BPS indices: A Trilogy

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based on work with J. Manschot and A. Sen,

S. Alexandrov, G. Moore, A. Neitzke

Introduction I

- Gauge theories and string vacua with $\mathcal{N}=2$ supersymmetry on $\mathbb{R}^{3,1}$ have flat directions, parametrized by $u \in \mathcal{M}$. The number of bound states may depend on u in a complicated way.
- BPS bound states are much easier to control: due to $M=|Z(\gamma,u)|$, a BPS state with charge vector γ can only decay into a set of BPS constituents; moreover this decay can only happen on walls of marginal stability, where

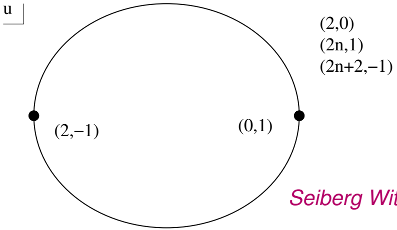
$$W(\gamma_1,\gamma_2)=\{u\,/\,\arg[Z(\gamma_1,u)]=\arg[Z(\gamma_2,u)]\}\subset\mathcal{M}$$

such that $\gamma=M_1\gamma_1+M_2\gamma_2$ for some positive integers M_1,M_2.

Cecotti Vafa 1992; Seiberg Witten 1994

Introduction II

- E.g., in $D = 4, \mathcal{N} = 2$ SYM with $G = SU(2)$ (Seiberg-Witten) on the Coulomb branch,



Seiberg Witten 1994; Bilal Ferrari 1996

- The BPS index is a trace over the Hilbert space of one-particle states $\mathcal{H}'_1(\gamma, u)$ (in center of mass frame),

$$\Omega(\gamma, u) = -\tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}'_1(\gamma,u)}(-1)^{2J_3}\,(2J_3)^2 \quad \in \mathbb{Z}\ ,$$

Being integer it is locally constant, but since it is sensitive to BPS states only, it may only jump across the walls $W(\gamma_1, \gamma_2)$.

Introduction III

- The BPS index should be contrasted with the Witten index, defined as a trace over the full Hilbert space $\mathcal{H}(u,\gamma)$,

$$\varpi(R,\gamma,u) = -\tfrac{1}{2}\mathrm{Tr}_{\mathcal{H}(u,\gamma)}(-1)^{2J_3}(2J_3)^2\, e^{-2\pi R H} \quad \notin \mathbb{Z}!$$

- In center of mass frame, the Hamiltonian has a discrete spectrum starting at the BPS bound $E_0 = |Z(\gamma,u)|$, and a continuum of multi-particle states opening at $E_c > E_0$. These non-BPS states can still contribute to the Witten index, since the spectral densities of bosonic and fermionic states need not be equal.

Kaul Rajaraman 1983, Akhoury Comtet 1984, Troost 2010, ...

Introduction IV

- Unlike the BPS index $\Omega(\gamma, u)$, the Witten index $\varpi(R, \gamma, u)$ is smooth across walls of marginal stability. It is a universal function of the BPS indices $\{\Omega(\alpha_i, u)\}$, such that discontinuities cancel:

$$\varpi(R, \gamma, u) = R\, \Omega(\gamma, u)\, |Z_\gamma(u)|\, K_1(2\pi R|Z_\gamma(u)|) + \sum_{\substack{\gamma_1, \gamma_2 \\ \gamma=\gamma_1+\gamma_2}} \Omega(\gamma_1)\, \Omega(\gamma_2)\, J_{\gamma_1,\gamma_2}(R, u) + \dots$$

Alexandrov Moore Neitzke BP 2014, BP 2015

- Indeed, $\varpi(R, \gamma, u)$ is computed by a path integral on $\mathbb{R}^3 \times S^1(R)$ with periodic boundary conditions (with an insertion of a 4-fermion vertex to soak up fermionic zero-modes). Path integrals may diverge when new modes become massless, but this is not the case on the walls.

Introduction V

- The jump of the BPS index $\Omega(\gamma, u)$ across $W(\gamma_1, \gamma_2)$ reflects the decay of the bound state of charge $\gamma = M\gamma_1 + N\gamma_2$ into BPS constituents of charge $\alpha_i = M_i\gamma_1 + N_i\gamma_2$ with $\sum(M_i, N_i) = (M, N)$.
- The discontinuity of $\Omega(\gamma, u)$ can be computed from the BPS indices $\Omega(\alpha_i)$ on either side of the wall, and from the index $g(\{\alpha_i\})$ of the SUSY quantum mechanics describing the constituents. The latter is computable using localization.

Denef 2002; Manschot BP Sen 2010-13

- Although the BPS constituents with charge α_i are stable across the wall $W(\gamma_1, \gamma_2)$, they may decay across some other wall $W(\gamma_1', \gamma_2')$ into other constituents with charge $\alpha_j' = M_j'\gamma_1' + N_j'\gamma_2'$, and so on.

- This raises the question: can one identify a set of elementary constituents which never decay, from which the full BPS spectrum may be reconstructed at any point in moduli space ?
- In Seiberg-Witten theory: the monopole and dyon satisfy this condition. In supergravity, single-center black holes are absolutely stable.
- Can the BPS index $\Omega(\gamma, u)$ away from the wall be computed from a set of 'stable' indices $\Omega_S(\gamma)$, and if so, what do these indices count ?
- We shall investigate this question in the context of $\mathcal{N} = 4$ quiver quantum mechanics, which describes certain D-brane bound states.

Outline

1. Revisiting the two-body problem
2. Quantum mechanics of n dyons
3. The Coulomb branch formula for the BPS index
4. Applications to quiver quantum mechanics

Outline

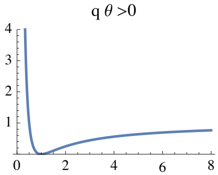
Quantum mechanics of two BPS dyons I

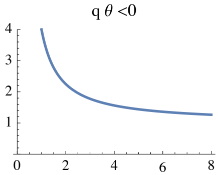
- At large distances, the interactions between two mutually-non local BPS dyons are described by $\mathcal{N} = 4$ supersymmetric quantum mechanics,

$$\boxed{H = \frac{1}{2m}\,(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\,\vec{B}\cdot\vec{\sigma}\otimes(1_2 - \sigma_3) + \frac{1}{2m}\left(\zeta - \frac{q}{r}\right)^2}$$

$$q = \frac{1}{2}\langle\gamma_1, \gamma_2\rangle\ ,\ \ \frac{\zeta^2}{2m} = |Z_{\gamma_1}| + |Z_{\gamma_2}| - |Z_{\gamma_1+\gamma_2}|$$

$$m = \frac{|Z_{\gamma_1}||Z_{\gamma_2}|}{|Z_{\gamma_1}|+|Z_{\gamma_2}|}$$
$$\vec{\nabla}\wedge\vec{A} = \vec{B} = \frac{\vec{r}}{r^3}$$





Quantum mechanics of two BPS dyons II

- H describes two bosonic degrees of freedom with helicity $h = 0$, and one helicity $h = \pm 1/2$ fermionic doublet with gyromagnetic ratio $g = 4$.

D'Hoker Vinet 1985; Denef 2002; Avery Michelson 2007;Lee Yi 2011

- H commutes with 4 supercharges,

$$Q_4 = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 & -\mathrm{i}\left(\zeta - \frac{q}{r}\right) + \vec{\sigma}\cdot(\vec{p} - q\vec{A}) \\ \mathrm{i}\left(\zeta - \frac{q}{r}\right) + \vec{\sigma}\cdot(\vec{p} - q\vec{A}) & 0 \end{pmatrix}$$

$$\{Q_m, Q_n\} = 2H\,\delta_{mn}$$

- When $q\zeta > 0$, H has a BPS ground state with degeneracy $2|q|$, and a number of non-BPS bound states. In addition, for any sign of $q\zeta$, it has a continuum of scattering states.

Non-relativistic electron-monopole problem I

- Going to a basis of monopole spherical harmonics, the Schrödinger equation with energy $E = k^2/(2m)$ becomes

$$\left[-\frac{1}{r}\partial_r^2 \, r + \frac{\nu^2 - q^2 - \frac{1}{4}}{r^2} + \left(\zeta - \frac{q}{r} \right)^2 \right] \Psi(r) = k^2 \Psi \,,$$

where

$$\nu = j + h + \tfrac{1}{2} \,, \quad j = |q| + h + \ell \,, \ell \in \mathbb{N} \,.$$

- Supersymmetric bound states exist for $q\zeta > 0$, $h = -\frac{1}{2}$, $\ell = 0$, and form a multiplet of spin $j = |q| - \frac{1}{2}$, with multiplicity $2|q| = |\langle \gamma_1, \gamma_2 \rangle|$.

Denef 2002

Non-relativistic electron-monopole problem II

- The S-matrix for partial waves is similar to that of H-atom,

$$S_\nu(k) = \frac{\Gamma\left(\frac{1}{2}+\nu+\mathrm{i}\frac{q\zeta}{\sqrt{k^2-\zeta^2}}\right)}{\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}\frac{q\zeta}{\sqrt{k^2-\zeta^2}}\right)}$$

BP 2015

- The contribution of the continuum to $\mathrm{Tr}(-1)^F\, e^{-2\pi RH}$ is thus

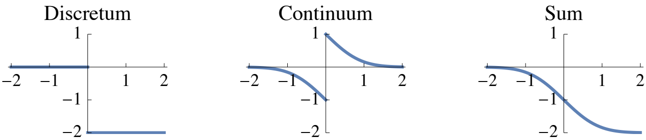
$$\sum_{h=0^2,\pm\frac{1}{2}}(-1)^{2h}\sum_{\ell=0}^{\infty}\int_{k=\zeta}^{\infty}\frac{\mathrm{d}k\,\partial_k}{2\pi\mathrm{i}}\log\frac{\Gamma\left(|q|+\ell+2h+1+\mathrm{i}\frac{q\zeta}{\sqrt{k^2-\zeta^2}}\right)}{\Gamma\left(|q|+\ell+2h+1-\mathrm{i}\frac{q\zeta}{\sqrt{k^2-\zeta^2}}\right)}\,e^{-\frac{\pi Rk^2}{m}}$$

Non-relativistic electron-monopole problem III

- Terms with $\ell > 0$ cancel, leaving the contribution from $\ell = 0$ only:

$$\mathrm{Tr}(-1)^F\, e^{-2\pi RH} = -\,|2q|\,\Theta(q\zeta) - \frac{2q\zeta}{\pi}\int_{k=|\zeta|}^{\infty} \frac{\mathrm{d}k}{k\sqrt{k^2-\zeta^2}}\, e^{-\frac{\pi R k^2}{m}}$$

$$= -\,2|q|\,\Theta(q\zeta) + |q|\,\mathrm{sgn}(q\zeta)\,\mathrm{Erfc}\left(|\zeta|\sqrt{\frac{\pi R}{m}}\right)$$



- This reproduces the two-particle contribution $J_{\gamma_1,\gamma_2}(R,u)$ in the formula for $\varpi(R,\gamma,u)$ in the vicinity of the wall. Further away from the wall, the dyons become relativistic...

Wall-crossing formula I

- This leads to the primitive wall-crossing formula:

$$\Delta\Omega(\gamma_1+\gamma_2) = \pm|\langle\gamma_1,\gamma_2\rangle|\,\Omega(\gamma_1)\,\Omega(\gamma_2)$$

Denef Moore 2007

- More generally if $\gamma = M\gamma_1 + N\gamma_2$, then

$$\Delta\bar\Omega(\gamma) = \sum_{n\geq 2} \sum_{\substack{\gamma=\alpha_1+\dots+\alpha_n \\ \alpha_i=M_i\gamma_1+N_i\gamma_2}} \frac{g(\{\alpha_i\})}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^n \bar\Omega(\alpha_i)$$

where $g(\{\alpha_i\})$ is the BPS index of the quantum mechanics of n dyons, while the symmetry factor $|\mathrm{Aut}(\{\alpha_i\})|$ and the replacement

$$\bar\Omega(\gamma) = \sum_{d|\gamma} \frac{1}{d^2}\Omega(\gamma/d)$$

Manschot BP Sen 2010

takes care of Bose-Fermi statistics.

Outline

Quiver quantum mechanics I

- The quantum mechanics of n non-relativistic dyons with charges α_i is described by $\mathcal{N}=4$ quiver quantum mechanics: if all α_i's are distinct, this is a 0+1-dimensional gauge theory with n Abelian vector multiplets and, for all i,j such that $\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle > 0$, α_{ij} chiral multiplets with charge $(1,-1)$ under $U(1)_i \times U(1)_j$.
- If $\{\alpha_i\}$ consists of N_1 copies of $\beta_1, \dots, N_r$ copies of β_r with all β_j distinct, then the gauge group is $\prod_{j=1\dots r} U(N_j)$ and there are β_{ij} chiral multiplets in the representation $(N_i, \bar{N}_j)$ whenever $\beta_{ij} > 0$.
- The Fayet-Iliopoulos parameters depend on the moduli u via $c_i = 2\,\mathrm{Im}\left[e^{-i\phi} Z(\alpha_i, u)\right]$ where $\phi = \arg[Z(\sum_i \alpha_i, u)]$. If the quiver has closed oriented loops, there is also a superpotential W.

Quiver quantum mechanics II

- In the Abelian case, the classical vacua decompose into a Higgs branch, where the chiral multiplets have a vev and the vector multiplets are massive, and the Coulomb branch, where the opposite happens.
- On the Coulomb branch, after integrating the massive chiral multiplets, supersymmetric vacua are solutions of

$$\forall i \ : \ \sum_{j \neq i} \frac{\alpha_{ij}}{|\vec{r}_i - \vec{r}_j|} = c_i(u)$$

The same equations describe multi-centered supersymmetric solutions in $\mathcal{N} = 2$ supergravity !

Denef 2000, Denef Bates 2003

Geometric quantization on the Coulomb branch I

- For fixed charges α_i and moduli u, the space of solutions modulo overall translations is a symplectic manifold $\mathcal{M}_n$ of dimension $2n-2$, invariant under $SO(3)$:

$$\omega = \frac{1}{2}\sum_{i<j} \alpha_{ij}\, \sin\theta_{ij}\, \mathrm{d}\theta_{ij}\wedge \mathrm{d}\phi_{ij}\ , \quad \vec{J} = \frac{1}{2}\sum_{i<j} \alpha_{ij}\, \frac{\vec{r}_{ij}}{|r_{ij}|}$$

de Boer El Showk Messamah Van den Bleeken 2008

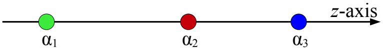
- For charges $\alpha_i = M_i\gamma_1 + N_i\gamma_2$ with $M_1, M_2 > 0$, the quiver has no loop, and $\mathcal{M}_n$ is compact !
- Given a symplectic manifold $(\mathcal{M},\omega)$, geometric quantization produces a graded Hilbert space $\mathcal{H}$, the space of harmonic spinors for the Dirac operator D coupled to ω. The refined index $g(\{\alpha_i\}, y) \equiv \mathrm{Tr}(-y)^{2J_3}$ in the SUSY quantum mechanics is equal to the equivariant index of D.

The Coulomb index from localization I

- Since $\mathcal{M}_n$ admits a $U(1)$ action, the equivariant index can be computed by localization.

Atiyah Bott, Berline Vergne

- For any n, the fixed points of the action of J_3 are collinear multi-centered configurations along the z-axis:



$$\forall i\,, \quad \sum_{j\neq i} \frac{\alpha_{ij}}{|z_i - z_j|} = c_i\,, \qquad J_3 = \frac{1}{2}\sum_{i<j} \alpha_{ij}\,\mathrm{sign}(z_j - z_i)\,.$$

- These fixed points are isolated, and labelled by permutations σ:

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{(y-y^{-1})^{n-1}} \sum_{\sigma} s(\sigma)\, y^{\sum_{i<j}\alpha_{\sigma(i)\sigma(j)}}\,, \qquad s(\sigma) = 0, \pm 1$$

Manschot, BP, Sen 2010

The Coulomb index from localization II

- E.g. for $n = 2$, $\mathcal{M}_2 = S^2$, $J_3 = \alpha_{12} \cos\theta$:

$$g(\{\alpha_1, \alpha_2\}, y) = \frac{(-1)^{\alpha_{12}}}{1/y - y} \left(\underbrace{y^{+\alpha_{12}}}_{\text{North pole}} - \underbrace{y^{-\alpha_{12}}}_{\text{South pole}} \right) \xrightarrow{y \to 1} \pm \alpha_{12}$$

- E.g. for $n = 3$ with $\alpha_{12} > \alpha_{23}$, there are 4 collinear configurations:

$$g(\{\alpha_i\}, y) = \frac{(-1)^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}{(y-1/y)^2} \times \left[\underbrace{y^{\alpha_{13}+\alpha_{23}+\alpha_{12}}}_{(123)} - \underbrace{y^{-\alpha_{13}-\alpha_{23}+\alpha_{12}}}_{(312)} - \underbrace{y^{\alpha_{13}+\alpha_{23}-\alpha_{12}}}_{(213)} + \underbrace{y^{-\alpha_{13}-\alpha_{23}-\alpha_{12}}}_{(321)} \right]$$

$$\xrightarrow{y \to 1} \pm \langle \alpha_1, \alpha_2 \rangle \, \langle \alpha_1 + \alpha_2, \alpha_3 \rangle$$

The Coulomb index from localization III

- The same formula can be obtained can be obtained by quantizing the Higgs branch. Allowing for non-Abelian gauge groups, one arrives at the general refined wall-crossing formula:

$$\Delta\bar\Omega(\gamma,y)=\sum_{n\geq 2}\sum_{\substack{\gamma=\alpha_1+\dots+\alpha_n\\ \alpha_i=M_i\gamma_1+N_i\gamma_2}}\frac{g(\{\alpha_i\},y)}{|\mathrm{Aut}(\{\alpha_i\})|}\prod_{i=1}^n\bar\Omega(\alpha_i,y)$$

  where

$$\bar\Omega(\gamma,y)\equiv\sum_{d|\gamma}\frac{1}{d}\frac{y-1/y}{y^d-y^{-d}}\Omega(\gamma/d,y^d)$$

  Manschot BP Sen 2011

- This turns out to be equivalent to the Kontsevich-Soibelman and Joyce-Song formulae, but the physical interpretation is transparent !

Outline

The Coulomb branch formula I

- The wall-crossing formula suggests that it should be possible to express the BPS index $\Omega(\gamma; u; y)$ as a sum of bound states of a set of elementary, absolutely stable constituents, carrying index $\Omega_S(\alpha_i, y)$, independent of u. Naively,

$$\bar{\Omega}(\gamma; u, y) = \sum_{\gamma=\sum \alpha_i} \frac{g(\{\alpha_i\}, \{c_i\}; y)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_i \bar{\Omega}_S(\alpha_i, y)$$

The subscript S stands for 'stable', or historically, 'single-centered'.

Manschot BP Sen 2011

- Things are not quite so simple, because for general $\{\alpha_i\}$, the BPS phase space $\mathcal{M}_n$ may be non-compact due to scaling solutions. The sum over collinear solutions $g(\{\alpha_i\}, \{c_i\}; y)$ is then not necessarily a symmetric Laurent polynomial.

Scaling solutions I

- E.g., consider the 3-node quiver

$$\begin{tikzcd} & 1 \arrow[dr, "a"] & \\ 3 \arrow[ur, "c"] & & 2 \arrow[ll, "b"] \end{tikzcd} \qquad \begin{array}{l} 0 < a < b+c \\ 0 < b < c+a \\ 0 < c < a+b \end{array}$$

There exist scaling solutions of Denef's equations

$$\frac{a}{r_{12}} - \frac{c}{r_{13}} = c_1 , \quad \frac{b}{r_{23}} - \frac{a}{r_{12}} = c_2 , \quad \frac{c}{r_{31}} - \frac{b}{r_{23}} = c_3 ,$$

with $r_{12} \sim a\,\epsilon, r_{23} \sim b\,\epsilon, r_{13} \sim c\,\epsilon, \vec{J}^2 \sim \epsilon^2$ as $\epsilon \to 0$.

Scaling solutions II

- For $c_1, c_2 > 0$, the only collinear configurations are (123) and (321), leading to a rational function rather than a Laurent polynomial,

$$g(\{\alpha_i\}, y) = \frac{(-1)^{a+b+c}(y^{a+b-c} + y^{-a-b+c})}{(y-1/y)^2}$$

The Coulomb branch formula I

- We propose to repair this by replacing $\Omega_S(\alpha; y)$ on the r.h.s. by

$$\Omega_S(\alpha; y) \to \Omega_S(\alpha; y) + \sum_{\substack{\{\beta_i \in \Gamma\}, \{m_i \in \mathbb{Z}\} \\ m_i \geq 1, \sum_i m_i \beta_i = \alpha}} H(\{\beta_i\}; \{m_i\}; y) \prod_i \Omega_S(\beta_i; y^{m_i})$$

- $H(\{\beta_i\}; \{m_i\}; y)$ incorporates contributions from scaling solutions. It is determined recursively by the conditions
  - H is symmetric under $y \to 1/y$,
  - H vanishes at $y \to 0$,
  - the coefficient of $\prod_i \Omega_S(\beta_i; y^{m_i})$ in the expression for $\Omega(\sum_i m_i \beta_i; y)$ is a Laurent polynomial in y.

The formula is implemented in MATHEMATICA: `CoulombHiggs.m`

Manschot BP Sen 1302.5498; 1404.7154

The Coulomb branch formula II

- Returning to previous example, the prescription gives

$$H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) = \begin{cases} -\frac{2}{(y-y^{-1})^2} & , a+b+c \text{ even} \\ \frac{y+y^{-1}}{(y-y^{-1})^2} & , a+b+c \text{ odd} \end{cases}$$

so the claim is that the index of the Abelian 3-node quiver decomposes into

$$\begin{aligned}\Omega(\gamma_1+\gamma_2+\gamma_3,\{c_i\},y) = & \; g(\{\gamma_1,\gamma_2,\gamma_3\},\{c_i\},y) \\ & + H(\{\gamma_1,\gamma_2,\gamma_3\};\{1,1,1\};y) \\ & + \Omega_S(\gamma_1+\gamma_2+\gamma_3;y)\,.\end{aligned}$$

The Coulomb branch formula III

- Inverting the formula, one can express $\Omega_S(\gamma, y)$ in terms of $\Omega(\alpha, u, y)$ with $\alpha \leq \gamma$ in some chamber. The main claim is that the result does not depend on the chamber, and is a symmetric Laurent polynomial with integer coefficients.
- The analogy with single-centered black holes suggests that the states counted by $\Omega_S(\gamma, y)$ should carry zero angular momentum, i.e. $\Omega_S(\gamma, y)$ should be independent of y. If so, the formula becomes predictive !
- Actually, things are more subtle: $\Omega_S(\gamma)$ should be independent of y, the fugacity for the angular momentum J_3, but may depend on t, the fugacity for the R-symmetry charge I_3. The formula then gives $\Omega(\gamma, u, y, t)$ in terms of $\Omega_S(\gamma, t)$.This reduces to the protected spin character when $t = y$.

Outline

The exact index of quiver quantum mechanics I

- In the context of quiver quantum mechanics, $\Omega(\gamma, y, t, \{c_i\})$. can be computed from the cohomology of the Higgs branch:

$$\Omega(\gamma; c_i; y, t) = \sum_{p,q=0}^{2d} h_{p,q}(\mathcal{M}_H)\,(-y)^{p+q-d}\, t^{p-q}$$

- For quivers without closed loop and γ primitive, the Hodge numbers were computed by Reineke. The result agrees with the Coulomb branch formula upon setting $\Omega_S(\gamma) = 1$ when $\gamma = \gamma_i$, and zero otherwise !
- For quivers with loops, the Hodge numbers can sometimes be computed by hand. E.g. the 3-node Abelian quiver above, $\Omega_S(\gamma_1 + \gamma_2 + \gamma_3; y)$ turns out to be independent of (y, t) and grow exponentially with (a, b, c) !

Bena Berkooz El Showk de Boer van den Bleeken 2012

The exact index of quiver quantum mechanics II

- For general non-Abelian quivers with loop and generic superpotential, the protected spin character $\Omega(\gamma; c_i; y = t)$ can be computed using localization:

$$\Omega(\gamma; c_i, y = t) = \frac{1}{|W|} \sum_p \mathrm{JK-Res}_{(p,c_i)} \left[Z(u,y) \, \mathrm{d}^r u \right]$$

  where $Z(u,y)$ is a product of one-loop factors associated to vector and chiral multiplets, p runs over all isolated intersections of singular hyperplanes, and $\mathrm{JK-Res}_{(p,c_i)}$ is the Jeffrey-Kirwan residue at p for stability condition c_i.

  Benini Eager Hori Tachikawa 13; Hori Kim Yi 14

- All examples studied so far support the formula, with $\Omega_S(\gamma, t)$ being non-zero when γ has support on nodes joined by a loop, and being independent of y.

  MPS, Lee Wang Yi 12-13

Conclusion I

- In this talk, we have encountered three different types of indices:
  1. the BPS index $\Omega(\gamma, u)$, which counts single-particle states, but jumps across walls of marginal stability;
  2. the Witten index $\varpi(R, \gamma, u)$, which counts multi-particule states and is smooth across the walls, but depends on the temperature $1/R$;
  3. the stable index $\Omega_S(\gamma)$, which is independent of u and R, and seems to count absolutely stable, 'single-centered' states.
- In the context of string vacua, $\Omega_S(\gamma)$ should count states associated to single-centered black holes, and is the microscopic counterpart to the 'quantum entropy function'.
- Finding all possible decompositions $\gamma = \sum \alpha_i$ corresponding to regular multi-centered solutions is a tall order, only partly addressed by the 'split attractor flow conjecture' of Denef and Moore.

Conclusion II

- In the context of quiver quantum mechanics, the prescription to compute $\Omega_S(\gamma)$ from $\Omega(\gamma, u)$ is perfectly well defined, albeit cumbersome. Can one compute $\Omega_S(\gamma)$ directly from the residue formula, and prove the Coulomb branch formula ?
- $\Omega_S(\gamma)$ seems to capture the additional 'pure Higgs' degrees of freedom needed to reconcile the BPS spectrum on the Higgs branch with the Coulomb branch picture. Can one characterize these states from the Higgs branch point of view ?
- From the point of view of algebraic geometry, this seems to give a powerful structural decomposition of the cohomology of the moduli space of quiver representations. In particular, it gives a way to compute the Hodge polynomial in all chambers from the χ_y genus in any chamber !

Thank you for your attention !