Exact Path Integral for 3D Quantum Gravity

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based on the paper collaborated with Norihiro Iizuka and Akinori Tanaka:
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Introduction
Quantum Gravity:

Difficulties:

• Perturbatively unrenormalizable
• Un-boundedness of action
• Summations over different topologies
Dual CFT
(for QG with asymptotic AdS)

Renormalizable, Positive definite action,
No geometries (for finite N).

However, no proof of the equivalence
In this talk, we will try to compute QG partition function from bulk theory.

$$Z_{\text{gravity}} = \int \mathcal{D}g_{\mu\nu} e^{-S_{\text{gravity}}}$$

(Is it possible?)
3d pure gravity
(with negative cosmological constant)

\[ S_{gravity} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left( R + \frac{2}{\ell^2} \right) \]

Simplest theory with “black holes”
(although no gravitons)

They are called BTZ black holes,
which have horizons and nonzero entropies

Banados-Teitelboim-Zanelli
We will use following relations:

3d pure gravity

\[\begin{array}{c}
\text{3d Chern-Simons theory} \\
\text{3d SUSY Chern-Simons}
\end{array}\]
3d SUSY Chern-Simons theory

We can use localization technique to compute exact partition function (with some assumptions)

The result for \( c = 24 \) is

\[
Z_{\text{gravity}} = J(q) J(\bar{q})
\]

which is \( Z \) for Frenkel-Lepowsky-Meurman’s CFT, as Witten proposed before!
Plan

• Introduction

• Review of 3d gravity

• Previous results

• Exact partition function using localization
Review of 3d gravity
3d pure gravity

What we want to compute is

$$Z_{gravity} = \int Dg_{\mu \nu} e^{-S_{gravity}}$$

with

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^3 x \sqrt{g} \left( R + \frac{2}{\ell} \right) + S_{GH} + S_c$$

where

$$\ell \text{ is AdS scale}$$

$$S_{GH} = \frac{1}{8\pi G_N} \int d^2 x \sqrt{h} K$$ is Gibbons-Hawking boundary term

$$K \text{ extrinsic curvature, and } h \text{ the boundary metric}$$

$$S_c = -\frac{1}{8\pi G_N} \int d^2 x \sqrt{h}$$
Central charge of dual CFT from classical 3d gravity

Asymptotic symmetry of 3d QG with asymptotic AdS_3 is Virasoro algebra with the central charge:

\[ c = \frac{3\ell}{2G_N} \]

Using this and Cardy formula, we can reproduce entropy of BTZ black hole (leading order).

Brown-Henneaux

Strominger
Introducing $SL(2, \mathbb{C})$ gauge field $A$,

$$
\left( \omega^a_\mu + \frac{i}{\ell} e^a_\mu \right) \frac{i}{2} \sigma_a dx^\mu = A,
\left( \omega^a_\mu - \frac{i}{\ell} e^a_\mu \right) \frac{i}{2} \sigma_a dx^\mu = \bar{A}
$$

we can show

$$
\int_{\mathcal{M}} d^3x \; e \left( R(e, \omega) + \frac{2}{\ell^2} \right) + \int_{\partial \mathcal{M}} e^a \omega^a = i\ell \left( S_{CS}[A] - S_{CS}[\bar{A}] \right)
$$

where $S_{CS}[A] := \int_M \text{Tr} \left( AdA + \frac{2}{3} A^3 \right)$
3d pure QG $\iff$ 3D Chern-Simons (classical)

Achucarro-Townsend, Witten

Therefore,

$$S_{\text{gravity}} = \frac{ik}{4\pi} S_{\text{CS}}[A] - \frac{ik}{4\pi} S_{\text{CS}}[\bar{A}] \equiv S_{\text{gauge}}$$

where $k \equiv \frac{\ell}{4G_N}$

Charge quantization condition: $k/4 \in \mathbb{Z}$

Witten

$$c = 6k \in 24\mathbb{Z}$$
Previous results
What we will compute:

Partition function of Euclidian 3d gravity with asymptoric AdS with torus boundary.

complex structure $\tau$

for AdS soliton, BTZ black holes
Important facts of the theory:

only solutions of e.o.m. are AdS space and BTZ black holes.

all of them are topologically solid torus (\(= D^2 \times S^1 \)), but, contractable cycles of torus are different.

non-contractable cycle

for AdS soliton, BTZ black holes

contractable cycle

surface (boundary)
Important facts of the theory:

Moreover, they are related by SL(2,Z) action and the on-shell action is

\[ e^{-S} = Z_{c,d} \overline{Z}_{c,d}, \quad Z_{c,d} \equiv e^{-2\pi i \frac{c}{24} \frac{a\tau + b}{c\tau + d}} \]

where \( c \geq 0 \) and \( (c, d)_{GCD} = 1 \)

for AdS soliton, BTZ black holes
3d pure QG $\leftrightarrow$ 2d extremal CFT

From CS description, Witten suggest that partition function of 3d gravity may be holomorphically factorized:

$$Z_{gravity} = Z(q)Z(\bar{q})$$

where $q = \exp(2\pi i \tau)$

For $c=24$, there is unique CFT by FLM with

$$Z(q) = J(q) = \frac{1}{q} + 196884q + \cdots$$

$J(q)$ is modular invariant has pole at $q = 0$

Frenkel-Lepowsky-Meurman
J-function as a (classical) partition function

Pah-integrals over different topologies

\[ Z_{\text{gravity}} = \int \mathcal{D}g_{\mu\nu} e^{-S_{\text{gravity}}} \to \sum_{\text{topology}} \int \mathcal{D}g_{\mu\nu}^{\text{sector}} e^{-S_{\text{gravity}}} \]

Assuming only solutions of e.o.m. contribute

\[ \to \sum_{\text{topology}} e^{-S_{\text{gravity}}}|_{\text{on-shell}} = \sum_{c \geq 0, \quad (c,d)=1} Z_{c,d} \bar{Z}_{c,d} \]

Including ”non-geometric” contributions

\[ \to Z(q) \bar{Z}(\bar{q}) \quad \text{where} \quad Z(q) = \sum_{c \geq 0, \quad (c,d)=1} Z_{c,d} \]
J-function as a (classical) partition function

\[ Z(q) = \sum_{\substack{c \geq 0, \\ (c,d)=1}} Z_{c,d}(\tau) \]

Adding a constant as a regularization (Rademacher sum)

\[ \rightarrow Z_{0,1} + \sum_{\substack{c>0, \\ (c,d)=1}} \left( Z_{c,d}(\tau) - Z_{c,d}(\infty) \right) = R^{-\frac{c}{24}} (q) \]

where \( R^{(m)}(q) \equiv e^{2\pi i m \tau} + \sum_{\substack{c>0, \\ (c,d)=1}} \left( e^{2\pi i m \frac{a \tau + b}{c \tau + d}} - e^{2\pi i m \frac{a}{c}} \right) \)

In particular, \( R^{-1}(q) = J(q) \)
J-function as a (classical) partition function

However,

we do not know

why there is no loop corrections,

nor,

why including non-geometric contributions
Partition function (classical + 1-loop)

Perturbatively, 3d gravity is 1-loop exact, and for AdS background we have

$$Z(q) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} (1 - q^n),$$

where

$$\prod_{n=2}^{\infty} (1 - q^n)$$ is from Virasoro descendents of vacuum
Partition function (classical + 1-loop)

Thus, naively

\[ Z^{\text{gravity}} = \sum_{c \geq 0, (c,d) = 1} Z_{c,d} \overline{Z}_{c,d} \]

\[ Z_{c,d} = \text{modular transform of } Z_{0,1} = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} (1 - q^n), \]

However, coefficient of \( q^n \) in \( Z^{\text{gravity}} \) can be non-integer or negative.

Thus, \( Z^{\text{gravity}} \neq \text{CFT partition function} \)
Partition function (classical + 1-loop)  

Even if we assume holomorphic factorized one

\[ Z_{\text{gravity}} = \left( \sum_{c \geq 0, \atop (c,d)=1} Z_{c,d} \right) \left( \sum_{c \geq 0, \atop (c,d)=1} \overline{Z}_{c,d} \right) \]

We cannot have J-function for \( c=24 \) because of 

\[ \prod_{n=2}^{\infty} (1 - q^n) \]
So, there are many mysteries...
Path-integral of 3D Gravity using localization

Iizuka-Tanaka-ST
We will use following relations:

3d pure gravity

\[ \text{3d Chern-Simons theory} \]

\[ \text{3d SUSY Chern-Simons} \]
3d SUSY Chern-Simons theory

Let us consider

3d $\mathcal{N} = 2$ vector multiplet $V = (A, \sigma, D, \bar{\lambda}, \lambda)$

with the SUSY action

$$S_{SCS}[V] = S_{CS}[A] + \int d^3x \sqrt{g} \ Tr\left(-\bar{\lambda}\lambda + 2D\sigma\right)$$

where additional fermion and bosons are auxiliary fields.

Therefore,

$$\int \mathcal{D}A \ e^{-S_{CS}[A]} \approx \int \mathcal{D}V \ e^{-S_{SCS}[V]}$$
SUSY CS theory on solid torus

The solid torus

\[ ds^2 = d\theta^2 + \cos^2 \theta \left( d\varphi^2 + \tan^2 \theta dt_{E}^2 \right) \]

for \( 0 \leq \theta \leq \theta_0 < \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi, 0 \leq t_E \leq 2\pi, \)

where \( \theta = \theta_0 \) is a boundary torus with purely imaginary \( \tau = i\beta = i \tan \theta_0 \)

Following Dirichlet boundary condition keeps SUSY:

\[ A_\varphi \rightarrow a_\varphi, \quad A_{tE} \rightarrow a_{tE}, \quad \sigma \rightarrow 0 \]

\[ \lambda \rightarrow e^{-i(\varphi - t_E)\gamma\theta\bar{\lambda}} \]
Partition functions are related as

\[ Z_{\text{gravity}} \]

\[ = \int \mathcal{D}g_{\mu\nu} e^{-S_{\text{gravity}}} \rightarrow \sum \int \mathcal{D}g_{\mu\nu} e^{-S_{\text{gravity}}} \]

\[ \rightarrow \int (\sum \mathcal{D}A)(\sum \mathcal{D}\bar{A}) e^{-\frac{ik}{4\pi} S_{CS}[A] + \frac{ik}{4\pi} S_{CS}[\bar{A}]} \]

\[ \rightarrow \int (\sum \mathcal{D}V)(\sum \mathcal{D}\bar{V}) e^{-\frac{ik}{4\pi} S_{CS}[V] + \frac{ik}{4\pi} S_{CS}[\bar{V}]} \]

where we assumed \( \sum \mathcal{D}g_{\mu\nu}^{\text{sector}} = \left( \sum_{c \geq 0, (c,d)=1} \mathcal{D}A \right) \left( \sum_{c \geq 0, (c,d)=1} \mathcal{D}\bar{A} \right) \)

This will be justified by the localization
Using localization technique

\[ \int \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]} = \int \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]+t S_{SYM}[V]} \]

\[ \to_{t \to \infty} Z_{\text{classical}} \times Z_{\text{one-loop}} \]

\[ Z_{\text{classical}} = e^{ik\pi Tr(a_\varphi a_{tE})} \]

\[ Z_{\text{one-loop}} = \prod_{m \in \mathbb{Z}} \left( m - \alpha(a_\varphi) \right) \]

\[ = e^{i\pi \alpha(a_\varphi)} - e^{-i\pi \alpha(a_\varphi)} \]
Saddle points equation:

$$F = 0, \quad \sigma = 0$$

this corresponds to e.o.m. of gravity

definition (BTZ) is

$$a_\varphi = \frac{1}{2i\beta} \sigma_3 = \frac{1}{2\tau} \sigma_3 \quad a_{t_E} = \frac{1}{2} \sigma_3$$

Note that

the solutions of e.o.m. should have
solid torus topology.
The partition function:

\[ Z_{(c,d)} = \int \mathcal{D}V e^{-\frac{ik}{4\pi} \langle SSQS[V] \rangle} = e^{ik\pi Tr(a_\varphi a_{tE})} \times \left( e^{i\pi \alpha(a_\varphi)} - e^{-i\pi \alpha(a_\varphi)} \right) \]

\[ Z_{(0,1)} = q^{-k_{eff}} (1 - q) \text{ where } k_{eff} = \frac{(k + 2)}{4} \]

Other contributions are obtained by SL(2,Z) action

\[ Z_{(c,d)} = e^{-2\pi i k_{eff} \frac{a\tau + b}{c\tau + d}} - e^{-2\pi i k_{eff} \frac{a\tau + b}{c\tau + d}} \]
Rademacher sum:

\[ Z_{hol}[q] \equiv Z_{(0,1)}(\tau) + \sum_{\substack{c>0, \\ (c,d)=1}} \left( Z_{c,d}(\tau) - Z_{c,d}(\infty) \right) \]

\[ = R^{(-k_{eff})}(q) - R^{(-k_{eff}+1)}(q) \]

\[ R^{(m)}(q) \equiv e^{2\pi im\tau} + \sum_{\substack{c>0, \\ (c,d)=1}} \left( e^{2\pi im \frac{a\tau+b}{c\tau+d}} - e^{2\pi im \frac{a}{c}} \right) \]

For \( c=24 \), we have \( Z_{0,1} = \frac{1}{q} - 1 \)

\[ Z_{hol}[q] = J(q) + \text{const} \]

\[ Z_{\text{gravity}} = J(q)J(\bar{q}) \]
How about the Virasoro descendents modes?

We have seen that

\[ Z_{0,1} = q^{-k_{eff}} (1 - q) \text{ where } k_{eff} = (k + 2)/4 \]

which can be rewritten as

\[ Z_{0,1} = Z_{B-fermion} \times q^{-k_{eff}} \prod_{n=2}^{\infty} (1 - q^n)^{-1}, \]

\[ Z_{B-fermion} \equiv \prod_{n=1}^{\infty} (1 - q^n) \]

This is the 1-loop result of gravity
SUSY CS has extra fermion modes:

Using doubling trick: \( x \equiv \theta_0 - \theta \)

we will define

\[
\psi_1(x) = \lambda_1(x)\theta(x) + e^{-i(\varphi-t_E)}\bar{\lambda}_1(-x)\theta(-x)
\]

\[
\psi_2(x) = \bar{\lambda}_2(x)\theta(x) - e^{+i(\varphi-t_E)}\lambda_2(-x)\theta(-x)
\]

Then, the mass term is

\[
\bar{\lambda}(x)\lambda(x) = \lambda_1(x)\bar{\lambda}_2(x) - \lambda_2(x)\bar{\lambda}_1(x)
\]

\[
= \psi_1(x)\psi_2(x) - \psi_1(-x)\psi_2(-x) \equiv \text{sign}(x)\psi_1(x)\psi_2(x)
\]

domain wall type mass term,
massless “edge” modes on \( x=0 \)
SUSY CS has extra fermion modes:

These fermion modes strictly localized at the boundary and then completely decoupled from other modes for $t \to 0$

Thus, we can think the theory essentially describe 3d pure gravity
Conclusion

• “exact” partition function of 3d pure gravity

• Using SUSY extension and localization
Fin.