

A Landscape of Field Theories

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Based on work to appear (shortly!) with Daniel Robbins and my student, Travis Maxfield.

Today, I want to convey a single idea about field theory.

The idea precisely involves "Geometry in Gauge Theories and String Theory" so it actually fits the workshop theme well.

It's still in the stage of clarifying final issues so I'll be less concerned about details and more concerned with describing the idea.

In the past few years, there have been programs underway to associate field theories with geometries in at least two ways:

Approach 1:

M-theory, F-theory or string theory on local or global geometries can give rise to field theories, perhaps coupled to gravity.

i.e. F-theory on a local elliptic CY 3-fold giving rise to many (1,0) theories.

This approach is known as “geometric engineering.”

Approach 2:

The second approach is the one we will consider today.

Take a field theory in some dimension. A case of prime interest to us today is the (2,0) theory in $D=6$.

This is the theory on coincident M5-branes in M-theory. For a single M5-brane, the theory is free consisting of the following bosonic fields:

$$h_3 = *h_3, \quad h_3 = db_2$$

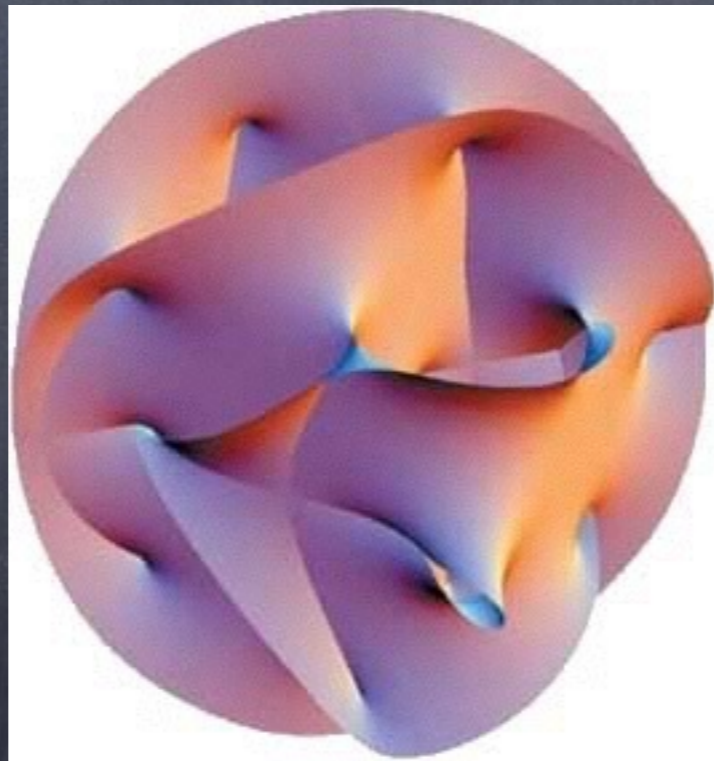
$$\phi^i, \quad i = 1, \dots, 5$$

$$Spin(5, 1) \times Spin(5)_R$$

The $(2,0)$ theory has maximal supersymmetry.

Compactify the theory on a geometry, perhaps with punctures or boundaries, to construct a lower-dimensional field theory.

For example:



$$(2, 0) \text{ on } \mathcal{M}_4 \rightarrow D = 2 \text{ QFT}$$

Let's do an example. Consider the free (2,0) theory compactified on a K3 surface.

$$b_2 \rightarrow \alpha_{\pm}^i \omega_i^{\pm}, \quad \omega_i^{\pm} \in H^2(K3, \mathbb{Z})$$

This gives 22 D=2 scalars.

These bosons are **chiral** with 3 right-moving scalars from the 3 SD 2-forms and 19 left-moving scalars from the 19 ASD 2-forms. The scalars are associated to the lattice,

$$\Gamma^{(3,3)} \oplus \Gamma^{(0,16)}$$

and parametrize the moduli space:

$$\frac{O(3, 19)}{O(3) \times O(19) \times O(\Gamma^{3,19})}$$

This is the heterotic string on a 3-torus with (0,8) world-sheet supersymmetry.

It is a free theory in D=2.

M-theory on $K3$



Heterotic on T^3

A wrapped M5 gives the heterotic string, while a wrapped NS5 in heterotic gives the M2-brane.

As we just saw, depending on the properties of the geometry, the resulting lower-dimensional field theory could be supersymmetric, chiral etc.

Compactifications to $D=0$ of a Euclidean field theory give partitions functions.

For the case of supersymmetric partition functions, the very powerful tool of localization provides a way to obtain exact results about compactified supersymmetric field theories.

(Pestun,...)

We've seen the usefulness of this computational tool repeatedly in this workshop.

The natural question to ask is: which geometries preserve supersymmetry?

From a string perspective, one class of such geometries can be found by studying branes wrapping supersymmetric cycles, like the case we just studied.

However, there is a more general answer which includes spaces like spheres.

To see how this comes about, one couples a SUSY field theory to background SUGRA.

However, the background SUGRA need not be on-shell!

As an example, take an $N=1$ $D=4$ field theory and couple it to $N=1$ SUGRA. The choice of SUGRA matters and there is more than one choice. The different choices differ in the structure of their auxiliary fields.

(Festuccia, Seiberg,...)

Allowing the auxiliary fields in the SUGRA theory to be off-shell, one imposes the condition that the gravitino variation vanishes:

$$\begin{aligned}\delta\Psi_{\mu}^{\alpha} &= -2\nabla_{\mu}\zeta^{\alpha} + \frac{i}{3} \left(M(\epsilon\sigma_{\mu}\bar{\zeta})^{\alpha} + 2b_{\mu}\zeta^{\alpha} + 2b^{\nu}(\zeta\sigma_{\nu\mu})^{\alpha} \right) \\ \delta\bar{\Psi}_{\mu\dot{\alpha}} &= -2\nabla_{\mu}\bar{\zeta}_{\dot{\alpha}} - \frac{i}{3} \left(\bar{M}(\zeta\sigma_{\mu})_{\dot{\alpha}} + 2b_{\mu}\bar{\zeta}_{\dot{\alpha}} + 2b^{\nu}(\bar{\zeta}\bar{\sigma}_{\nu\mu})_{\dot{\alpha}} \right)\end{aligned}$$

Now one finds lots of solutions like AdS₄ with

$$M = \bar{M} = -\frac{3}{r}, \quad b_{\mu} = 0.$$

Or a sphere $S^3 \times \mathbb{R}$,

$$M = \overline{M} = b_i = 0, \quad b_0 = -\frac{3}{r}.$$

In general, a background $(g, M, \overline{M}, b_\mu)$ with vanishing gravitino variation implies some unbroken supersymmetry.

This approach is systematic and explains why the extra couplings needed in the Lagrangian of the theory on a background \mathcal{M} terminates at order 2,

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L}_{\mathcal{M}}^{(0)} + \delta\mathcal{L}_{\mathcal{M}} = \sum_{n=0}^2 \frac{1}{r^n} \mathcal{L}_{\mathcal{M}}^{(n)}.$$

So far, I've told you nothing new, but recounted a beautiful story that is still being developed.

When there are additional global symmetries, like R-symmetries, one can turn on additional backgrounds.

Conventionally, studying a quantum field theory requires a choice of space-time metric and a choice of background for any global symmetries of the theory.

I want to argue that this is all part of a much larger story which, roughly, parallels the construction of the string landscape.

To see this generalization, let's study a specific example motivated from string theory.

Because the example comes from string theory, it will be the "on-shell" version of what should eventually be a complete "off-shell" story.

The String Landscape

Bulk Physics

The string landscape usually refers to the unproven belief that string theory possesses a huge number of metastable de Sitter vacua.

(Feng, March-Russell, S.S., Wilczek; Bousso, Polchinski;...)

Today I want to be more conservative and use the term "landscape" to refer to the huge number of supersymmetric and non-supersymmetric Minkowski flux vacua of M-theory and string theory.

I'll come back to some issues with the non-susy case momentarily.

Consider M-theory on $\mathbb{R}^{2,1} \times \mathcal{M}_8$.

At long wavelengths, we look for solutions of D=11 SUGRA plus higher derivative corrections.

To preserve supersymmetry, we demand that:

$$\delta\psi_\mu = \nabla_\mu \epsilon + \frac{1}{12} (\gamma_\mu G_4 - 3G_{4\mu}) \epsilon = 0.$$

This extension of the killing spinor condition to include flux leads to a study of "G-structures" as we heard in Dan Waldram's talk.

One class of solutions involves taking $\mathcal{M}_8 = CY_4$.

(Becker & Becker)

Let's focus on this class of solutions with a warped metric of the form:

$$ds^2 = \Delta(y)^{-1} \eta dx^2 + \Delta^{1/2}(y) ds^2_{\mathcal{M}_8}.$$

This is a conformally CY metric.

If the internal space is compact, there are no solutions of D=11 SUGRA.

Compact solutions do exist if we include higher derivative interactions in M-theory,

$$S_8 = \int \sqrt{-g} R^4 + \dots + C_3 \wedge X_8,$$

and solve the Gauss law constraint:

$$N_{M_2} + \frac{1}{2} \int \frac{G}{2\pi} \wedge \frac{G}{2\pi} = \frac{\chi(\mathcal{M}_8)}{24}.$$

Solving the SUGRA eom implies a constraint on the flux; namely,

$$G_4 \in H^{(4,0)} \oplus H^{(2,2)} \oplus H^{(0,4)}.$$

This generically breaks SUSY.

At large volume the breaking scale is below the KK scale. From a low-energy space-time SUGRA perspective, the breaking is by a space-time superpotential of the form:

$$W = \int \Omega_4 \wedge G_4.$$

(Gukov, Vafa, Witten)

I want to stress that for compact spaces, the SUSY broken backgrounds are not solutions of SUGRA.

They are also not solutions of SUGRA plus higher derivative corrections.

Higher order corrections to the space-time Kahler potential will generically break the no-scale symmetry of the theory. Here's an example,

$$K_{cl} = -3 \log(V) \rightarrow -3 \log(V + \xi).$$

These corrections to no-scale lead to a physical potential and a time-dependent background,

$$V = e^K \left(K^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 4|W|^2 \right).$$

So we have to be careful with the SUSY breaking case. On a non-compact space, the SUSY breaking flux can solve the D=11 SUGRA eom with a static solution.

For the moment, let's insist on preserving some supersymmetry. We have 2 cases:

(1) To preserve N=2 (4 real supercharges),

$$G_4 \in H^{(2,2)}(CY_4) \quad J \wedge G_4 = 0.$$

(2) To preserve N=1 (2 real supercharges),

$$G_4 \sim \alpha \left(J \wedge J + \frac{3}{2} \text{Re} \Omega^{(4,0)} \right) + G_4^{(2,2)}.$$

(Tsimpis)

Let me summarize the effect of turning on flux:

For a given topological G -flux, there are typically many choices of metric solving the equations of motion; the different choices can even preserve different amounts of supersymmetry.

Many moduli gain masses and there is a “landscape” of solutions.

(Dasgupta, Rajesh, S.S.)

The Field Theory Landscape

Brane Physics

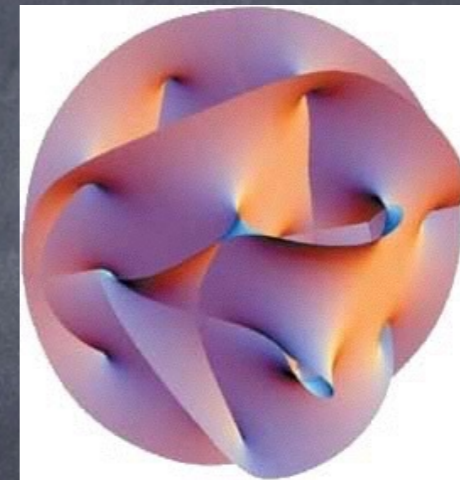
To understand the features of interest to us today, let's specialize further to an example that directly connects with our earlier discussion of reducing the $D=6$ $(2,0)$ theory on $K3$.

Consider M-theory on $\mathcal{M} = K3 \times K3$. (Dasgupta, Rajesh, S.S.)

This is one of the most heavily studied examples of a flux compactification. The geometry preserves N=4 SUSY.

We can relax the condition that the geometry is compact and also consider $\mathcal{M} = K3 \times ALE$.

$$G_4 \in H^4(\mathcal{M}, \mathbb{Z})$$



Think of the flux as supported on a 2-cycle of one K3 and a 2-cycle of the other K3/ALE.

To solve the equations of motion, the flux is again of specific Hodge type:

$$G_4 \in H^{(4,0)} \oplus H^{(0,4)} \oplus H^{(2,2)}$$

One can preserve $N=0$, $N=1$, $N=2$ and $N=4$ space-time SUSY, depending on the choice of flux and metric.

$$N=4: \quad G_4 \sim (1,1) \wedge (1,1)$$

$$N=2: \quad G_4 \sim (2,0) \wedge (0,2) + (1,1) \wedge (1,1)$$

Now wrap an M5-brane on the K3, as we did before. Note that,

$$G_4 = dC_3$$

and that the chiral tensor on the M5 couples to the flux via,

$$h_3 - C_3 \leftrightarrow F_2 - B_2.$$

Let's KK reduce on K3 to get a string as before.

$$C_3 \rightarrow A_{\pm}^i \omega_i^{\pm}, \quad \omega_i^{\pm} \in H^2(K3, \mathbb{Z}).$$

So the coupling on the M5-brane,

$$h_3 - C_3 \rightarrow \partial_{\pm} \alpha_{\pm}^i - A_{\pm}^i.$$

The chiral bosons are now interacting. The resulting D=2 theory preserves (0,2), or (0,4) world-sheet SUSY for the N=2 and N=4 cases.

The N=0 and N=1 cases are more mysterious but we suspect a (0,1) SUSY is preserved in both those cases.

The choice of flux changes the theory on the string. There is a landscape that follows directly from the string landscape.

A couple of comments:

(1) This is a new degree of freedom.

Putting a field theory on a curved space means coupling the stress-energy tensor to a background metric.

The flux corresponds to turning on operators in the field theory charged under: $Spin(5,1) \times Spin(5)_R$

This should lead to a kind of "G-structure" story for field theory, including the off-shell cases.

(2) The on-shell constraints imposed by string theory can almost certainly be relaxed.

We expect enormously more SUSY backgrounds of this type than from pure metric backgrounds. Any SUSY cycle of a flux vacuum should work in the "on-shell" case.

(3) One gets new partition functions, which are potentially computable by localization.

To see this in our example, compactify the resulting string on T^2 and compute the elliptic genus for models with at least (0,2) world-sheet SUSY.

This computation should be describable in the D=6 theory.

(4) There should be versions of this story reducing from $6 \rightarrow 4$, $6 \rightarrow 3$, $4 \rightarrow 2$ etc.

Hopefully we can learn much more about supersymmetric field theory!

(5) The construction involves coupling a brane to higher-dimensional SUGRA. Are all SUGRA preserving metrics and tensor field backgrounds describable this way, or are more general structures possible?

Can we rephrase this construction in the language of the $D=6$ (2,0) theory coupled to some $D=6$ SUGRA theory?

Let's return to the story and fill in a few details.

First: can we say something about the string we get from the wrapped M5?

$K3 \times ALE$



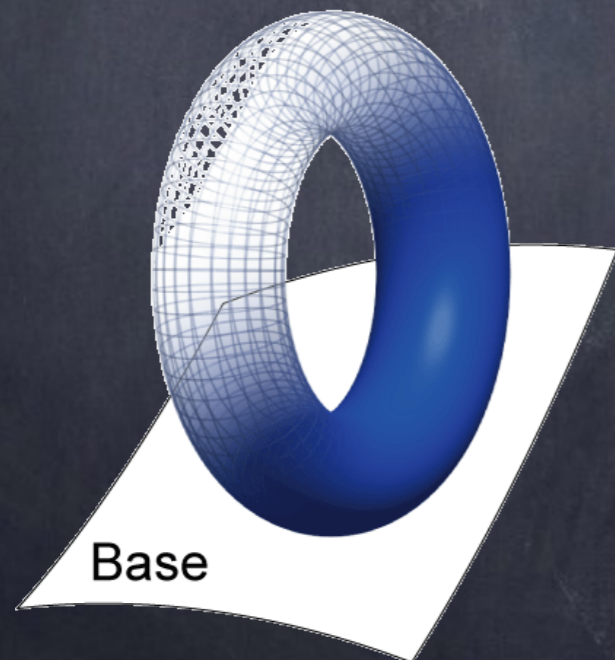
Heterotic string on T^3

Let's extend the duality.

A fiberwise application of duality gives a heterotic solution with metric,

$$ds_{\text{het}}^2 = \eta dx^2 + \Delta(y) ds_{ALE}^2 + \sum_{i=1}^3 |d\theta_i + A_i(y)|^2,$$

where the torus fibration is specified by the G-flux.



There is also a heterotic H_3 flux again determined by the M-theory G_4 flux. This is a torsional heterotic solution.

In $D=3$, such solutions are not well studied. In $D=4$, they are non-Kähler spaces satisfying:

$$\mathcal{H}_3 = i(\partial - \bar{\partial})J,$$

$$d\mathcal{H}_3 = \frac{\alpha'}{4}(\text{tr } R \wedge R - \text{tr } F \wedge F).$$

(Strominger)

The more general situation involves a supersymmetric 4-submanifold of a CY_4 .

In this case, the resulting string generalizes the MSW string. Its left and right-moving central charges should scale in an interesting way with M5-brane charge.

(Maldacena, Strominger, Witten)

The last point concerns the world-volume supersymmetry. The M5-brane has the following SUSY transformations in flat space:

$$\delta b_{\mu\nu} = -i\bar{\epsilon}\Gamma_{\mu\nu}\Theta$$

$$\delta\phi^i = -i\bar{\epsilon}\gamma^i\Theta$$

$$\delta\Theta = \left(\frac{1}{12}h^{\mu\nu\lambda}\Gamma_{\mu\nu\lambda} + \Gamma^\rho\partial_\rho\phi^i\gamma^i \right) \epsilon.$$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \quad \{\gamma^i, \gamma^j\} = 2\eta^{ij}.$$

It's easy to see how to couple a world-volume metric and a R-symmetry background. How does flux modify the SUSY transformations?

Approach 1:

Write down all couplings compatible with the symmetries of the problem and check closure of the SUSY algebra. A painful exercise!

Approach 2:

Use a Green-Schwarz-like description of the brane:

$$\mathcal{W}_{6|0} \rightarrow \mathcal{M}_{11|32}.$$

This is described by a superembedding, $Z^A(\sigma) = (x^M, \Theta^a)$.

We then gauge fix bulk diffeomorphisms and Kappa symmetry.

Supersymmetry acts on the superspace coordinates,

$$\delta\Theta^a = \epsilon^a, \quad \delta x^M = \bar{\epsilon}\Gamma^m\Theta.$$

Kappa symmetry reduces the 32 bulk fermions to 16,

$$\delta_\kappa\Theta = (1 + \Gamma_\kappa)\kappa, \quad \Gamma_\kappa^2 = 1.$$

All world-volume fields are pull-backs of superfields by Z .
For example,

$$C_{ijk}(Z) = Z^* C = \partial_i Z^A \partial_j Z^B \partial_k Z^c C_{ABC},$$

and each superfield has a theta expansion,

$$C_{MNP} = C_{MNP} + O(\Theta^2),$$

$$C_{aMN} = -\frac{i}{2} (\bar{\Theta} \Gamma_{MN})_a.$$

Let's skip much of the details and jump to the point where we fix static gauge,

$$X^M(\sigma) = (\sigma^\mu, \phi^i(\sigma)).$$

We then fix kappa symmetry and look for surviving world-volume supersymmetries. The variations now take the schematic form,

$$\delta\phi^i = \bar{\Theta}_- \gamma^i \epsilon_+,$$

$$\delta\mathcal{H}_{\mu\nu\lambda} = -3i\bar{\epsilon}_+ \bar{\sigma}_{[\mu\nu} \left(\nabla_{\lambda]}^G \Theta_- \right)_-$$

$$\delta\Theta_- = 3D_\mu \phi^i \bar{\sigma}^\mu \gamma_i \epsilon_+ - \frac{1}{4} \mathcal{H}_{\mu\nu\lambda} \bar{\sigma}_+^{\mu\nu\lambda} \epsilon_+,$$

It's important to note that,

$$\mathcal{H}_3 = db_2 - C_3$$

and that a "fluxed" covariant derivative appears,

$$\begin{aligned} (\nabla_\lambda^G \Theta_-)_- &= \partial_\lambda \Theta_- - \frac{1}{4} \delta_\lambda^\mu \omega_\mu^{\nu\rho} \bar{\sigma}_{\nu\rho} \Theta_- - \frac{1}{4} \delta_\lambda^\mu \omega_\mu^{ij} \gamma_{ij} \Theta_- \\ &+ \frac{1}{12} G_{\mu\nu\rho i} \left(\frac{1}{6} \delta_\lambda^\sigma \bar{\sigma}_\sigma^{\mu\nu\rho} \gamma^i \Theta_- - \delta_\lambda^\mu \bar{\sigma}^{\nu\rho} \gamma^i \Theta_- \right) \\ &+ \frac{1}{36} G_{\mu ijk} \left(\frac{1}{2} \delta_\lambda^\nu \bar{\sigma}^\nu_\mu \gamma^{ijk} \Theta_- - \delta_\lambda^\mu \gamma^{ijk} \Theta_- \right). \end{aligned}$$

We are in the process of checking closure conditions etc.
More to come soon...