Exact Path Integral for 3D Quantum Gravity

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based on the paper collaborated with Norihiro Iizuka and Akinori Tanaka: arXiv:1504.05991

Introduction

Quantum Gravity:

Difficulties:

- Perturbatively unrenormalizable
- Un-boundedness of action
- Summations over different topologies

Dual CFT
(for QG with asymptotic AdS)

Maldacena

Renormalizable, Positive definite action,
No geometries (for finite N).

However, no proof of the equivalence

In this talk,

we will try to compute QG partition function

from bulk theory.

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}}$$

(Is it possible?)

3d pure gravity
(with negative cosmological constant)

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right)$$

Simplest theory with "black holes"
(although no gravitons)

They are called BTZ black holes,
which have horizons and nonzero entropies

Banados-Teitelboim-Zanelli

We will use following relations:

3d pure gravity

⇕

3d Chern-Simons theory

⇕

3d SUSY Chern-Simons

3d SUSY Chern-Simons theory

We can use localization technique
to compute exact partition function
(with some assumptions)

The result for c=24 is

$$Z_{gravity} = J(q)J(\bar{q})$$

which is Z for Frenkel-Lepowsky-Meurman's CFT,
as Witten proposed before!

Plan

- **Introduction**
- **Review of 3d gravity**
- **Previous results**
- **Exact partition function using localization**

Review of 3d gravity

3d pure gravity

What we want to compute is

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}}$$

with

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell} \right) + S_{GH} + S_c$$

where

ℓ is AdS scale

$S_{GH} = \frac{1}{8\pi G_N} \int d^2x \sqrt{h} K$ is Gibbons-Hawking boundary term

K extrinsic curvature, and h the boundary metric

$$S_c = -\frac{1}{8\pi G_N} \int d^2x \sqrt{h}$$

Central charge of dual CFT from classical 3d gravity

Asymptotic symmetry of 3d QG with asymptotic AdS_3
is
Virasoro algebra with the central charge:

$$c = 3\ell/2G_N$$

Brown-Henneaux

Using this and Cardy formula,
we can reproduce
entropy of BTZ black hole (leading order).

Strominger

3d pure QG ⟺ 3D Chern-Simons (classical)

Achucarro-Townsend, Witten

Introducing $SL(2,\mathbf{C})$ gauge field A,

$$\left(\omega_\mu^a + \frac{i}{\ell} e_\mu^a\right) \frac{i}{2} \sigma_a dx^\mu = A \, , \left(\omega_\mu^a - \frac{i}{\ell} e_\mu^a\right) \frac{i}{2} \sigma_a dx^\mu = \bar{A}$$

where σ_a is Pauli matrix

we can show

$$\int_{\mathcal{M}} d^3x \; e\left(R(e,\omega) + \frac{2}{\ell^2}\right) + \int_{\partial\mathcal{M}} e^a \omega^a \; = i\ell\left(S_{CS}[\mathcal{A}] - S_{CS}[\overline{\mathcal{A}}]\right)$$

where $S_{CS}[A] := \int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$

3d pure QG ⟺ 3D Chern-Simons (classical)

Achucarro-Townsend, Witten

Therefore,

$$S_{gravity} = \frac{ik}{4\pi} S_{CS}[A] - \frac{ik}{4\pi} S_{CS}[\bar{A}] \equiv S_{gauge}$$

$$\text{where} \quad k \equiv \frac{\ell}{4G_N}$$

$$\text{Charge quantization condition:} \quad k/4 \in \mathbf{Z}$$

Witten

$$\Longrightarrow \quad c = 6k \in 24\mathbf{Z}$$

Previous results

What we will compute:

Partition function of Euclidian 3d gravity with asymptoric AdS with torus boundary.

complex structure τ

solid torus

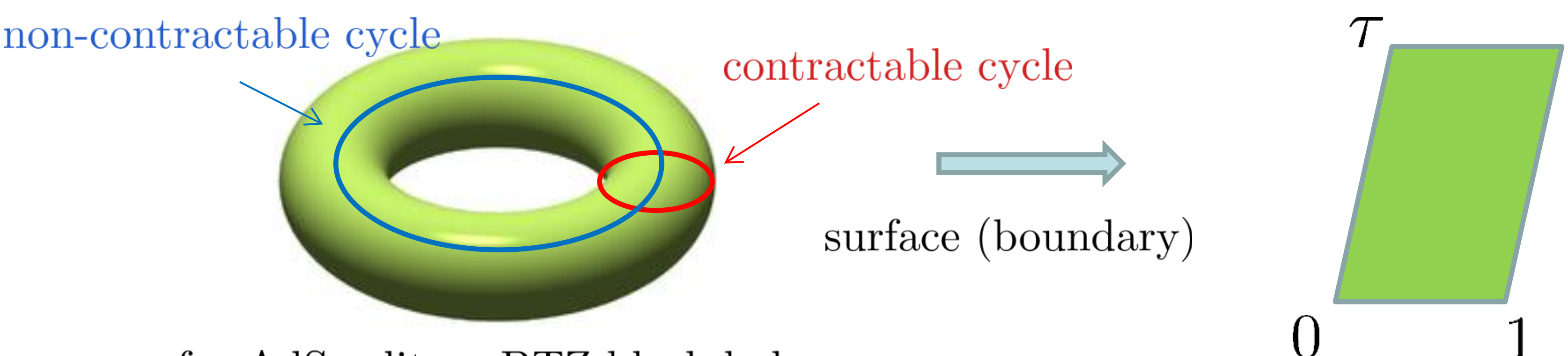
surface (boundary)

for AdS soliton, BTZ black holes

Important facts of the theory:

only solutions of e.o.m. are
AdS space and BTZ black holes.

all of them are topologically solid torus $(= D^2 \times S^1)$,
but, contractable cycles of torus are different.



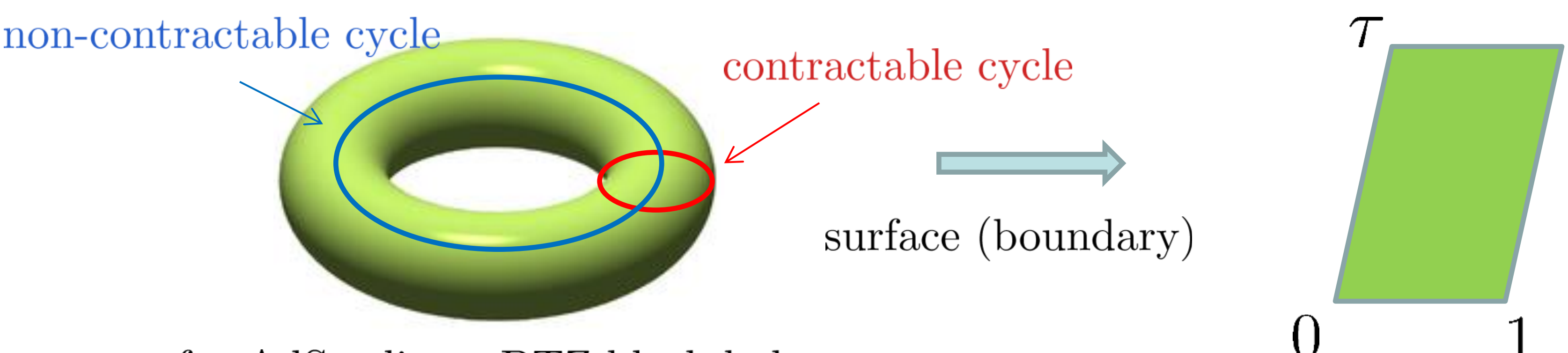
for AdS soliton, BTZ black holes

Important facts of the theory:

Moreover, they are related by SL(2,Z) action
and the on-shell action is

$$e^{-S} = Z_{c,d}\overline{Z}_{c,d}, \quad Z_{c,d} \equiv e^{-2\pi i \frac{c}{24}\frac{a\tau+b}{c\tau+d}}$$

where $c \geq 0$ and $(c,d)_{GCD} = 1$



for AdS soliton, BTZ black holes

3d pure QG ⟺ 2d extremal CFT

Witten

From CS description, Witten suggest that partition function of 3d gravity may be holomorphically factorized:

$$Z_{gravity} = Z(q)Z(\bar{q})$$

where $q = \exp(2\pi i\tau)$

For c=24, there is unique CFT by FLM with

$$Z(q) = J(q) = \frac{1}{q} + 196884q + \cdots$$

$J(q)$ is modular invariant has pole at $q = 0$

Frenkel-Lepowsky-Meurman

J-function as a (classical) partition function

Manschot

Pah-integrals over different topologies

$$Z_{gravity} \qquad = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}} \to \sum_{topology} \int \mathcal{D}g_{\mu\nu}^{sector} e^{-S_{gravity}}$$

Assuming only solutions of e.o.m. contribute

$$\to \sum_{topology} e^{-S_{gravity}}|_{on-shell} = \sum_{\substack{c \geq 0, \\ (c,d)=1}} Z_{c,d} \overline{Z}_{c,d}$$

Including "non-geometric" contributions

$$\to Z(q)\overline{Z}(\bar{q}) \qquad \text{where } Z(q) = \sum_{\substack{c \geq 0, \\ (c,d)=1}} Z_{c,d}$$

J-function as a (classical) partition function

Manschot

$$Z(q) = \sum_{\substack{c \geq 0, \\ (c,d)=1}} Z_{c,d}(\tau)$$

Adding a constant as a regularization (Rademacher sum)

$$\to Z_{0,1} + \sum_{\substack{c > 0, \\ (c,d)=1}} \left(Z_{c,d}(\tau) - Z_{c,d}(\infty)\right) = R^{-\frac{c}{24}}(q)$$

$$\text{where} \quad R^{(m)}(q) \equiv e^{2\pi i m \tau} + \sum_{\substack{c > 0, \\ (c,d)=1}} \left(e^{2\pi i m \frac{a\tau + b}{c\tau + d}} - e^{2\pi i m \frac{a}{c}} \right)$$

In particular, $R^{(-1)}(q) = J(q)$

J-function as a (classical) partition function

Manschot

However,

we do not know

why there is no loop corrections,
nor,
why including non-geometric contributions

Partition function (classical + 1-loop)

Maloney-Witten

Perturbatively, 3d gravity is 1-loop exact, and for AdS background we have

$$Z(q) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} (1 - q^n) ,$$

where

$\prod_{n=2}^{\infty} (1 - q^n)$ is from Virasoro decendants of vacuum

Witten

Giombi-Maloney-Yin

Partition function (classical + 1-loop)

Maloney-Witten

Thus, naively

$$Z^{gravity} = \sum_{\substack{c \geq 0, \\ (c,d)=1}} Z_{c,d} \overline{Z}_{c,d}$$

$$Z_{c,d} = \text{modular transform of } Z_{0,1} = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} (1 - q^n) \; ,$$

However,

coefficient of q^n in $Z^{gravity}$ can be non-integer or negative.

Thus, $Z^{gravity} \neq$ CFT partition function

Partition function (classical + 1-loop)

Maloney-Witten

Even if we assume holomorphic factorized one

$$Z^{gravity} = \left(\sum_{\substack{c\geq 0,\\ (c,d)=1}} Z_{c,d} \right) \left(\sum_{\substack{c\geq 0,\\ (c,d)=1}} \overline{Z}_{c,d} \right)$$

We can not have J-function for c=24

because of $\prod_{n=2}^{\infty}(1-q^n)$

So, there are many mysteries…

Path-integral of 3D Gravity using localization

Iizuka-Tanaka-ST

We will use following relations:

3d pure gravity

⇕

3d Chern-Simons theory

⇕

3d SUSY Chern-Simons

3d SUSY Chern-Simons theory

Let us consider

$$\text{3d } \mathcal{N}=2 \text{ vector multiplet } V=(A,\sigma,D,\overline{\lambda},\lambda)$$

with the SUSY action

$$S_{SCS}[V]=S_{CS}[A]+\int d^3x\sqrt{g}\ \mathrm{Tr}\Big(-\overline{\lambda}\lambda+2D\sigma\Big)$$

where additional fermion and bosons are auxiliary fields.

Therefore,

$$\int \mathcal{D}A\, e^{-S_{CS}[A]} \approx \int \mathcal{D}V\, e^{-S_{SCS}[V]}$$

SUSY CS theory on solid torus

The solid torus

$$ds^2 = d\theta^2 + \cos^2\theta\left(d\varphi^2 + \tan^2\theta dt_E^2\right)$$

for $0 \leq \theta \leq \theta_0 < \pi/2\,, 0 \leq \varphi \leq 2\pi\,, 0 \leq t_E \leq 2\pi,$

where $\theta = \theta_0$ is a boundary torus with purely imaginary $\tau = i\beta = i\tan\theta_0$

Following Dirichlet boundary condition keeps SUSY:

$$A_\varphi \to a_\varphi, \quad A_{t_E} \to a_{t_E}, \quad \sigma \to 0$$

$$\lambda \to e^{-i(\varphi - t_E)}\gamma^\theta\overline{\lambda}$$

Partition functions are related as

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}} \to \sum \int \mathcal{D}g_{\mu\nu}^{sector} e^{-S_{gravity}}$$

$$\to \int (\sum \mathcal{D}A)(\sum \mathcal{D}\bar{A}) e^{-\frac{ik}{4\pi}S_{CS}[A]+\frac{ik}{4\pi}S_{CS}[\bar{A}]}$$

$$\to \int \left(\sum \mathcal{D}V\right)\left(\sum \mathcal{D}\bar{V}\right) e^{-\frac{ik}{4\pi}S_{SCS}[V]+\frac{ik}{4\pi}S_{SCS}[\bar{V}]}$$

where we assumed $\sum \mathcal{D}g_{\mu\nu}^{sector} = \left(\sum_{\substack{c\geq 0,\\(c,d)=1}} \mathcal{D}A\right)\left(\sum_{\substack{c\geq 0,\\(c,d)=1}} \mathcal{D}\bar{A}\right)$

This will be justified by the localization

Using localization technique

Sugishita-ST

$$\int \mathcal{D}V e^{-\frac{ik}{4\pi}S_{SCS}[V]} = \int \mathcal{D}V e^{-\frac{ik}{4\pi}S_{SCS}[V]+tS_{SYM}[V]}$$

$$\rightarrow_{t\rightarrow\infty} Z_{classical} \times Z_{one-loop}$$

$$Z_{classical} = e^{ik\pi Tr(a_\varphi a_{t_E})}$$

$$Z_{one-loop} = \prod_{m\in\mathbf{Z}} \Big(m - \alpha(a_\varphi)\Big)$$

$$= e^{i\pi\alpha(a_\varphi)} - e^{-i\pi\alpha(a_\varphi)}$$

Saddle points equation:

$$F = 0,\ \sigma = 0$$

this corresponds to e.o.m. of gravity

the solution (BTZ) is

$$a_\varphi = \frac{1}{2i\beta}\sigma_3 = \frac{1}{2\tau}\sigma_3\,, \quad a_{t_E} = \frac{1}{2}\sigma_3$$

Note that
the solutions of e.o.m. should have
solid torus topology.

The partition function:

$$Z_{(c,d)} = \int \mathcal{D}V e^{-\frac{ik}{4\pi}(S_{SCS}[V])} = e^{ik\pi Tr(a_\varphi a_{t_E})} \times \left(e^{i\pi\alpha(a_\varphi)} - e^{-i\pi\alpha(a_\varphi)}\right)$$

$$Z_{(0,1)} = q^{-k_{eff}}(1-q) \text{ where } k_{eff} = (k+2)/4$$

Other contributions are obtained by SL(2,Z) action

$$Z_{(c,d)} = e^{-2\pi i k_{eff}\frac{a\tau+b}{c\tau+d}} - e^{-2\pi i k_{eff}\frac{a\tau+b}{c\tau+d}}$$

Rademacher sum:

$$Z_{hol}[q] \equiv Z_{(0,1)}(\tau) + \sum_{\substack{c>0,\\(c,d)=1}} \Big(Z_{c,d}(\tau) - Z_{c,d}(\infty) \Big)$$

$$= R^{(-k_{eff})}(q) - R^{(-k_{eff}+1)}(q)$$

$$R^{(m)}(q) \equiv e^{2\pi i m\tau} + \sum_{\substack{c>0,\\(c,d)=1}} \Big(e^{2\pi i m \frac{a\tau+b}{c\tau+d}} - e^{2\pi i m \frac{a}{c}} \Big)$$

For c=24, we have $Z_{0,1} = \frac{1}{q} - 1$

⇒ $$Z_{hol}[q] = J(q) + cosnt$$

$$Z_{gravity} = J(q)J(\bar{q})$$

How about the Virasoro decendants modes?

We have seen that

$$Z_{0,1} = q^{-k_{eff}}(1-q) \text{ where } k_{eff} = (k+2)/4$$

which can be rewritten as

$$Z_{0,1} = Z_{B-fermion} \times q^{-k_{eff}} \Pi_{n=2}^{\infty}(1-q^n)^{-1},$$

$$Z_{B-fermion} \equiv \Pi_{n=1}^{\infty}(1-q^n)$$

This is the 1-loop result of gravity

SUSY CS has extra fermion modes:

Using doubling trick: $x \equiv \theta_0 - \theta$
we will define

$$\psi_1(x) = \lambda_1(x)\theta(x) + e^{-i(\varphi - t_E)}\bar{\lambda}_1(-x)\theta(-x)$$

$$\psi_2(x) = \bar{\lambda}_2(x)\theta(x) - e^{+i(\varphi - t_E)}\lambda_2(-x)\theta(-x)$$

Then, the mass term is

$$\begin{aligned}\overline{\lambda}(x)\lambda(x) &= \lambda_1(x)\bar{\lambda}_2(x) - \lambda_2(x)\bar{\lambda}_1(x) \\ &= \psi_1(x)\psi_2(x) - \psi_1(-x)\psi_2(-x) \equiv \mathrm{sign}(x)\psi_1(x)\psi_2(x)\end{aligned}$$

domain wall type mass term,
massless "edge" modes on x=0

SUSY CS has extra fermion modes:

These fermion modes strictly localized at the boundary and then completely decoupled from other modes for $t \to 0$

Thus, we can think the theory essentially describe 3d pure gravity

Conclusion

- **“exact” partition function of 3d pure gravity**
- **Using SUSY extension and localziation**

Fin.