Strings and Branes are Waves and Monopoles or both

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Outline

Motivation

One way of looking at Double Field theory is of a lift of the NS-NS sector of supergravity to a purely *geometric* theory. That is, it is a sort of Kaluza Klein theory that gives ordinary gravity and 2-form gauge theory under reduction. Its local symmetries must be a combination of diffeomorphisms with the gauge tranformations of the 2-form potentials and its action and equations of motion must contain the usual SUGRA ones once one removes dependences on any extra dimensions we have added.

Kaluza Klein modes

- Start with massless, thus null states in the full theory, with momentum directed in the extra dimensions
- These states from the perspective of the reduced theory have mass and charge
- The mass will be given by momentum in KK direction

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M-theory Example

- Null wave solution in M-theory gives D0-brane
- D0-brane is momentum mode in 11th direction
- Mass and charge given by momentum - BPS state

Standard Solutions

- wave

▸ D0-Brane

$$ds^2 = -H^{-1}dt^2 + H\left[dz - (H^{-1} - 1)dt\right]^2 + d\vec{y}^2_{(D-2)}$$
$$B_{\mu\nu} = 0, \qquad e^{-2\phi} = e^{-2\phi_0}$$

- F1-string

$$ds^2 = -H^{-1}\left[dt^2 - dz^2\right] + d\vec{y}^2_{(D-2)}$$
$$B_{tz} = -(H^{-1} - 1), \qquad e^{-2\phi} = He^{-2\phi_0}$$

- Harmonic Function

$$H = 1 + \frac{h}{|\vec{y}_{(D-2)}|^{D-4}}, \qquad \nabla^2 H = 0$$

Introduction to Double Field Theory

Double Field Theory

[Hull and Zweibach, also later work of Park]

- Bosonic NS-NS sector: $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ
- Makes $O(D, D; R)$ a manifest symmetry of the action
- Metric and B-field on equal footing - geometric unification

Double the dimension of space but require a global $O(D, D)$ structure

- $O(D, D)$ structure $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Geometric Framework

Doubling the dimension of space to $2D$

- Introduce new coordinates $\tilde{x}_\mu$
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Unification of two concepts

- Metric and B-field $\to$ generalized metric
- Diffeos and gauge transformations $\to$ generalized diffeos
- Generated by generalized Lie derivative

The Doubled Formalism

Generalized coordinates

- Combine x^μ and $\tilde{x}_\mu$ into

$$X^M = (x^\mu, \tilde{x}_\mu)$$

- $\mu = 1, \ldots, D$ and $M = 1, \ldots, 2D$

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Generalized metric

- Combine metric $g_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ into

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu} & B_{\mu\rho} g^{\rho\nu} \\ -g^{\mu\sigma} B_{\sigma\nu} & g^{\mu\nu} \end{pmatrix}$$

- Rescale the dilaton $\quad e^{-2d} = \sqrt{g} e^{-2\phi}$

The DFT Action

The action integral

$$S = \int \mathrm{d}^{2D} X e^{-2d} R$$

The *generalized Ricci scalar*

$$\begin{aligned} R = {} & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ & + 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\ & - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \end{aligned}$$

Equations of Motion

Since $\mathcal{H}$ is constrained, get projected EoMs

$$P_{MN}{}^{KL}K_{KL} = 0$$

where

$$K_{MN} = \delta R / \delta \mathcal{H}^{MN}$$

$$P_{MN}{}^{KL} = \frac{1}{2}(\delta_M{}^{(K}\delta_N{}^{L)} - \mathcal{H}_{MP}\eta^{P(K}\eta_{NQ}\mathcal{H}^{L)Q})$$

Dilaton equation

$$R = 0$$

The DFT Wave Solution

$$X^M = (t, z, y^m, \tilde{t}, \tilde{z}, \tilde{y}_m)$$

Generalized metric

$$\begin{aligned} \mathrm{d}s^2 &= \mathcal{H}_{MN}\mathrm{d}X^M\mathrm{d}X^N \\ &= (H-2)\left[\mathrm{d}t^2 - \mathrm{d}z^2\right] - H\left[\mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{z}^2\right] \\ &\quad + 2(H-1)\left[\mathrm{d}t\mathrm{d}\tilde{z} + \mathrm{d}\tilde{t}\mathrm{d}z\right] \\ &\quad + \delta_{mn}\mathrm{d}y^m\mathrm{d}y^n + \delta^{mn}\mathrm{d}\tilde{y}_m\mathrm{d}\tilde{y}_n \end{aligned}$$

Rescaled dilaton

$$d = const.$$

The DFT Wave Solution

Properties

- Null
- Carries momentum in $\tilde{z}$ direction
- Interprete as null wave in DFT
- Smeared over dual directions $\rightarrow$ obeys section condition

Reducing the Solution

Examine from the point of view of the reduced theory

- Get fundamental string solution
- Extended along z
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If z and $\tilde{z}$ are exchanged

- Get pp-wave in z direction
- Expected as wave and string are T-dual

Key Result

The fundamental string is a massless wave in doubled space with momentum in a dual direction.

Goldstone Mode Analysis

Zero modes

- Symmetry breaking
- Moduli $\rightarrow$ collective coordinates
- Generated by large gauge transformations / diffeos
- Make local on worldvolume $\rightarrow$ get zero modes

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Number of modes

- String: $D-2$ modes
- Doubled wave / string: ???

Constructing the Zero Modes

Transformations of $\mathcal{H}$ and d

$$h_{MN} = \mathcal{L}_\xi \mathcal{H}_{MN} \qquad \lambda = \mathcal{L}_\xi d$$

- gauge parameter $\xi^M = (0, H^\alpha \hat{\phi}^m, 0, H^\beta \hat{\tilde{\phi}}_m)$
- $\hat{\phi}^m$ and $\hat{\tilde{\phi}}_m$ are constant moduli

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Allow dependece on $x^a = (t, z)$ to get zero modes

$$\hat{\phi}^m \to \phi^m(x) \qquad \hat{\tilde{\phi}}_m \to \tilde{\phi}^m(x)$$

Equations of motion

- Insert into DFT EoMs (two derivatives, first order)
- Find $\Box\phi = 0$ and $\Box\tilde{\phi} = 0$
- Also get self-duality relation for $\Phi^M = (0, \phi^m, 0, \tilde{\phi}_m)$

$$\mathcal{H}_{MN}\mathrm{d}\Phi^N = \eta_{MN} \star \mathrm{d}\Phi^N$$

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Duality symmetric string in doubled space (Tseytlin)

- Can be written as (anti-)chiral equation for $\psi_\pm = \phi \pm \tilde{\phi}$

$$\mathrm{d}\psi_\pm = \pm \star \mathrm{d}\psi_\pm$$

Summary

Wave solution in DFT

- Solution unifies pp-wave and F1-string (T-duals)
- Momentum mode in dual direction gives fundamental string

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Goldstone modes

- Find chiral zero modes of the wave solution
- Gives the correct degrees of freedom for the string in doubled space with manifest $O(d,d)$

Extension to M-Theory

Extended theories

- Make U-duality manifest
- Include brane wrapping directions
- Geometrically unify metric and C-field(s)

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Example: $SL(5)$

- Duality group for M-theory in 4 dimensions x^μ
- Combine with 6 wrapping directions $y_{\mu\nu}$
- Wave in extended space gives M2-brane

Extend space to include *dual* membrane winding modes, $y_{\mu\nu}$ along with usual x^μ coordinates. No longer a simple doubling. Now the generalised tangent space is:

$$\Lambda^1(M) \oplus \Lambda^{*2}(M)\,. \tag{7.1}$$

The metric for the Sl_5 case is given by:

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2} C_a{}^{ef} C_{bef} & \frac{1}{\sqrt{2}} C_a{}^{kl} \\ \frac{1}{\sqrt{2}} C^{mn}{}_b & g^{mn,kl} \end{pmatrix}, \tag{7.2}$$

where $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$ and has the effect of raising an antisymmetric pair of indices.

We can construct the Lagrangian with all the right properties:

$$\begin{aligned} L &= \Bigg(\frac{1}{12} M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL})(\partial_L M_{MK}) \\ &+ \frac{1}{12} M^{MN} (M^{KL} \partial_M M_{KL})(M^{RS} \partial_N M_{RS}) \\ &+ \frac{1}{4} M^{MN} M^{PQ} (M^{RS} \partial_P M_{RS})(\partial_M M_{NQ}) \Bigg) \end{aligned}$$

The SL5 Wave Solution

Properties

- Wave solution as before with:
- momentum in y_{zw} direction
- this is a null wave in extended geometry
- Smeared over dual direction obeying section condition
- Interpretation in reduced theory is as a membrane stretched over the zw directions!

Other duality groups. eg $SO(5,5)$

$$\Lambda^1(M) \to \Lambda^{*2}(M) \oplus \Lambda^{*5}(M) \tag{7.3}$$

So we have coordinates

$$Z^I = (x^a, y_{ab}, y_{abcde}) \tag{7.4}$$

with $a = 1..5$, $ab = 6..15$, $abcde = 16$. Thus the space is 16 dimensional corresponding to the **16** of SO(5,5). The y_{abcde} correspond to fivebrane winding mode.

The SO(5,5) generalized metric is (upper case latin indices run from 1 to 16):

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2}C_a{}^{ef}C_{bef} + \frac{1}{16}X_aX_b & \frac{1}{\sqrt{2}}C_a{}^{mn} + \frac{1}{4\sqrt{2}}X_aV^{mn} & \frac{1}{4}X_a \\ \frac{1}{\sqrt{2}}C^{kl}{}_b + \frac{1}{4\sqrt{2}}V^{kl}X_b & g^{kl,mn} + \frac{1}{2}V^{kl}V^{mn} & \frac{1}{\sqrt{2}}V^{kl} \\ \frac{1}{4}X_b & \frac{1}{\sqrt{2}}V^{mn} & 1 \end{pmatrix} \tag{7.5}$$

where we have defined:

$$V^{ab} = \frac{1}{6}\eta^{abcde}C_{cde}\,, \tag{7.6}$$

with η^{abcde} being the totally antisymmetric permutation symbol (it is only a tensor density and thus distinguished from the usual ϵ^{abcde} symbol) and

$$X_a = V^{de}C_{dea}\,. \tag{7.7}$$

We can attempt to reconstruct the dynamical theory out of this generalized metric. We have the following Lagrangian with manifest $SO(5,5)$,

$$L = \frac{1}{16}M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2}M^{MN}(\partial_N M^{KL})(\partial_L M_{MK}) + \frac{3}{128}M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}) - \frac{1}{8}M^{MN}M^{PQ}(M^{RS}\partial_P M_{RS})(\partial_M M_{NQ}) \quad (7.8)$$

where $\partial_M = \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_{ab}}, \frac{\partial}{\partial z}\right)$.

By now it is no surprise that a null wave in the y_{abcde} direction is an M5-brane stretched in the $abcde$ directions.
But is there another way to view this?

Fivebranes and monopoles

In Kaluza-Klein theory, once we have waves along the KK directions and see that these allow electric charges we can ask how to produce a monopoles. The gives us the Kaluza-Klein monopole, essentially a nontrivial bundle that is an S^1 over S^2 with total space S^3.
In terms of M-theory, this is the D6 brane.
Can we do the same trick for DFT or other extended geometries and find monopole like solutions?

Yes:
in DFT the monopole whose KK circle is $\tilde{z}$ is an NS5-brane from the reduced perspetive.
In exceptional case, the monopole whose KK circle is in the y_{ab} direction is an M5-brane.
The monopole whose KK circle is in the y_{abcde} direction is the M2-brane.

Monopoles

- KK-monopole

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2_{(d-5)} + H^{-1}\left[\mathrm{d}z + A_i \mathrm{d}y^i\right]^2 + H\mathrm{d}\vec{y}^2_{(3)}$$

$$\partial_{[i}A_{j]} = \frac{1}{2}\epsilon_{ij}{}^{k}\partial_k H, \qquad e^{-2\phi} = e^{-2\phi}$$

This is a description of the solution in a local patch. The usual Dirac patching must take place to allow a nontrivial flux.
We now take this KK-monopole solution but this time we examine it as a solution to DFT (or the exceptional geometry equivalent). Again as with the wave, we treat DFT as a Kaluza Klein type theory. What will change is how we identify the fibre in the monopole. In this solution the fibre is denoted with the z coordinate. What this coordinate will be in terms of DFT is what will determine the interpretation of the solution in terms of usual supergravity.

As promised, placing the fibre as a winding direction and then extracting the usual spacetime metric and b-field yields the NS5 brane:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2_{(d-5)} + H\mathrm{d}\vec{y}^2_{(4)}, \quad B_{zi} = A_i, \quad e^{-2\phi} = H^{-1}e^{-2\phi_0}$$

H is, as usual, a Harmonic function.

$$H(r) = 1 + \frac{g}{r^2}$$

The section condition implies that it cannot be a function of more than half the coordinates in DFT.
Thus it is at best smeared over half the number of dimensions but localised in the other half. In fact it may be localised completely. That is the usual NS5 brane solution.

T-duality

We can also consider the solution where one smears the brane in an additional direction thus adding an additional isometry. This results in the Harmonic function depending on fewer coordinates eg. Let $r^2 = |w|^2 + z^2$. Then, smearing implies

$$H(w,z) = 1 + \frac{g}{|w|^2 + z^2} \to H(w) = 1 + \frac{g}{|w|^2}$$

and there is a new isometry in the z direction. Now there is no dependence on either z or $\tilde{z}$ directions. Thus there is an ambiguity in how one identifies spacetime and the usual supergravity description. That ambiguity is **T-duality**.

- Recall the $O(d, d; \mathbb{R})$ symmetry in DFT is not T-duality.
- This continous symmetry is the local symmetry that comes of combining diffeomorphisms with two form gauge transformations.
- The section condition that restricts the dependence of the fields effectively halves the number of dimensions.
- When we solve the section condition there is generically a canonical choice of our interpretation of what is spacetime that is given by the Lagrangain submanifold (the maximally isotropic subspace) defined by the section condition solution.
- When there are isometries then there is not a canonical choice of section condition and the Lagrangian submanifold is ambigous.
- Different choices gives us T-duality related frames.

What happens if one takes a noncanonical choice of Lagrangian submanfold?

- Provided the section condition is obeyed then everything is consistent.
- However, with a noncanonical choice of Lagrangian submanifold then there is no supergravity interpretation.
- The theory looks like supergravity but the solution has dependence on winding mode coordinates that have no spacetime interpretation eg. $H(w, \tilde{z})$.

Remarkably, this has been seen before. A gauged linear sigma model may be used to calculate the effect of world sheet instantons on T-duality for the NS5-KKmonopole system. The results of these corrections is to localise the solution in winding mode space. (This was first reported by Jensen.)

Summary

- In DFT and other extended exceptional geometries, waves with momentum along the novel directions are strings or branes.
- Monopoles whose fibre direction is along one of those novel directions is also a brane but S-dual to the one given by a wave with monentum in those directions.
- Thus all branes in exceptional geomerties are simulataneously waves and monopoles.

More avenues for research

- A Goldstone analysis for these monople/brane solutions
- note the way chirality entered the Tseytlin string, a natural extension to higher dimensions would be the fivebrane self-dual two form.
- waves and monopoles are nonsingular, a careful analysis reveals that the solutions described here are non-singular in DFT even though they are singular in the reduced theory.
- do black branes have any special properties from this perspective?
- the connection to world sheet instanton corrections is intruiging but as yet unexplained.

- One can ask whether the solution picks out a duality frame. This happens in normal $S^1(R)$ compactifications. Although we are free to pick duality frame, the solution picks out a frame since there will be an energy (or coupling) associated to the different frames.
- For $R >> l_s$ we chose the normal momentum frame. For $R << l_s$ we choose the winding mode frame.
- The frame is chosen by this, for $R \approx l_S$ we cannot choose, there is no natural frame.
- What happens in the DFT wave/F1 solution?

- For the DFT description of the F1/wave:
- At infinity, we have complete ambiguity in how we choose the frame. However as we approach the core of the solution there is a prefered frame. To see this we need to diagonalise the solution and determine the "large directions". This gives the twisted light cone.
- Thus for DFT the fundamental string is ambiguous at infinity (it can be a wave or the string) but at the core it is a wave in the twisted space!
- Some of this is not a surprise. One always suspected that the winding modes would remove singularities and that the core of the fundamental string solution would not be singular. In DFT one sees this directly through the calculations. The core is non-singular and a particular duality frame is picked out.

- For the exceptional geometries we truncated the theory.
- There is a nontruncated version developed by Hohm and Sambtleben
- In this theory, the branes are simultaneously waves and monpoles.
- In fact there is a single Self-Dual solution that covers all 1/2 BPS brane solutions.
- This is the gravitational version of the self-dual string. Through picking its orientation one then changes the duality frame. In the reduced theory it is either electric or magnetic. The hidden symmetry comes from the geometry of the space on which the self-dual string wraps.

We no longer truncate the theory. For concreteness lets consider the E_7 theory. We still have the M^{56} of the truncated theory with its generalised metric $\mathcal{M}_{IJ}$ but now also $g_{\mu\nu}$ of the previously truncated 4d space and crucially the vector fields $\mathcal{A}_\mu^I$ that mixes these spaces.

The action is given by:

$$S = \int \mathrm{d}^4x \mathrm{d}^{56}Y e \Big[\hat{R} + \frac{1}{48} g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}^{MN} \mathcal{D}_\nu \mathcal{M}_{MN} \\ - \frac{1}{8} \mathcal{M}_{MN} \mathcal{F}^{\mu\nu\ M} \mathcal{F}_{\mu\nu}{}^N - V(\mathcal{M}_{MN}, g_{\mu\nu}) + e^{-1} \mathcal{L}_{\text{top}}\Big] \,.$$

The first term is a covariantized Einstein-Hilbert term

$$\mathcal{L}_{\mathrm{EH}} = e\hat{R} = e e_{\underline{a}}{}^{\mu} e_{\underline{b}}{}^{\nu} \hat{R}_{\mu\nu}{}^{\underline{ab}} \,\text{where}\, \hat{R}_{\mu\nu}{}^{\underline{ab}} \equiv R_{\mu\nu}{}^{\underline{ab}}[\omega] + \mathcal{F}_{\mu\nu}{}^{M} e^{\underline{a}\rho} \partial_M e_{\rho}{}^{\underline{b}} \,.$$

The second term is a kinetic term for the generalized metric $\mathcal{M}_{MN}$ which takes the form of a non-linear gauged sigma model with target space $E_7/SU(8)$. The third term is a Yang-Mills-type kinetic term for the gauge vectors ${\mathcal{A}_{\mu}}^{M}$. The fourth term is the "potential" V as before

$$V = -\frac{1}{48}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}$$
$$-\frac{1}{2}\mathfrak{g}^{-1}\partial_M\mathfrak{g}\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}\mathfrak{g}^{-1}\partial_M\mathfrak{g}\mathfrak{g}^{-1}\partial_N\mathfrak{g} - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu}$$

where $\mathfrak{g} = \det g_{\mu\nu}$. The last term is a topological Chern-Simons-like term which is required for consistency.

The final ingredient of the theory are the twisted self-duality equations for the 56 EFT gauge vectors ${\mathcal{A}_\mu}^M$

$$
{\mathcal{F}_{\mu\nu}}^M = \frac{1}{2} e \epsilon_{\mu\nu\rho\sigma} \Omega^{MN} \mathcal{M}_{NK} \mathcal{F}^{\rho\sigma\ K}
$$

which relate the 28 “electric” vectors to the 28 “magnetic” ones. This self-duality relation is a crucial property of the E_7 EFT and is essential for the results presented here.

We will seek solutions that are pointlike in the 4-d space and use our intuition from the previous truncated case so we have both wave and monopole simultaneously:

$$g_{\mu\nu} = \mathrm{diag}[-H^{-1/2}, H^{1/2}\delta_{ij}]\,, \qquad H(r) = 1 + \frac{h}{r}$$

The EFT vector potential ${\mathcal{A}_\mu}^M$ of our solution has "electric" and "magnetic" components (from the four dimensional specetime persepctive) that are given respectively by

$$
{\mathcal{A}_t}^M = \frac{H-1}{H} a^M \qquad \text{and} \qquad {\mathcal{A}_i}^M = A_i \tilde{a}^M \,,
$$

where A_i is a potential of the magnetic field. The magnetic potential obeys a BPS-like condition where its curl is given by the gradient of the harmonic function that appears in the metric

$$
\vec{\nabla} \times \vec{A} = \vec{\nabla} H \qquad \text{or} \qquad \partial_{[i} A_{j]} = \frac{1}{2} {\epsilon_{ij}}^k \partial_k H \,.
$$

The vector a^M in the extended space points in one of the 56 extended directions. This is the direction of propagation of a wave.
The dual vector $\tilde{a}^M$ denotes the direction dual to a^M given by $a^M = \Omega^{MN}\mathcal{M}_{NK}\tilde{a}^K$.
One can check explicitly that $\mathcal{A}_\mu{}^M$ satisfies the twisted self-duality equation.
We then wish to interpret this solution in terms of wrapped branes in ordinary supergravity. The choice of the direction of a^M is what determines how is object is charged in terms of usual supergravity fields.
All the 1/2 BPS maybe be recovered by choossing a different direction for a^M.

We can write this solution in an interesting way when one think of it as a full 60 dimensional solution: the three constituents of the solution, the external metric $g_{\mu\nu}$, the extended internal metric $\mathcal{M}_{MN}$ and the vector potential ${\mathcal{A}_\mu}^M$ are combined in the usual Kaluza-Klein fashion to form a 60-dimensional metric

$$\mathcal{H}_{\hat{M}\hat{N}} = \begin{pmatrix} g_{\mu\nu} + {\mathcal{A}_\mu}^M {\mathcal{A}_\nu}^N \mathcal{M}_{MN} & {\mathcal{A}_\mu}^M \mathcal{M}_{MN} \\ \mathcal{M}_{MN} {\mathcal{A}_\nu}^N & \mathcal{M}_{MN} \end{pmatrix}$$

where $g_{\mu\nu}$ and $\mathcal{A}_\mu^M$ is given above and

$$\mathcal{M}_{MN} = \mathrm{diag}[H^{3/2}, H^{1/2}\delta_{27}, H^{-1/2}\delta_{27}, H^{-3/2}]\,. \qquad (10.1)$$

This maybe written as follows:

$$\mathcal{H}_{\hat{M}\hat{N}} = \begin{pmatrix} H^{1/2}\mathcal{H}^{\text{wave}}_{AB} & 0 \\ 0 & H^{-1/2}\mathcal{H}^{\text{mono}}_{\bar{A}\bar{B}} \end{pmatrix} \tag{10.2}$$

where to top left block is simply the metric of a wave with 27 transverse dimensions and the bottom right block is the metric of a monopole, also with 27 transverse dimensions.
$H(r)$ is the harmonic function as before.

At infinity $H = 1$ and the solution is equally wave and monopole however as we move to the core of the solution at $r = 0$, the wave dominates.

In summary, looking at the usual solutions of supergravity from this perspective gives a new view point and there are many more directions for new research, such as blackbranes, nongeometric branes and other types of supergravity solution.

Let us return to the Null wave solutions and understand these in the context of supersymmetry and how the section condition arises. For dimension d there is a coset G/H; one then does the following:

1. Construct spinors of H and write down the associated Clifford algebra, Γ^i.
2. Form a representation of G using a sum of antisymmetrised products of Γ^i to give Γ^I.
3. Combine the momenta and central charges to form a representation of G which we think of as the generalised momenta P_I.
4. Rewrite the superalgebra in terms of only the generalised momenta P_I and the set of generalised gamma matrices Γ^I.
5. Demand (11.9) is saturated for the *massless* representation to give constraints on P_I.
6. This constraint should be the same as that required by the closure of the algebra of generalised Lie derviatives, also known as the section condition.

The example of Sl_5:
First introduce the **10** of Sl_5:

$$\begin{aligned} z^{[ij]} &= z^{a5} = x^a \\ &= z^{ab} = \frac{1}{2}\eta^{abcd} y_{cd} \ . \end{aligned} \tag{11.1}$$

Where η^{abcd} is the alternating symbol, ie. $\eta^{1234} = 1$. $i, j = 1..5$ and the coordinates $z^{[ij]}$ are in the **10** of $Sl(5)$ which we denote with the index $I, J = 1..10$. Associated to these coordinates will be a set of translation generators or generalised momenta:

$$P_{[ij]} = P_I \ . \tag{11.2}$$

We will need to work with the coset:

$$sl(5)/so(2,3)\,. \tag{11.3}$$

We will now introduce spinors of the local group H, $SO(2,3)$, with spinor index $\alpha = 1..4$. Consequently we then have the associated gamma matrices:

$$(\Gamma^i)^\alpha{}_\beta \ , \qquad i = 1..5 \tag{11.4}$$

which form the Clifford algebra for $SO(2,3)$. Along with this we have the charge conjugation matrix $(C)_{\alpha\beta}$ and its inverse $(C^{-1})^{\alpha\beta}$ with which we can lower and raise spinor indices repectively through left multiplication.

From these $SO(2,3)$ Γ matrices we can form the appropriate representation of the global group G, Sl_5. The set of antisymmetrised products of the Γ^i matrices:

$$(\Gamma^{[ij]})^{\alpha}{}_{\beta} = (\Gamma^I)^{\alpha}{}_{\beta}\,, \qquad I = 1..10 \tag{11.5}$$

are in the **10** of sl_5. To compare with the usual supersymmetry algebra they be decomposed into $SO(1,3)$ Γ matrices just as we did with the coordinates described by (11.2):

$$\Gamma^I = \Gamma^{[ij]} = (\Gamma^{[a5]}, \Gamma^{[cd]})\,. \tag{11.6}$$

Similarly, the generalised momentum decomposes as:

$$P_I = P_{[ij]} = (P_{a5}, \frac{1}{2}\eta_{abcd}Z^{cd})\,. \tag{11.7}$$

With the obvious identification of $P_{a5} = P_a$ being momenta in the usual four dimensional spacetime and the set $\{Z^{cd}\}$ labels momenta in the novel extended directions.

Now we wish to be able to form a supersymmetry algebra using this set of generalised gamma matrices, $\{\Gamma^I\}$, the set of generalised momenta P_I and the supercharges, Q_α. No central charges are required; the bosonic sector has only the generators of the generalised Poincare group. Thus the complete superalgebra is given by:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^I)_{\alpha\beta} P_I \,, \qquad [Q_\alpha, P_I] = 0 \,, \qquad [P_I, P_J] = 0 \,. \tag{11.8}$$

Where C is charge conjugation matrix for $so(2,3)$ spinors.

We will proceed exactly as in the usual superalgebra case when one wishes to examine the massless representations ie. where the quadratic Casimr of momentum vanishes, and show that they form "short multiplets" .
We calculate the square of the susy algebra which is positive definite:

$$(C^{-1}\Gamma^{I})^{\alpha\beta}\,(C\Gamma^{J})_{\beta\gamma}P_{I}P_{J} \geq 0\,. \tag{11.9}$$

Then by demanding that this bound is saturated then we have a quadratic constraint on the generalised momenta, P^{I}.

We will now determine this constraint on the generalised momenta by substituting the decomposition (11.6) into (11.9) and demand the bound is saturated.
This produces (surpressing spinor indices):

$$
\begin{aligned}
(C^{-1}\Gamma^I)\,(C\Gamma^J)P_I P_J &= 2(\eta_{ab}P^a P^b - Z_{ab}Z_{cd}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}))\mathbb{I} \\
&+ 4P^a Z_{ab}\Gamma^b \\
&+ 2Z_{ab}Z_{cd}\Gamma^{abcd}\,.
\end{aligned}
\tag{11.10}
$$

Demanding that this is zero, we need each line to vanish sperately. This means the constraints in terms of the four dimensional momenta and central charges are:

$$P_a P^a = Z_{ab}Z^{ab}\,, \qquad P^a Z_{ab} = 0\,, \qquad Z_{ab}Z_{cd}\epsilon^{abcd} = 0\,. \tag{11.11}$$

The first term is the standard BPS condition requiring the mass be equal to the central charge and the second two equations are the quadratic constraints required for the state to be 1/2 BPS.

In terms of the Sl_5 generalised momenta, $P_{[ij]}$ these equations become:

$$P_{[ij]}P^{[ij]} = P_I P^I = 0\,, \tag{11.12}$$

$$\epsilon^{ijklm}P_{[ij]}P_{[kl]} = 0\,. \tag{11.13}$$

The first equation (11.13) implies that the state is massless in from the point of view of the extended space Poincare algebra. Thus in extended geometry the usual BPS states are massless. Supersymmetry works because the massless multiplet of SO(2,3) has the same number of degrees of freedon as the massive multiplet in SO(1,4).

The second equation (11.13) is precisely the *physical section condition* that we need to impose so that the local symmetry algebra of the extended geometry ie. the algebra of generalised Lie derivatives closes.

Other Solutions

- D0-brane

Back to Standard Solutions

$$\mathrm{d}s^2 = -H^{-1}\mathrm{d}t^2 + \mathrm{d}\vec{y}^2_{(d-1)}, \qquad A_t = -(H^{-1} - 1)$$

- KK-monopole

Back to Other Solutions

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2_{(d-5)} + H^{-1}\left[\mathrm{d}z + A_i \mathrm{d}y^i\right]^2 + H\mathrm{d}\vec{y}^2_{(3)}$$

$$\partial_{[i}A_{j]} = \frac{1}{2}\epsilon_{ij}{}^k\partial_k H, \qquad e^{-2\phi} = e^{-2\phi}$$

- NS5-brane

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2_{(d-5)} + H\mathrm{d}\vec{y}^2_{(4)}, \quad B_{zi} = A_i, \quad e^{-2\phi} = H^{-1}e^{-2\phi_0}$$