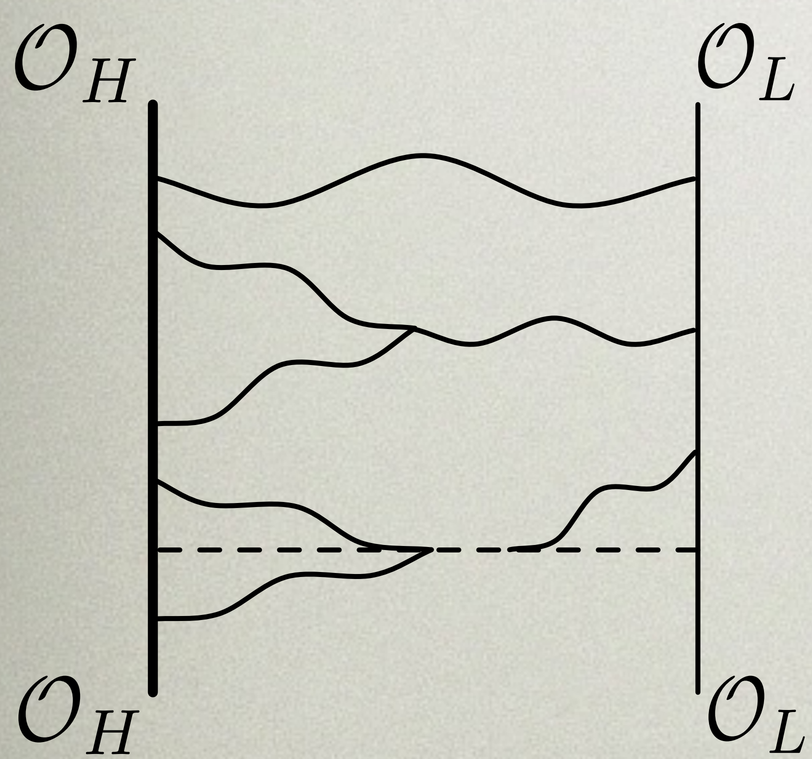
$$\approx \langle \mathcal{O}_L(1)\mathcal{O}_L(z)\rangle_{T_H} \ ?$$

A thermal 2-pt function at Hawking temperature?

Is it **periodic in Euclidean time**?

A New Perspective on CFT Thermodynamics

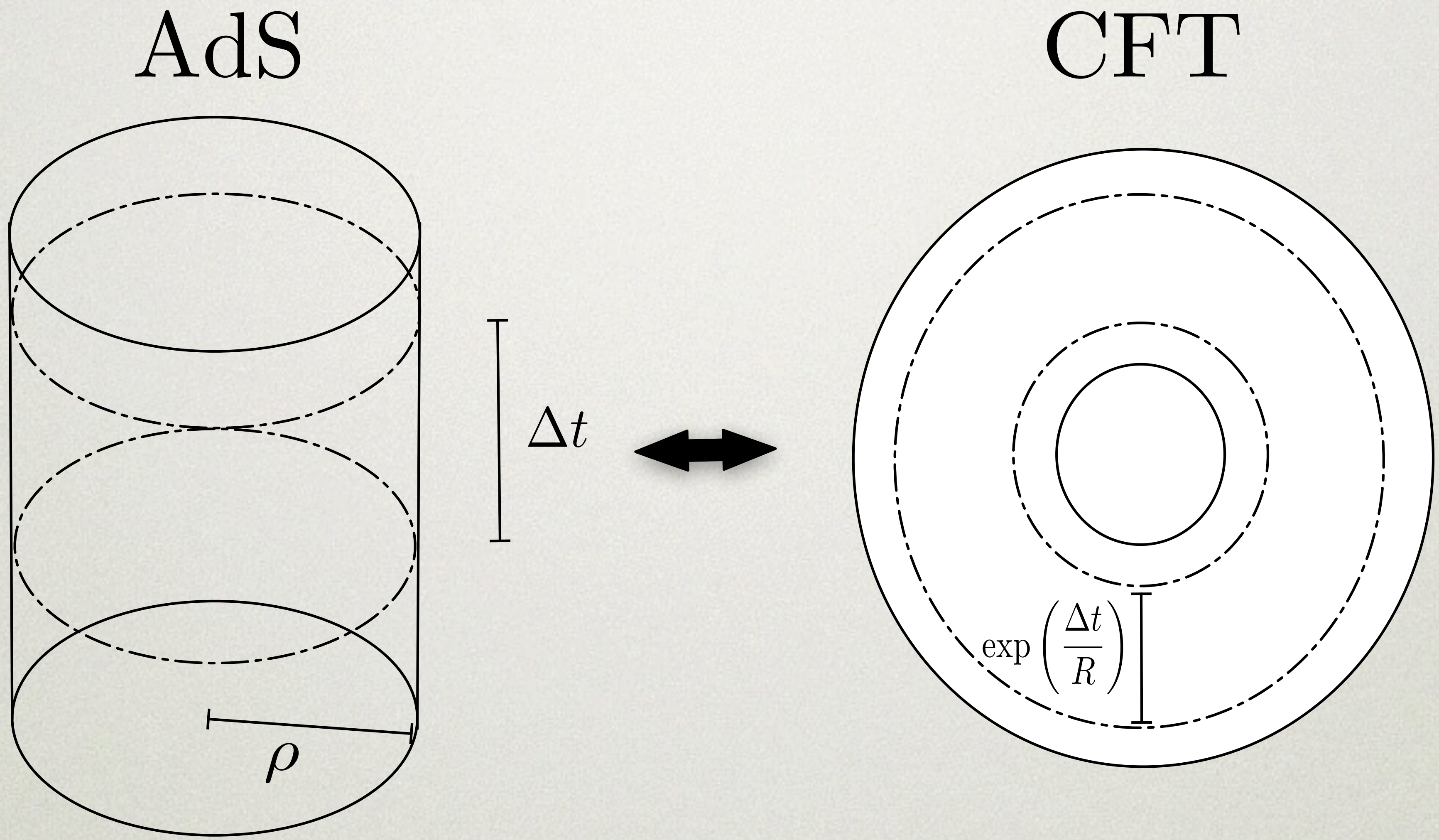
$$\langle \psi_H | \mathcal{O}_L \mathcal{O}_L | \psi_H \rangle = \langle \mathcal{O}_H \mathcal{O}_H \left(\sum_\alpha |\alpha\rangle\langle\alpha| \right) \mathcal{O}_L \mathcal{O}_L \rangle$$



We view thermodynamics as a direct result of interactions between the heavy `bath' and light probe. This can be made precise in CFTs, via **crossing symmetry**. Are stress tensors ``universal thermalizers"?

Review of AdS/CFT Kinematics

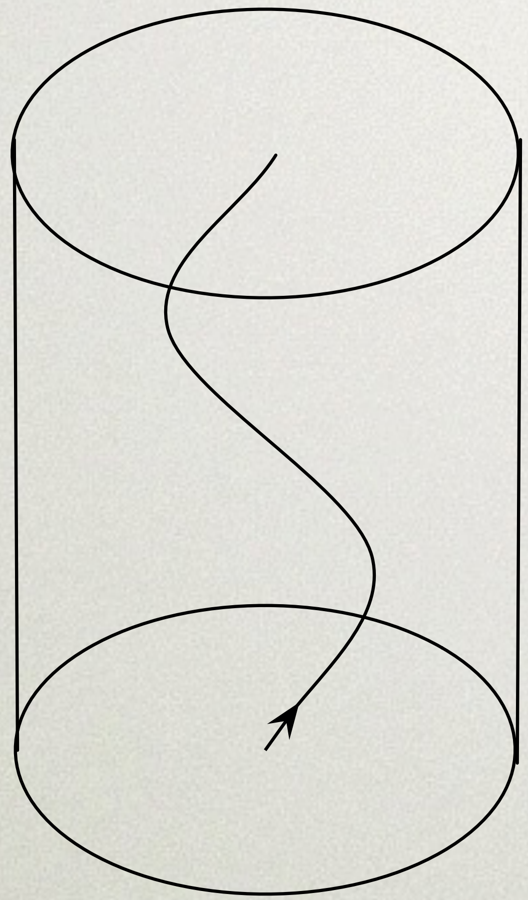
Energies and Dimensions in AdS/CFT



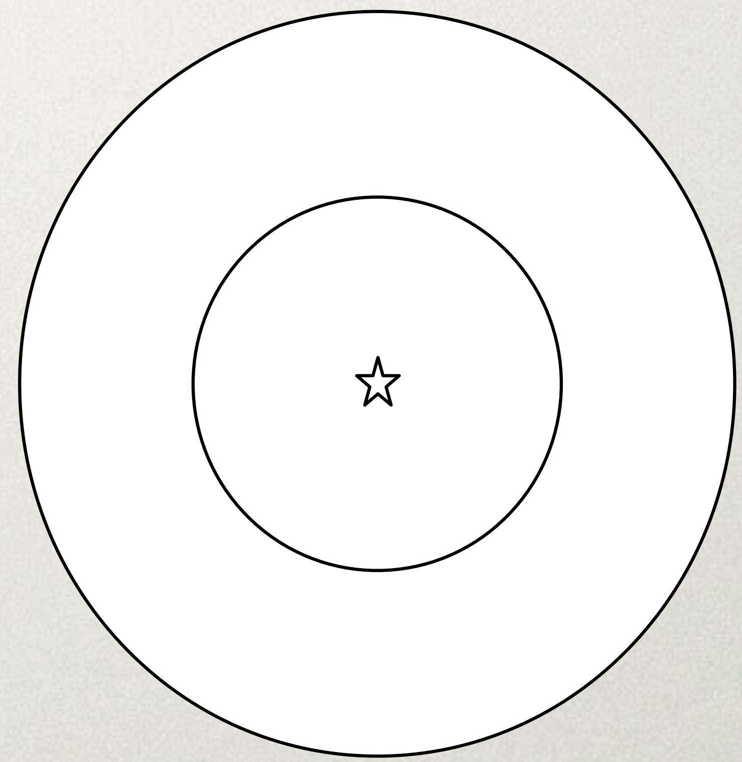
$$H_{AdS} = D_{CFT}$$

AdS Motion = Conformal Representation Theory

AdS



CFT



$$\psi_{n,\ell}(t, \rho, \Omega)$$

Center of Mass
for Excited State

$$\left(\partial^{2n}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\mathcal{O}\right)|0\rangle$$

Descendant of a
Primary

Gravity and the CFT Stress Tensor

In AdS/CFT, graviton states created by stress tensor.

$$g_{\mu\nu}(X) \leftrightarrow T_{\mu\nu}(x)$$

Graviton vertices must be universal:

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(0)T_{\mu\nu}(1)\rangle \sim \frac{\Delta_H}{\sqrt{c}}$$

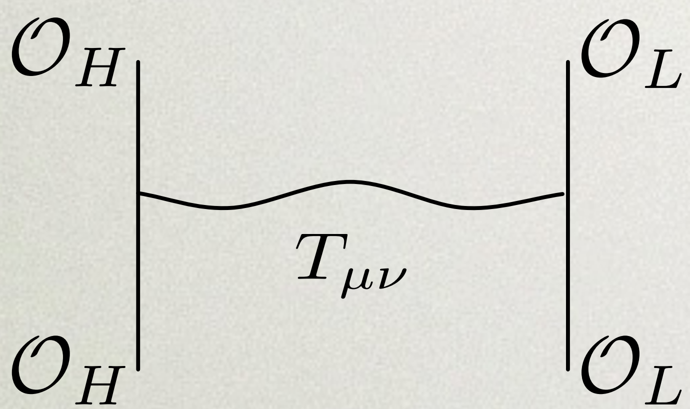
This comes from the stress tensor normalization

$$\langle T_{\mu\nu}T_{\alpha\beta}\rangle \sim c$$

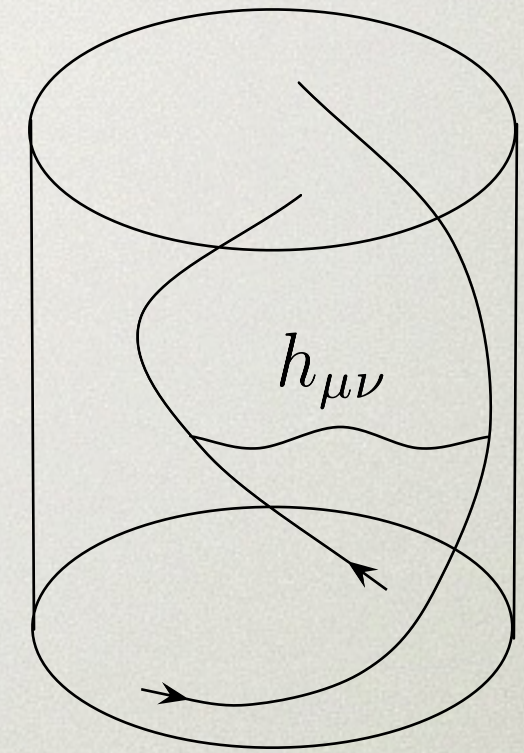
plus Ward identities for conformal symmetries.

Universality of Gravitional Potentials

So stress tensor exchange looks like…



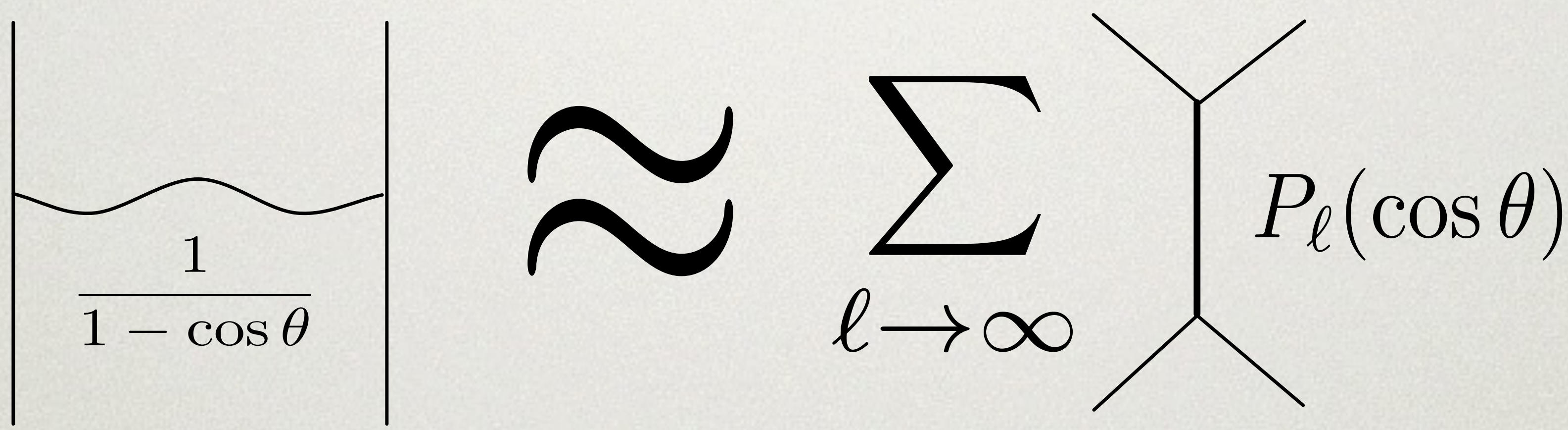
$$\sim \frac{\Delta_H \Delta_L}{c} \sim$$



Using the **Bootstrap**, can derive universal long-distance gravitational force for all CFTs.

The Idea of the Proof: A Scattering Analogy

Free propagation and massless exchange require large amplitude at large ℓ, e.g.

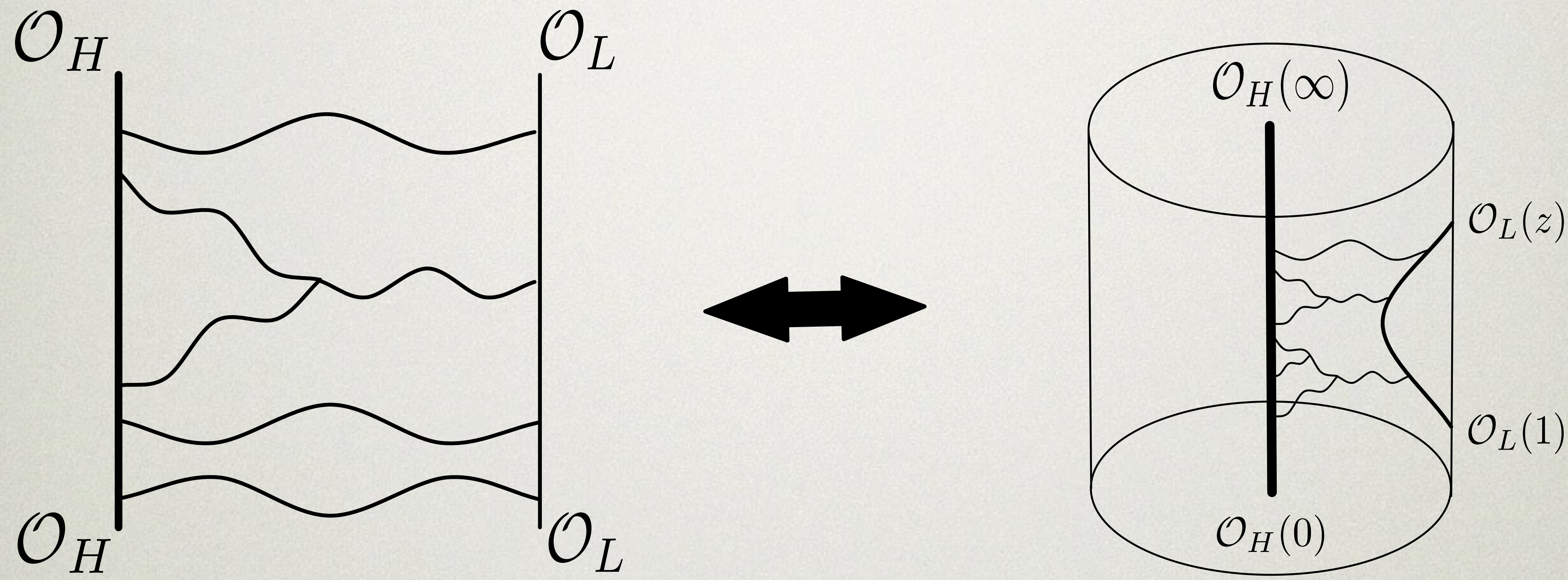


Completely analogous CFT phenomenon. Implies existence and energy of large ℓ states.

Partial Wave Amplitudes —> Conformal Partial Waves

t-channel singularity —> lightcone OPE singularity

Full Non-Linear Gravity from the CFT?



Want to sum over all multi-stress tensor operator exchanges in the CFT to reproduce the full AdS gravitational field.

Let's focus on **2d CFTs at large central charge**.

Virasoro Conformal Blocks and Multi-Graviton Exchange

Gravitons and Virasoro

Graviton states in a 2d CFT or 3d Gravity…

$$g_{\mu\nu}(X) \leftrightarrow T_{\mu\nu}(x)$$

In 2d, the stress tensor is purely (anti-)holomorphic

$$T(z) = \sum_n z^{-2-n} L_n$$

States created by acting at the origin, so

$$|\text{grav}\rangle = T(0)|0\rangle = L_{-2}|0\rangle$$

Virasoro Algebra

All graviton correlators and interactions are fixed by the Virasoro algebra (ie by symmetry):

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}$$

We will be interested in large central charge:

$$c = \frac{3}{2G}$$

Virasoro approximately just decoupled oscillators…

Representations at Large Central Charge

$$[L_n, L_m] \approx \frac{c}{12} n(n^2 - 1)\delta_{n,-m}$$

Just have independent oscillators. States are

$$L_{-m_1}^{k_1} L_{-m_2}^{k_2} \cdots L_{-m_n}^{k_n} |h\rangle$$

Refer to as \`gravitons'. These are orthogonal when m = 2,3,… and norm is

$$\mathcal{N}_{m_i,k_i} \propto c^{k_1+\cdots+k_n}$$

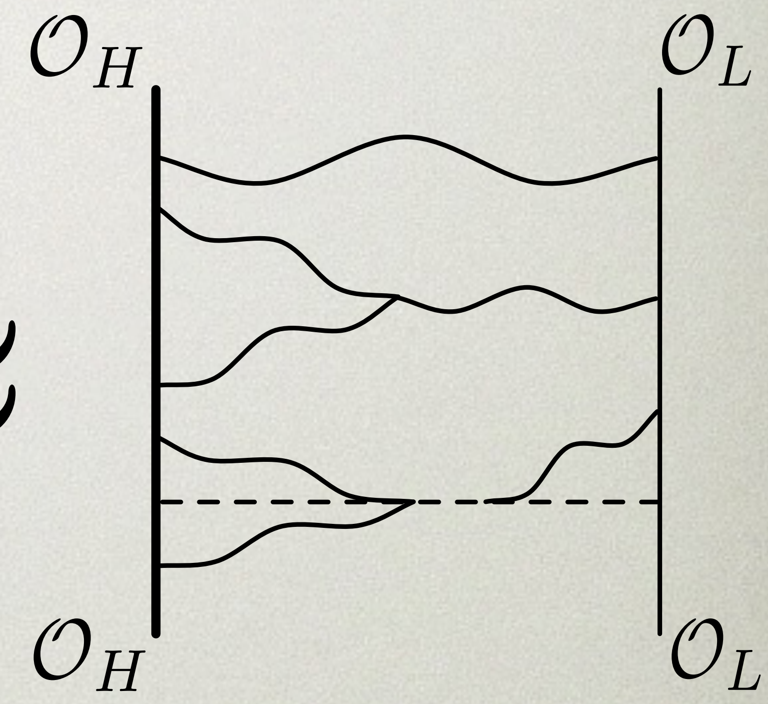
What does this mean for the exchange of gravitons?

Virasoro Blocks at Large Central Charge

Virasoro blocks, ie all graviton exchanges in AdS.

(projector)

$$\left\langle \mathcal{O}_H \mathcal{O}_H \left(\sum_{\{m_i,k_i\}} \frac{L_{-m_1}^{k_1} \cdots L_{-m_n}^{k_n} |h\rangle\langle h| L_{m_n}^{k_n} \cdots L_{m_1}^{k_1}}{\mathcal{N}_{\{m_i,k_i\}}} \right) \mathcal{O}_L \mathcal{O}_L \right\rangle \approx$$



At large central charge (other parameters fixed!) can just sum over Virasoro generators directly.

Virasoro Blocks at Large Central Charge

When all dimensions are fixed, just get

$$\left\langle \mathcal{O}_H \mathcal{O}_H \left(\sum_k \frac{L_{-1}^k |h\rangle\langle h| L_1^k}{\mathcal{N}_k} \right) \mathcal{O}_L \mathcal{O}_L \right\rangle$$

This is just the global conformal block!

For definiteness, each k is the expansion of:

$$g_h(1-z) = (1-z)^h {}_2F_1(h, h, 2h, 1-z)$$

It corresponds to motion in AdS, but **no graviton states are created at infinite central charge**.

How do `Gravitons' couple to Operators?

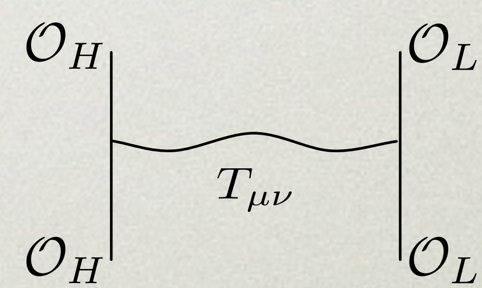
Formally, gravitons couple to operators via

$$[L_m, \mathcal{O}(z)] = h_i(1+m)z^m\mathcal{O}(z) + z^{1+m}\partial_z\mathcal{O}(z)$$

Follows from Ward identity for stress tensor. Thus

$$\frac{\langle \mathcal{O}\mathcal{O}L_{-m}\rangle}{\langle \mathcal{O}\mathcal{O}\rangle} \sim h_{\mathcal{O}}$$

or



$$\sim \frac{\Delta_H \Delta_L}{c}$$

Large central charge could be compensated by a large operator dimension.

Ex: Virasoro Blocks in a 'Newtonian' Limit

Only produce gravitons when operator dimensions grow in the limit of large central charge.

$$[L_m, \mathcal{O}(z)] = h_i(1+m)z^m \mathcal{O}(z) + z^{1+m}\partial_z \mathcal{O}(z)$$

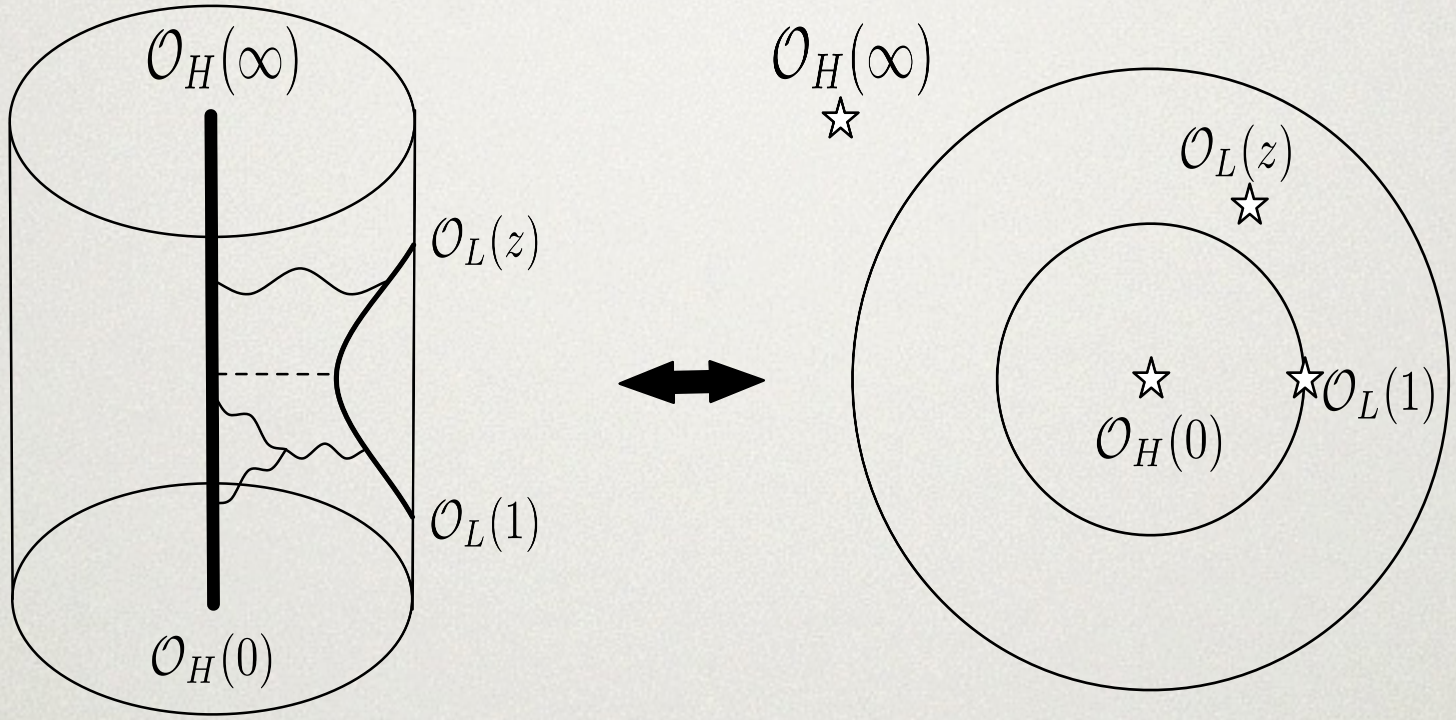
For example, in the `Newtonian' limit

$$c \to \infty \quad \text{with} \quad \frac{h_H h_L}{c} \text{ fixed}$$

this has no graviton-graviton interactions, but objects do have **gravitational binding energy**.

Virasoro Blocks in a Heavy-Light Semi-Classical Limit

Heavy-Light Semi-Classical Limit



$$h_H, \bar{h}_H \propto c \gg h_L, \bar{h}_L$$

$$c \to \infty$$

How to proceed??

Heavy-Light Semi-Classical Limit

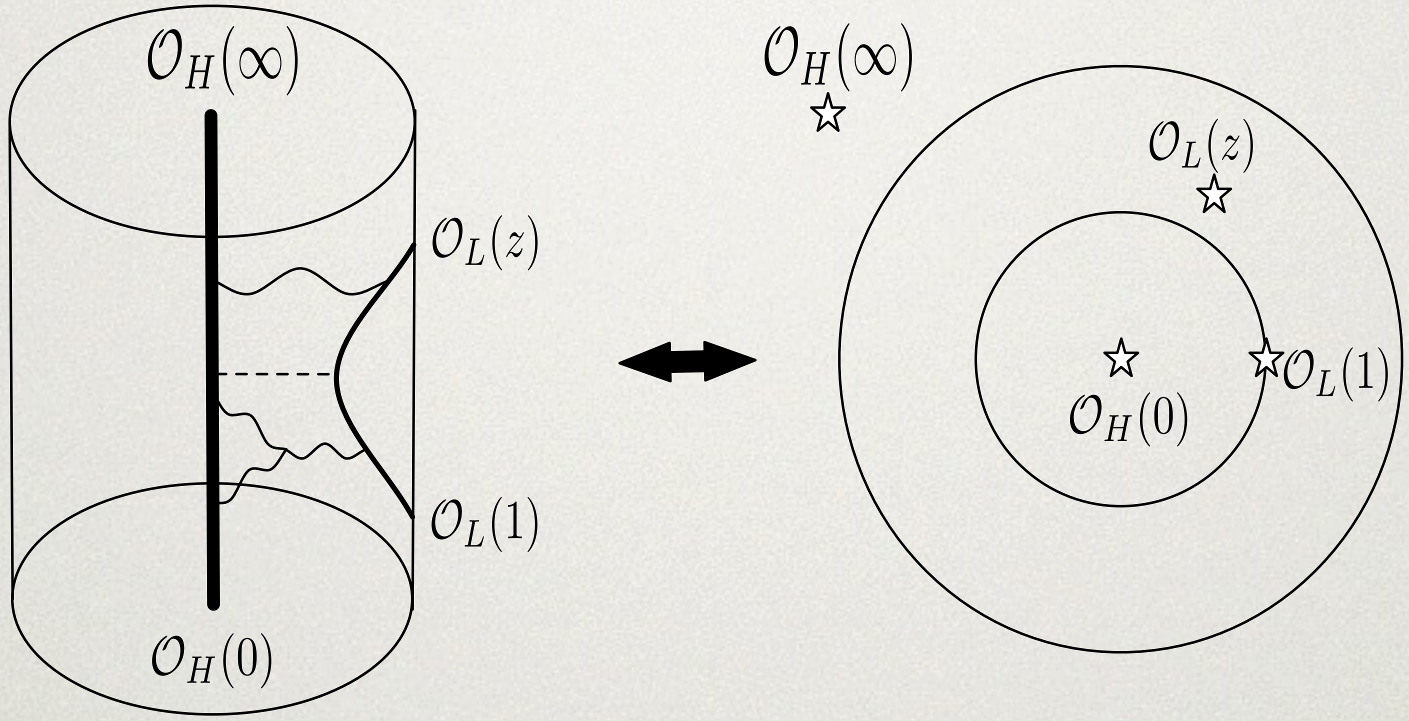
Three Different Methods (known)

I. `Monodromy Method′ — based on Liouville theory

II. `Hawking from Catalan′ — Direct Summation

III. `Background field Method′ — Most Powerful…

Idea of the Method



Main Idea: Black Hole creates a classical background. Can we **find this background directly in the CFT, and expand about it?**

Find `gravitons' as perturbations of BTZ background.

Why the Stress Tensor Isn't a Primary Operator

Under a conformal transformation

$$z \to w(z)$$

Primary operators are supposed to transform to

$$\mathcal{O}(w) = [z'(w)]^h \mathcal{O}(z(w))$$

a primary in the new coordinates/metric.

Now let's study the CFT stress tensor under a conformal transformation…

Why the Stress Tensor Isn't a Primary Operator

After conformal trans, CFT lives in a **new metric**

$$ds^2 = dz d\bar{z} \to dw d\bar{z} = w'(z) dz d\bar{z}$$

Stress tensor will get a **VEV** due to curvature:

$$\langle T_{\mu\nu} \rangle_{g_{ab}} \neq 0$$

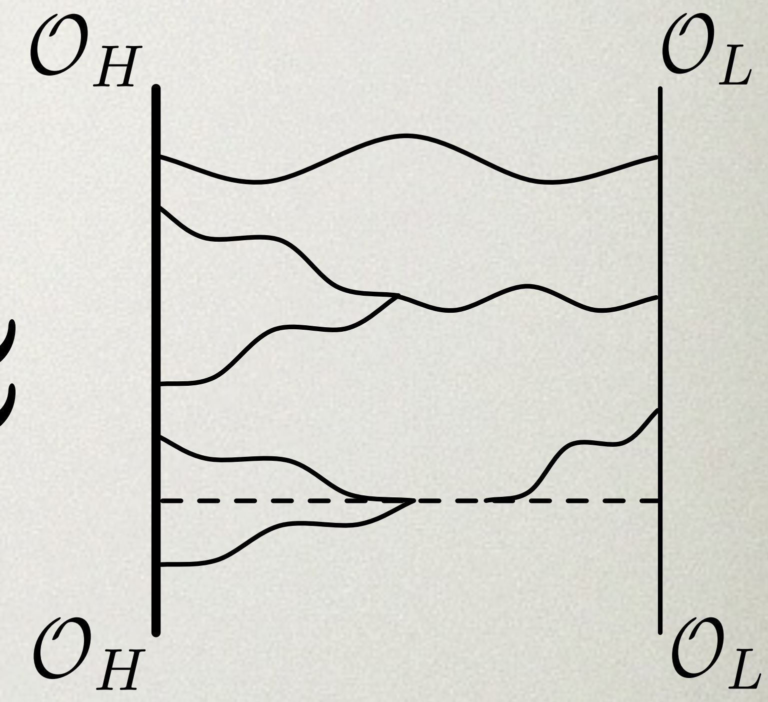
This is why it's not a primary operator.

$$T(z) \to [z'(w)]^2 T(z(w)) + S(z(w), w)$$

Second term, `Schwarzian Derivative', is the VEV.

What can we do with a Background for the CFT?

$$\left\langle \mathcal{O}_H \mathcal{O}_H \left(\sum_{\{m_i,k_i\}} \frac{L_{-m_1}^{k_1} \cdots L_{-m_n}^{k_n} |h\rangle\langle h| L_{m_n}^{k_n} \cdots L_{m_1}^{k_1}}{\mathcal{N}_{\{m_i,k_i\}}} \right) \mathcal{O}_L \mathcal{O}_L \right\rangle \approx$$



All complication from powers of **heavy** dimension:

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)T(z)|h\rangle = C_{HHh}\left(\frac{h_H}{(1-z)^2} + \frac{h}{z^2(1-z)}\right)$$

In new background for CFT, cancel this against VEV…

What can we do with a Background for the CFT?

We transform to coordinates

$$1 - w = (1 - z)^{\alpha} \quad \text{with} \quad \alpha = \sqrt{1 - \frac{24 h_H}{c}}$$

Using the transformation rule, we find

$$\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) T(w) | h \rangle = C_{HHh} h \frac{1 - z(w)}{z^2(w)}$$

All dependence on the heavy operator dimension has **cancelled** once we put the CFT in a background!

Relation to AdS

Deficit/BTZ metric can be written as

$$ds^2 = \frac{1}{\cos^2\kappa}\left(d\kappa^2 + \alpha\bar{\alpha}\frac{dz d\bar{z}}{z\bar{z}}\right) + \frac{1}{4}\left(\alpha\frac{dz}{z} - \bar{\alpha}\frac{d\bar{z}}{\bar{z}}\right)^2$$

Thus z and w related in the same way as pure AdS and BTZ.

Either obtain a deficit angle or temperature:

$$T_H = \frac{\sqrt{\frac{24h_H}{c} - 1}}{2\pi}$$

New Virasoro: Gravitons About BTZ Background

Series expand stress tensor in new coordinates:

$$T(w) = \sum_n w^{-2-n} L'_n$$

New, distinct, Virasoro generators.
Same Virasoro algebra because the stress tensor OPE take the same form in new coords.

Now we can write the sum over the Virasoro irreducible rep using new generators…

Block Computation in New Coordinates

We can compute block using new generators:

$$\left\langle \mathcal{O}_H \mathcal{O}_H \left(\sum_{\{m_i, k_i\}} \frac{L'^{k_1}_{-m_1} \cdots L'^{k_n}_{-m_n} |h\rangle\langle h'| L'^{k_n}_{m_n} \cdots L'^{k_1}_{m_1}}{\mathcal{N}_{\{m_i, k_i\}}} \right) \mathcal{O}_L \mathcal{O}_L \right\rangle$$

All L'_n with $|n| > 1$ act trivially at large c!

This follows because the normalizations of gravitons still proportional to central charge, but no large dimensions in `vertices'.

Obtain Heavy-Light Virasoro Blocks

Heavy-Light Virasoro blocks are just **global blocks in new coordinates.**

$$\mathcal{V}_h(1-w) = (1-w)^{h-2h_L} {}_2F_1(h,h,2h,1-w)$$

No `gravitons' are created at large central charge except those already in the classical background.

Note that all analysis was entirely in CFT, based only on Virasoro and large c.

Vacuum Block and Thermality

Above the BTZ threshold, we seem to have **periodicity** in Euclidean time, ie **thermality**, as Virasoro blocks depend on Euclidean time via:

$$1 - w = (1 - z)^{2\pi i T_H} = e^{2\pi i T_H t_E}$$

because above the BTZ threshold, we have

$$\alpha \equiv \sqrt{1 - \frac{24 h_H}{c}} = 2\pi i T_H$$

If we look at \`pure graviton exchange' we find…

Vacuum Block and Thermality

The universal vacuum Virasoro block is:

$$\mathcal{V}(t) = \left(\frac{\pi T_H}{\sin(\pi T_H t)} \right)^{2h_L}$$

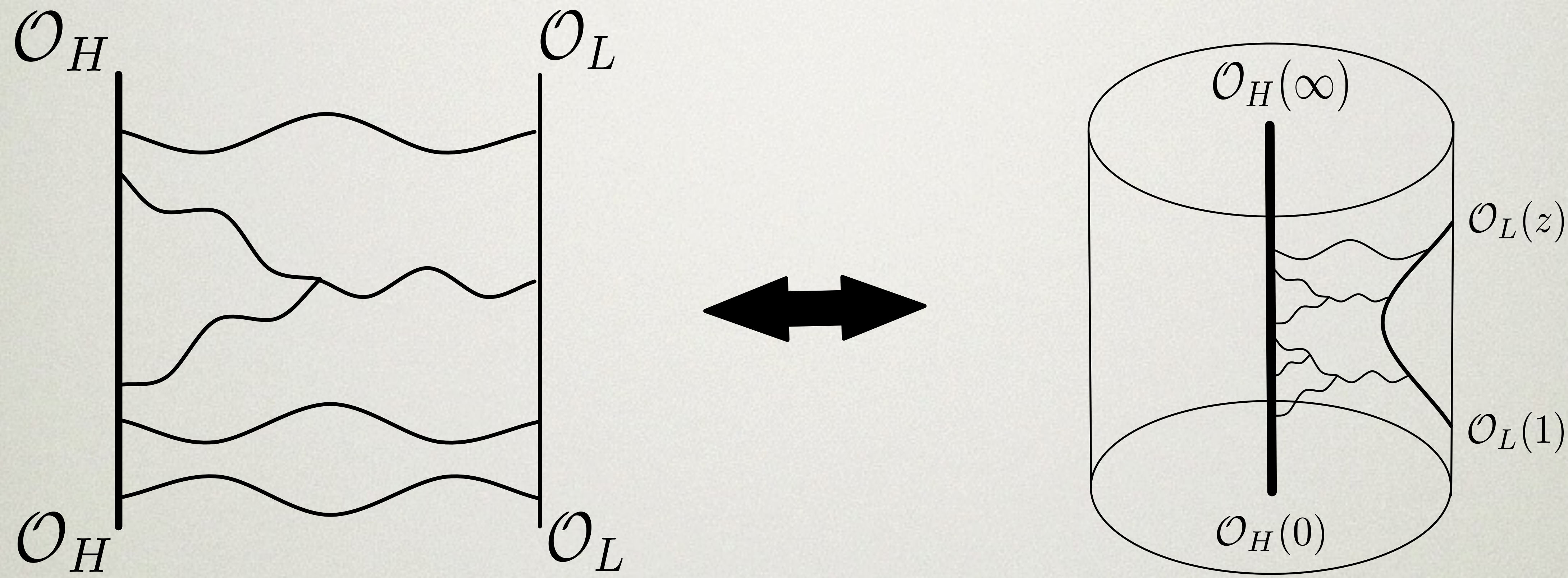
after we transform from the plane
to the cylinder via:

$$z \to 1 - e^t$$

Precisely what we expect for a thermal correlator!

Universality of OPE Data from Thermodynamics in General Dimensions?

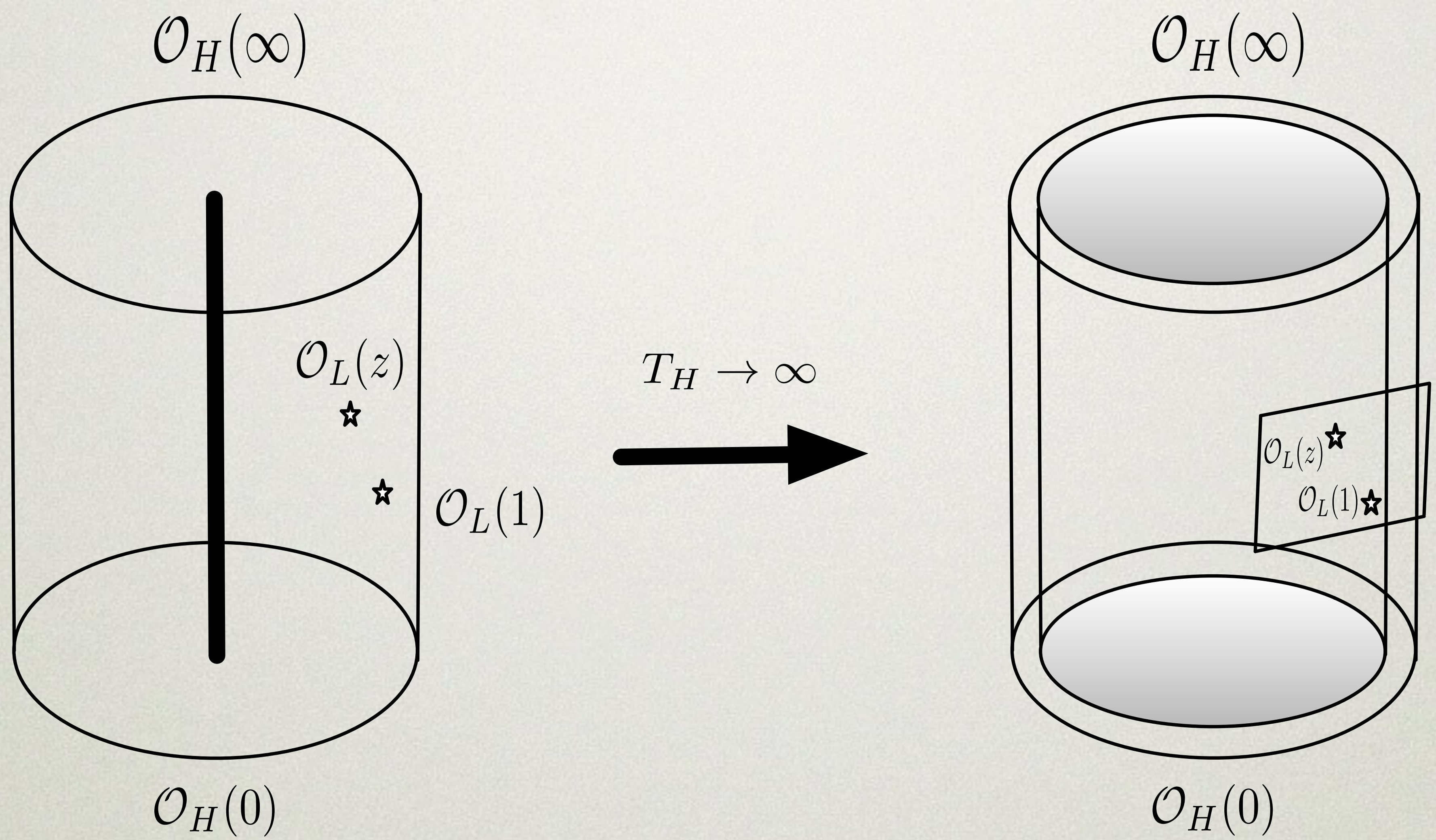
Heavy-Light Correlator in AdS/CFT with d>2



Which states propagate between heavy and light to `thermalize' the light operator?

$$T_{\mu\nu},\ T_{\mu\nu}T_{\alpha\beta},\ T\partial^n T, \cdots, T^k, \cdots$$

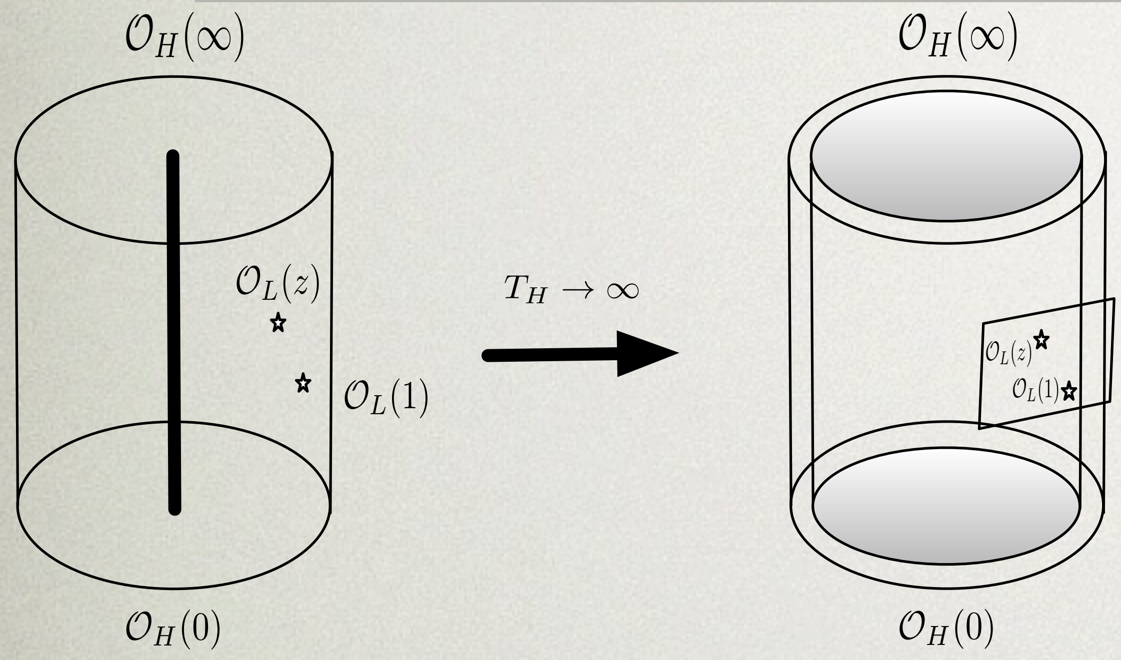
Simplifies Further in High Temperature Limit



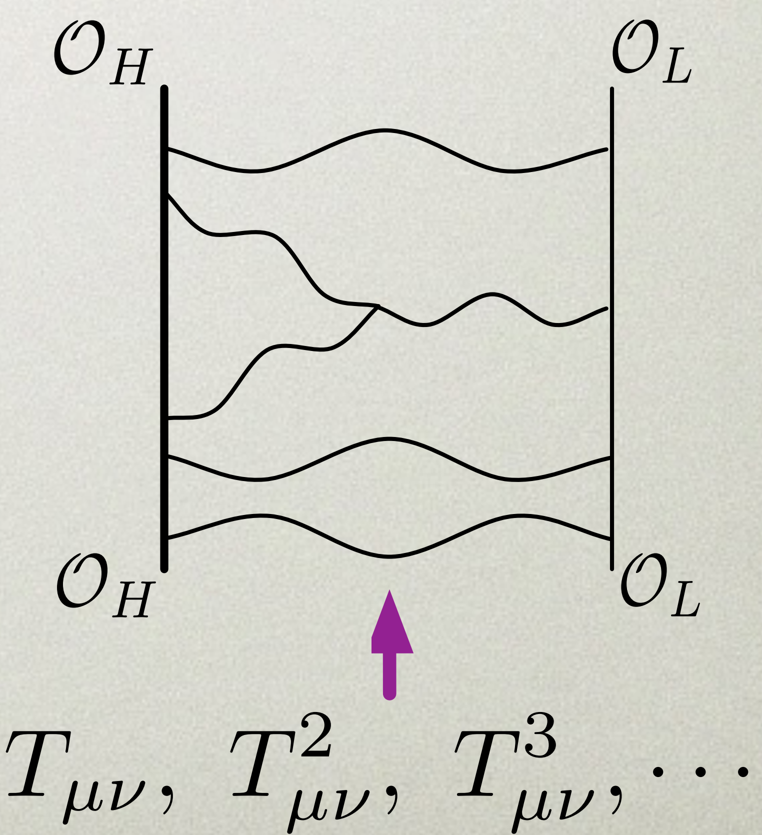
Measure distances in CFT in units of temperature:

$$(1-z) \to \frac{t}{T_H}, \quad T_H \to \infty$$

High Temperature Limit



Only operators without derivatives contribute.



OPE Coefficients must have universal properties to recover thermodynamics:

$$\langle \mathcal{O}\mathcal{O}T^k \rangle$$

More specifically…

Universal Relations for OPE Coefficients

If we define the OPE coefficients as

$$\langle \mathcal{O}\mathcal{O}T^k \rangle = \frac{C_k}{\sqrt{k!}}$$

Periodicity in Euclidean time of the correlator

$$\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}(t)\mathcal{O}(0) \rangle = \sum_k \frac{C_k}{k!} t^k$$

implies relations among OPE coefficients (any k):

$$C_k = \sum_{m=1}^{\infty} \frac{C_{k+m}}{m!}$$

Infinitely many relations for each operator!

Certainly not a consequence of Ward identity, etc…

Thermodynamic Constraints on Holographic CFT Data?

$$\langle \mathcal{O}\mathcal{O}(T_{\mu\nu})^k \rangle \sim \sum_{m=0}^{\infty} \frac{\sqrt{k!(k+m)!}}{m!} \langle \mathcal{O}\mathcal{O}(T_{\mu\nu})^{k+m} \rangle$$

- For all, or only almost all operators?
- Derivable from other constraints (e.g. the bootstrap) or **is this a new constraint?**
- Constraints remain at finite central charge?
- Can this help to explain if/why it's hard to UV complete quantum gravity?

Future Directions

- New perspectives on Virasoro Blocks, including generalizations and $1/c$ corrections (in progress)
- Connections between OPE Data and Thermality… for which theories, in what limits?
- Do thermodynamics constraints on OPE Data explain why it's difficult to UV Complete Gravity?

EXTRA SLIDES

Exact Result for Four Different Operators

$$(1-w)^{(h_L+\delta_L)(1-\frac{1}{\alpha})}\left(\frac{w}{\alpha}\right)^{h-2h_L}{}_2F_1\left(h-\frac{\delta_H}{\alpha}, h+\delta_L, 2h, w\right)$$

where we define

$$1-w=(1-z)^{\alpha} \quad \text{with} \quad \alpha=\sqrt{1-24\frac{h_H}{c}}.$$

Note the rescaling of the heavy dimension difference

$$h_{H_1}-h_{H_2}\equiv\delta_H\to\frac{\delta_H}{\alpha}$$

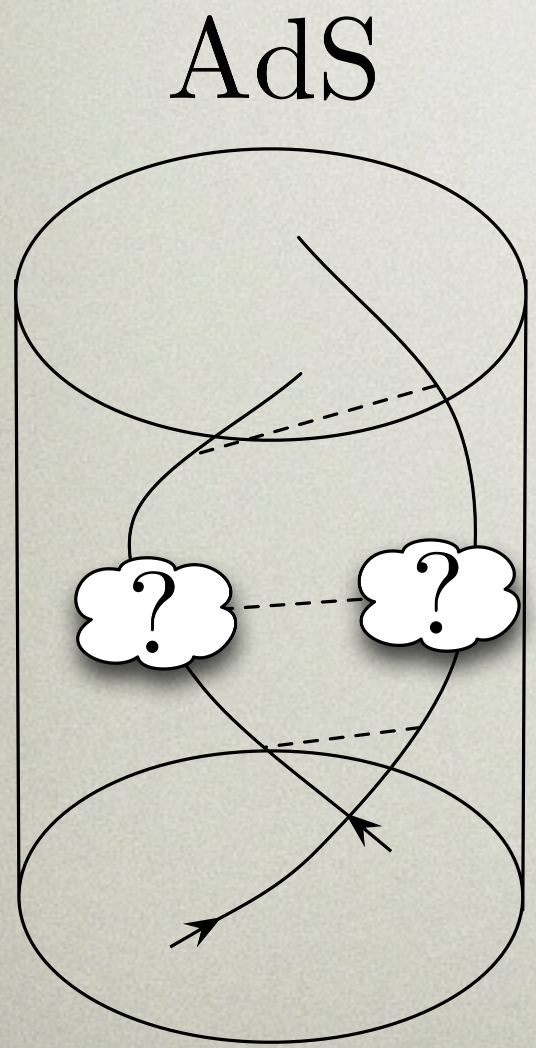
Interesting to find a clear physical interpretation.

What would we expect from 3d AdS Gravity?

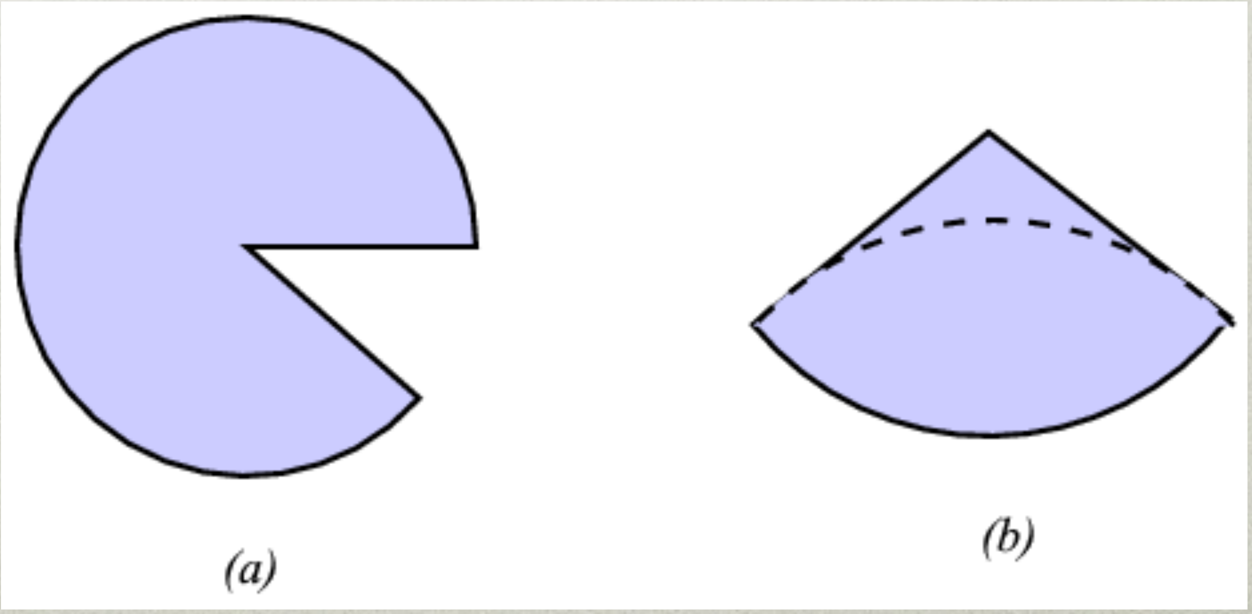
Deficit Angles in D=2

AdS Solution for a Sub-Planckian Object:

$$ds^2 = \cosh^2(\kappa)dt^2 - d\kappa^2 - (1 - 8GM)\sinh^2(\kappa)d\phi^2$$



In 2+1 dimensional AdS, expect deficit angles, detectable near infinity.

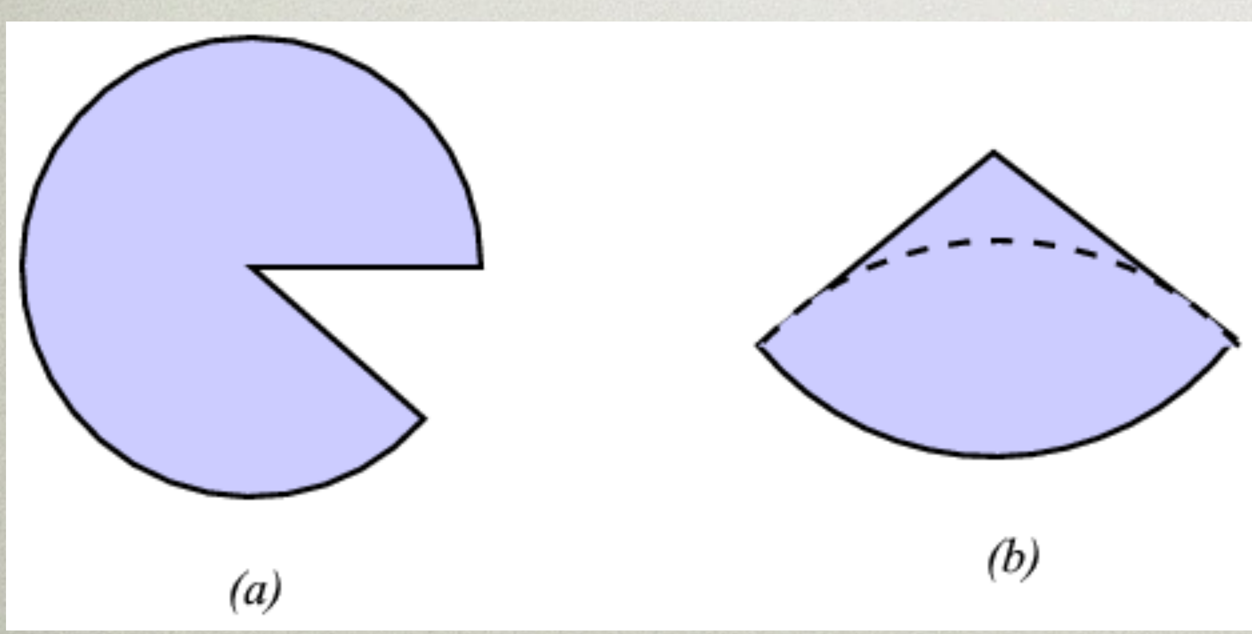


Deficit Angles in D=2

$$ds^2 = \cosh^2(\kappa)dt^2 - d\kappa^2 - (1-8GM)\sinh^2(\kappa)d\phi^2$$

Deficit angles are characterized by parameter

$$\alpha = \sqrt{1-8GM} = \sqrt{1-\frac{24h}{c}}$$



What if

$$h > \frac{c}{24}\,?$$

BTZ Black Holes

Deficit angle analytically continues to BTZ with

$$|\alpha| = 2\pi T_H$$

Signal: bulk and boundary correlators are **periodic in imaginary time.**

$$\langle \mathcal{O}_H | \mathcal{O}_A(it) \mathcal{O}_A(0) | \mathcal{O}_H \rangle = \frac{(\pi T_H)^{2h_A}}{\sinh^{2h_A}(\pi T_H t)}$$

But why should **all** sufficiently heavy states be BTZ black holes?

Heavy States Always BTZ Black Holes?

One reason this makes sense is that
around a BTZ black hole, there are no stable orbits!

Radial potential for orbits is

$$V(r) = \left(1 - \frac{M}{r^{d-2}} + r^2\right)\left(m^2 + \frac{\ell^2}{r^2}\right)$$

This is monotonic for d = 2.

Thus **all** geodesics `fall in' on an AdS timescale.
It's hard to avoid quickly forming a black hole.

CFT Probe of Black Holes?

New Virasoro: Some Technicalities (1)

First of all, we should note that:

$$\mathcal{L}_n|h\rangle = 0 \quad \text{for} \quad n \geq 1$$

because

$$1 - w = (1 - z)^\alpha$$

is analytic about the origin.

So primaries wrt old Virasoro
are also primary wrt new Virasoro.

New Virasoro: Some Technicalities (2)

$$\mathcal{L}_n^\dagger \neq \mathcal{L}_{-n}$$

Because inversions are non-trivial, or equivalently because conf. trans. non-analytic at infinity.

So how can we deal with adjoint states? Formally define:

$$\langle 0_w | \mathcal{L}_{-n} = 0$$

so we have adjoint states

$$\langle h_w | = \lim_{w \to \infty} \langle 0_w | w^{2h} \mathcal{O}(w)$$

New Virasoro: Some Technicalities (3)

Finally we can conclude that:

$$\langle h_w|\mathcal{L}_{m_n}^{k_n}\cdots\mathcal{L}_{m_1}^{k_1}|h\rangle = \langle h|L_{m_n}^{k_n}\cdots L_{m_1}^{k_1}|h\rangle$$

This follows because new and old generators both satisfy the Virasoro algebra, and also act on operators (in z or w, respectively) as determined by the OPE of the stress tensor with the primary operator in question, justifying:

$$\mathcal{V}_h(w) = \langle \mathcal{O}_{H_1}(\infty)\mathcal{O}_{H_2}(1)\left(\sum_k \frac{\mathcal{L}_{-1}^k|h\rangle\langle h_w|\mathcal{L}_1^k}{\langle h_w|\mathcal{L}_1^k\mathcal{L}_{-1}^k|h\rangle}\right)\mathcal{O}_{L_1}(w)\mathcal{O}_{L_2}(0)\rangle$$

New Virasoro: Some Technicalities (4)

$$\mathcal{V}_h(w) = \langle \mathcal{O}_{H_1}(\infty)\mathcal{O}_{H_2}(1) \left(\sum_k \frac{\mathcal{L}_{-1}^k |h\rangle\langle h_w| \mathcal{L}_1^k}{\langle h_w|\mathcal{L}_1^k \mathcal{L}_{-1}^k|h\rangle} \right) \mathcal{O}_{L_1}(w)\mathcal{O}_{L_2}(0)\rangle$$

Right correlator trivial; exactly as with old Virasoro.

Left correlator from a conformal transformation:

$$\begin{aligned} \langle \mathcal{O}_{H_1}(\infty)\mathcal{O}_{H_2}(1)\mathcal{L}_{-1}^k|h\rangle &= \lim_{w\to 0} \partial_w^k \langle \mathcal{O}_{H_1}(\infty)\mathcal{O}_{H_2}(1)\mathcal{O}_h(w)\rangle \\ &= \alpha^{-h} c_{H_1 H_2 h} \lim_{w\to 0} \partial_w^k (1-w)^{-h+\delta_H/\alpha} \end{aligned}$$

Using this relation gives the final result, as desired.