3d mirror symmetry as a canonical transformation

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`arXiv:1502.?????` - B. Assel, N.D. and J. Felix

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Outline

- Introduction and motivation
- Localization for 3d theories on S^3
- Solving the matrix model and Airy function behavior
- Fermi-gas formulation with masses and FI-parameters
- Mirror symmetry
- Fermi-gas for unitary D-quivers and Sp-unitary linear-quivers
- Summary

Introduction and motivation

- We will study mirror symmetry in $\mathcal{N} = 4$ theories in 3d.
- Let us look at the AGT correspondence for 4d $\mathcal{N} = 2$ theories for motivation

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- These theories have a rich structure of S-duality relating seemingly inequivalent theories.
- This can be identified with choice of pants decomposition of a Riemann surface. [Witten][Gaiotto]

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Is there a simple relation between the matrix integrals for S-dual theories?

- There are such integral identities, but they are very complicated.

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The lesson: Instead of proving the identities, find relation to another theory, where they are known.

Localization on S^3

[Kapustin, Willett, Yaakov]

- Consider any $\mathcal{N} = 4$ super Chern-Simons matter theory on S^3.
- Add to the action a Q-exact term of the form $t\,Q(\Psi Q \Psi)$.
- VEV of Q-invariant observables is unmodified by this insertion.
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- Each $U(N)$ vector multiplet will reduce an N dimensional integral where a Chern-Simons at level $k^{(a)}$ and FI parameter $\zeta^{(a)}$ gives

$$\int\prod_{i=1}^{N}d\lambda_i^{(a)}\,e^{\sum_i\left(2\pi i\zeta^{(a)}\lambda_i^{(a)}+i\pi k^{(a)}(\lambda_i^{(a)})^2\right)}\prod_{i<j}\left(2\sinh\pi\big(\lambda_i^{(a)}-\lambda_j^{(a)}\big)\right)^2$$

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- bifundamentals of mass m_a contribute

$$\frac{1}{\prod_{i,j} 2\cosh\pi\left(\lambda_i^{(a)} - \lambda_j^{(b)} + m_a\right)}$$

- In the case of ABJM theory we have a two node circular quiver with no fundamental matter, only massless bifundamentals, so one finds

$$Z=\frac{1}{(N!)^2}\int\prod_{i=1}^N d\lambda_i d\nu_i\, e^{\pi ik\sum_i(\lambda_i^2-\nu_i^2)}\,\frac{\prod_{i<j}\left(2\sinh\pi(\lambda_i-\lambda_j)\right)^2\left(2\sinh\pi(\nu_i-\nu_j)\right)^2}{\prod_{i,j}\left(2\cosh\pi(\lambda_i-\nu_j)\right)^2}$$

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- For a more general circular quiver with massless fundamentals

$$Z=\frac{1}{(N!)^n}\int\prod_{a=1}^{n}\prod_{i=1}^{N}d\lambda_i^{(a)}\,e^{\sum_{i=1}^{n}\left(2\pi i\zeta^{(a)}\lambda_i^{(a)}+\pi ik^{(a)}(\lambda_i^{(a)})^2\right)}$$

$$\times\prod_{a=1}^{n}\frac{1}{\prod_i\operatorname{ch}^{N_f^{(a)}}\lambda_i^{(a)}}\frac{\prod_{i<j}\operatorname{sh}^2\left(\lambda_i^{(a)}-\lambda_j^{(a)}\right)}{\prod_{i,j}\operatorname{ch}\left(\lambda_i^{(a)}-\lambda_j^{(a+1)}+m_a\right)}$$

using

$$\operatorname{sh}x\equiv2\sinh\pi x\,,\qquad\operatorname{ch}x\equiv2\cosh\pi x$$

Solving the matrix model

- The ABJM matrix model is very similar to that of pure Chern-Simons on a $S^3/\mathbb{Z}_2$, a lens space and can be solved exactly. [Aganagic,Klemm Mariño,Vafa][Halmagyi Yasnov][Drukker Mariño,Putrov]

- Explicit expression exist for the free energy in the genus expansion (large N) to the first few orders. At each order one finds a principle contribution and a series of instanton corrections (both world-sheet and D-brane instantons).

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- The leading behavior of the free energy at large N and $\lambda = N/k$ can be written as

$$F = \frac{F_0}{g_s^2} = -\frac{\pi\sqrt{2}}{3}k^2\hat{\lambda}^{3/2} = -\frac{\pi\sqrt{2}}{3}\sqrt{k}\,\hat{N}^{3/2}$$

$$\hat{\lambda} = \lambda - \frac{1}{24} - \frac{1}{3k^2} \qquad \hat{N} = N - \frac{k}{24} - \frac{1}{3k}$$

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- This can be done for quite general models.
- This matches the partition function of IIA on $AdS_4 \times X_6$

Universal Airy function behavior

- The genus expansion of ABJM theory satisfies a holomorphic anomaly equation.
- In the case of ABJM we know (recursively) the full all-genus partition function.
- Ignoring the instanton terms in the planar free energy F_0 the solution to this equation is remarkably simple [Fuji,Hirano Moriyama]

$$Z \approx \mathrm{Ai}\left[(c/k)^{-1/3}\hat{N}\right] \qquad c = \frac{2}{\pi^2}$$

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- The same is true for quite general theories with $\mathcal{N} = 3$ SUSY (with appropriate c and $\hat{N}$). [Mariño, Putrov]
- I will outline the derivation for a two node circular quiver and later a D-shaped quiver.
- The story for longer quivers is not significantly different.

- Considering a 2-node circular quiver with one fundamental field on each node, Fayet-Iliopoulos parameters ζ_i, bifundamental masses m_i and Chern-Simons levels $\pm k$, the matrix model is

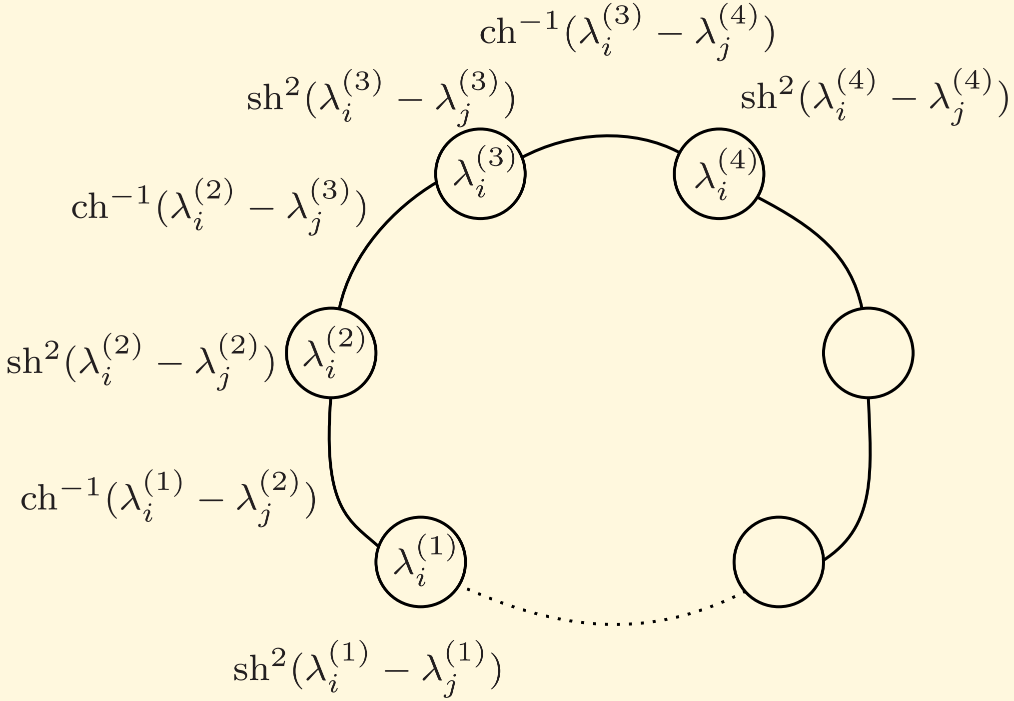
$$Z(N)=\frac{1}{(N!)^2}\int\prod_{i=1}^{N}d\lambda_i^{(1)}d\lambda_i^{(2)}\prod_{i=1}^{N}\frac{e^{2\pi i\zeta_1\lambda_i^{(1)}+\pi ik(\lambda_i^{(1)})^2}e^{2\pi i\zeta_2\lambda_i^{(2)}-\pi ik(\lambda_i^{(2)})^2}}{\operatorname{ch}\lambda_i^{(1)}\operatorname{ch}\lambda_i^{(2)}}$$
$$\times\frac{\prod_{i<j}\operatorname{sh}^2\left(\lambda_i^{(1)}-\lambda_j^{(1)}\right)\operatorname{sh}^2\left(\lambda_i^{(2)}-\lambda_j^{(2)}\right)}{\prod_{i,j}\operatorname{ch}\left(\lambda_i^{(1)}-\lambda_j^{(2)}+m_1\right)\operatorname{ch}\left(\lambda_i^{(2)}-\lambda_j^{(1)}+m_2\right)}$$

- The crucial step in rewriting this expression as a Fermi-gas partition function is the use of the Cauchy determinant identity

$$\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_j)}=\sum_{\sigma\in S_N}(-1)^\sigma\prod_{i=1}^{N}\frac{1}{(x_i-y_{\sigma(i)})}$$

- The partition function is then

$$Z(N)=\frac{1}{(N!)^2}\int\prod_{i=1}^{N}d\lambda_i^{(1)}d\lambda_i^{(2)}\prod_{i=1}^{N}\frac{e^{2\pi i\zeta_1\lambda_i^{(1)}+\pi ik(\lambda_i^{(1)})^2}e^{2\pi i\zeta_2\lambda_i^{(2)}-\pi ik(\lambda_i^{(2)})^2}}{\operatorname{ch}\lambda_i^{(1)}\operatorname{ch}\lambda_i^{(2)}}$$
$$\times\sum_{\sigma_1,\sigma_2\in S_N}(-1)^{\sigma_1+\sigma_2}\prod_{i=1}^{N}\frac{1}{\operatorname{ch}\left(\lambda_i^{(1)}-\lambda_{\sigma_1(i)}^{(2)}+m_1\right)}\prod_{i=1}^{N}\frac{1}{\operatorname{ch}\left(\lambda_i^{(2)}-\lambda_{\sigma_2(i)}^{(1)}+m_2\right)}$$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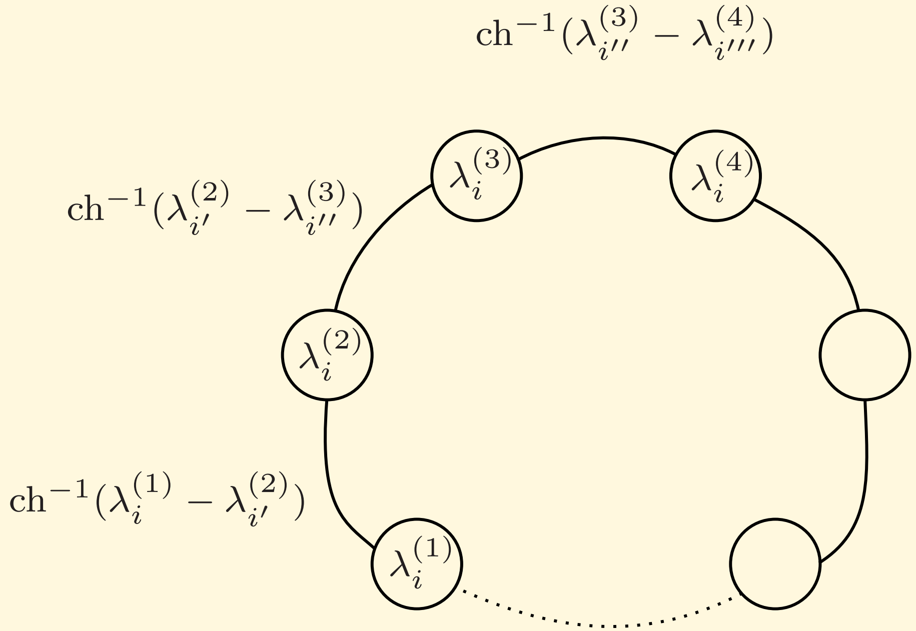
$\mathrm{ch}^{-1}(\lambda_{i''}^{(3)}-\lambda_{i'''}^{(4)})$
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- A relabelling of eigenvalues $\lambda_i^{(2)} \to \lambda_{\sigma_1^{-1}(i)}^{(2)}$ resolves one of the sums over permutations

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \prod_{i=1}^N d\lambda_i^{(1)} d\lambda_i^{(2)} \prod_{i=1}^N \frac{e^{2\pi i \zeta_1 \lambda_i^{(1)} + i\pi k (\lambda_i^{(1)})^2}}{\operatorname{ch} \lambda_i^{(1)}} \frac{1}{\operatorname{ch}\left(\lambda_i^{(1)} - \lambda_i^{(2)} + m_1\right)}$$

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- The kernel K can be considered the matrix element of the density operator $\hat{K}$ defined by

$$K(q_1,q_2) = \langle q_1|\hat{K}|q_2\rangle, \qquad \hat{K} = \frac{e^{2\pi i\zeta_1\hat{q}+\pi ik\hat{q}^2}}{\operatorname{ch}\hat{q}} \frac{e^{2\pi i m_1\hat{p}}}{\operatorname{ch}\hat{p}} \frac{e^{2\pi i\zeta_2\hat{q}-\pi ik\hat{q}^2}}{\operatorname{ch}\hat{q}} \frac{e^{2\pi i m_2\hat{p}}}{\operatorname{ch}\hat{p}}$$

- $\hat{p}$ and $\hat{q}$ are canonical conjugate variables: $[\hat{q},\hat{p}] = i\hbar$ with $\hbar = 1/2\pi$.

- and we use

$$f(\hat{q})|q\rangle = f(q)|q\rangle$$
$$e^{-2\pi i m\hat{p}} f(\hat{q}) e^{2\pi i m\hat{p}} = f(\hat{q}-m)$$
$$\langle q_1|\frac{1}{\operatorname{ch}\hat{p}}|q_2\rangle = \frac{1}{\operatorname{ch}(q_1-q_2)}$$

Grand potential and Airy function

- We can write

$$Z(N)=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma\int\prod_{i=1}^{N}d\lambda_i^{(1)}\,K\big(\lambda_i^{(1)},\lambda_{\sigma(i)}^{(1)}\big)=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma\prod_{l=\text{cycle of }\sigma}\operatorname{Tr}(\hat{K}^l)$$

- The resulting expressions are simpler if we switch to the grand-canonical partition function $z=e^\mu$

$$\Xi(z)=1+\sum_{N=1}^{\infty}z^N Z(N)=\det(1+z\hat{K})=\prod_n(1+e^{\mu-E_n})$$

where E_n are the eigenvalues of $\hat{H}=-\log\hat{K}$.

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$$J(\mu)=\log\Xi\approx\frac{c}{3k}\mu^3+\left(\frac{\pi^2 c}{3k}+n_0\right)\mu-A$$

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- Different statistical mechanical tools allow one to approximate the grand potential as

$$J(\mu)=\log\Xi\approx\frac{c}{3k}\mu^3+\left(\frac{\pi^2c}{3k}+n_0\right)\mu-A$$

- Then the canonical partition function can be derived from the canonical potential by

$$Z(N)=\frac{1}{2\pi i}\int d\mu\,e^{J(\mu)-N\mu}=\left(\frac{c}{k}\right)^{-1/3}e^A\,\mathrm{Ai}\left[\left(\frac{c}{k}\right)^{-1/3}\left(N-\frac{\pi^2c}{3k}-n_0\right)\right],$$

c and n_0 can be evaluated for any particular model. A depends more intimately on the instanton corrections and can be evaluated perturbatively.

Mirror symmetry

- If I set $k = 0$, this theory has two known mirror theories, related in the IIB brane construction by $SL(2, \mathbb{Z})$ transformations. [de Boer, Hori, Ooguri, Oz, Yin]
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- the density operator is

$$\hat{K}=\frac{e^{2\pi i\zeta_1\hat{q}}}{\operatorname{ch} q}\,\frac{e^{2\pi i m_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i\zeta_2\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{2\pi i m_2\hat{p}}}{\operatorname{ch}\hat{p}}$$

- The first of the known mirror theories is one with identical matter content but with mass and FI parameters exchanged

$$m_1\to\tilde{m}_1=-\zeta_1\,,\qquad m_2\to\tilde{m}_2=-\zeta_2\,,\qquad \zeta_1\to\tilde{\zeta}_1=m_2\,,\qquad \zeta_2\to\tilde{\zeta}_2=m_1$$

- At the level of the density function, this gives

$$\hat{K}^{(S)}=\frac{e^{2\pi i m_2\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2\pi i\zeta_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_1\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2\pi i\zeta_2\hat{p}}}{\operatorname{ch}\hat{p}}\sim\frac{e^{-2\pi i\zeta_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_1\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2\pi i\zeta_2\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_2\hat{q}}}{\operatorname{ch}\hat{q}}\,,$$

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- The first of the known mirror theories is one with identical matter content but with mass and FI parameters exchanged

$$m_1 \to \tilde{m}_1 = -\zeta_1\,, \qquad m_2 \to \tilde{m}_2 = -\zeta_2\,, \qquad \zeta_1 \to \tilde{\zeta}_1 = m_2\,, \qquad \zeta_2 \to \tilde{\zeta}_2 = m_1$$

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- We find that this density is the same as the original $\hat{K}$ under the replacement

$$\hat{p} \to \hat{q}\,, \qquad \hat{q} \to -\hat{p}$$

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- This was tested for the matrix model using integral identities. [Kapustin, Willett, Yaakov]
- the density operator is

$$\hat{K}=\frac{e^{2\pi i\zeta_1\hat{q}}}{\operatorname{ch} q}\,\frac{e^{2\pi i m_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i\zeta_2\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{2\pi i m_2\hat{p}}}{\operatorname{ch}\hat{p}}$$

- The first of the known mirror theories is one with identical matter content but with mass and FI parameters exchanged

$$m_1\to\tilde{m}_1=-\zeta_1\,,\qquad m_2\to\tilde{m}_2=-\zeta_2\,,\qquad \zeta_1\to\tilde{\zeta}_1=m_2\,,\qquad \zeta_2\to\tilde{\zeta}_2=m_1$$

- At the level of the density function, this gives

$$\hat{K}^{(S)}=\frac{e^{2\pi i m_2\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2\pi i\zeta_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_1\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2\pi i\zeta_2\hat{p}}}{\operatorname{ch}\hat{p}}\sim\frac{e^{-2\pi i\zeta_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_1\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2\pi i\zeta_2\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_2\hat{q}}}{\operatorname{ch}\hat{q}}\,,$$

- We find that this density is the same as the original $\hat{K}$ under the replacement

$$\hat{p}\to\hat{q}\,,\qquad \hat{q}\to-\hat{p}$$

$\boxed{\text{This is a linear canonical transformation on phase space!}}$

- To get the second mirror theory we apply

$$\hat{p} \to \hat{p} + \hat{q}\,, \qquad \hat{q} \to -\hat{p}$$

- The result is

$$\begin{aligned}\hat{K}^{(U)} &= \frac{e^{-2\pi i\zeta_1\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_1(\hat{p}+\hat{q})}}{\operatorname{ch}(\hat{p}+\hat{q})}\,\frac{e^{-2\pi i\zeta_2\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2\pi i m_2(\hat{p}+\hat{q})}}{\operatorname{ch}(\hat{p}+\hat{q})}\\ &= \frac{e^{-2\pi i\zeta_1\hat{p}}}{\operatorname{ch}\hat{p}}\,e^{-i\pi\hat{q}^2}\,\frac{e^{2\pi i m_1\hat{p}}}{\operatorname{ch}\hat{p}}\,e^{i\pi\hat{q}^2}\,\frac{e^{-2\pi i\zeta_2\hat{p}}}{\operatorname{ch}\hat{p}}\,e^{-i\pi\hat{q}^2}\,\frac{e^{2\pi i m_2\hat{p}}}{\operatorname{ch}\hat{p}}\,e^{i\pi\hat{q}^2}\end{aligned}$$

where we used the identity

$$e^{-\pi i\hat{q}^2}\,f(\hat{p})\,e^{\pi i\hat{q}^2} = f(\hat{p}+\hat{q})$$

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where we used the identity

$$e^{-\pi i\hat{q}^2}\,f(\hat{p})\,e^{\pi i\hat{q}^2} = f(\hat{p}+\hat{q})$$

- One can read off the corresponding quiver theory:
  - A four node circular quiver.
  - Alternating Chern-Simons levels $k = \pm 1$.
  - Vanishing FI parameters.
  - Bifundamental multiplets with masses $\{-\zeta_1, m_1, -\zeta_2, m_2\}$.

$SL(2,\mathbb{Z})$

- The S transformation

$$\hat{p} \to \hat{q}\,, \qquad \hat{q} \to -\hat{p}$$

  satisfies $S^2 = -1$.

- The U transformation

$$\hat{p} \to \hat{p} + \hat{q}\,, \qquad \hat{q} \to -\hat{p}$$

  satisfies $U^3 = -1$.

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- Defining $T = SU$, we find the usual $SL(2,\mathbb{Z})$ action of S, T.

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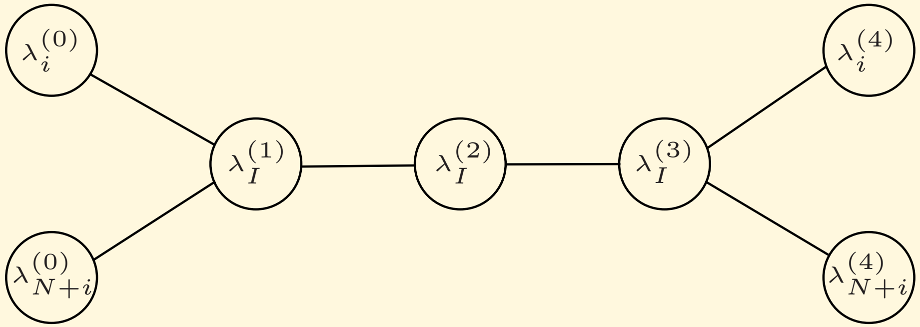
- Defining $T = SU$, we find the usual $SL(2,\mathbb{Z})$ action of S, T.
- Any other canonical transformation will give an equivalent operator, but most will not correspond to a Lagrangian thoery.

D-quivers and Sp-quivers



- The partition function for the D_{n+2}-quiver is given by

$$Z(N)=\frac{1}{N!^4(2N)!^{n-3}}\int\prod_{i=1}^{N}d\lambda_i^{(0)}d\lambda_{N+i}^{(0)}d\lambda_i^{(n)}d\lambda_{N+i}^{(n)}\prod_{a=1}^{n-1}\prod_{I=1}^{2N}d\lambda_I^{(a)}\,\frac{\prod_{a=1}^{n-1}\prod_{I<J}\mathrm{sh}^2\left(\lambda_I^{(a)}-\lambda_J^{(a)}\right)}{\prod_{a=1}^{n-2}\prod_{I,J}\mathrm{ch}\left(\lambda_I^{(a)}-\lambda_J^{(a+1)}\right)}$$
$$\frac{\prod_{i<j}\mathrm{sh}^2\left(\lambda_i^{(0)}-\lambda_j^{(0)}\right)\mathrm{sh}^2\left(\lambda_{N+i}^{(0)}-\lambda_{N+j}^{(0)}\right)\mathrm{sh}^2\left(\lambda_i^{(n)}-\lambda_j^{(n)}\right)\mathrm{sh}^2\left(\lambda_{N+i}^{(n)}-\lambda_{N+j}^{(n)}\right)}{\prod_{i,J}\mathrm{ch}\left(\lambda_i^{(0)}-\lambda_J^{(1)}\right)\mathrm{ch}\left(\lambda_{N+i}^{(0)}-\lambda_J^{(1)}\right)\mathrm{ch}\left(\lambda_i^{(n)}-\lambda_J^{(n-1)}\right)\mathrm{ch}\left(\lambda_{N+i}^{(n)}-\lambda_J^{(n-1)}\right)}$$

- Rewrite the contribution of the $U(N)$ (end) nodes

$$\frac{\prod_{i<j}\mathrm{sh}^2\left(\lambda_i^{(0)}-\lambda_j^{(0)}\right)\mathrm{sh}^2\left(\lambda_{N+i}^{(0)}-\lambda_{N+j}^{(0)}\right)}{\prod_{i,J}\mathrm{ch}\left(\lambda_i^{(0)}-\lambda_J^{(1)}\right)\mathrm{ch}\left(\lambda_{N+i}^{(0)}-\lambda_J^{(1)}\right)}$$
$$\to\quad\frac{\prod_{I<J}\mathrm{sh}\left(\lambda_I^{(0)}-\lambda_J^{(0)}\right)}{\prod_{I,J}\mathrm{ch}\left(\lambda_I^{(0)}-\lambda_J^{(1)}\right)}\,\frac{\prod_{i<j}\mathrm{sh}\left(\lambda_i^{(0)}-\lambda_j^{(0)}\right)\mathrm{sh}\left(\lambda_{N+i}^{(0)}-\lambda_{N+j}^{(0)}\right)}{\prod_{i,j}\mathrm{sh}\left(\lambda_i^{(0)}-\lambda_{N+j}^{(0)}\right)}$$

- Applying the Cauchy identity gives

$$Z(N) = \frac{1}{N!^4(2N)!^{n-3}} \sum_{\substack{\{\sigma\}\in S_N\\ \{\tau\}\in S_{2N}}} (-1)^{\sigma+\tau} \int \prod_{a=0}^{n} \prod_{I=1}^{2N} d\lambda_I^{(a)}$$

$$\prod_{i=1}^{N} \frac{1}{\mathrm{sh}\left(\lambda^{(0)}_{\sigma_1(i)} - \lambda^{(0)}_{N+i}\right)} \frac{1}{\mathrm{sh}\left(\lambda^{(n)}_{\sigma_2(i)} - \lambda^{(n)}_{N+i}\right)} \prod_{a=0}^{n-1} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda^{(a)}_{I} - \lambda^{(a+1)}_{\tau_a(I)}\right)}$$

- Relabelling the eigenvalues removes all the S_N and all but one of the S_{2N} permutations

$$Z(N) = \frac{1}{N!^2} \sum_{\tau\in S_{2N}} (-1)^{\tau} \int \prod_{a=0}^{n} \prod_{I=1}^{2N} d\lambda_I^{(a)} \prod_{i=1}^{N} \frac{1}{\mathrm{sh}\left(\lambda^{(0)}_{i} - \lambda^{(0)}_{R(i)}\right)} \frac{1}{\mathrm{sh}\left(\lambda^{(n)}_{i} - \lambda^{(n)}_{R(i)}\right)}$$

$$\left(\prod_{a=0}^{n-2} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda^{(a)}_{I} - \lambda^{(a+1)}_{I}\right)}\right) \frac{1}{\mathrm{ch}\left(\lambda^{(n-1)}_{I} - \lambda^{(n)}_{\tau(I)}\right)}$$

- We need to study the relation between the permutation τ and R acting by

$$R(I) = I \pm N$$

- an index I under a full cycle forward and back along the quiver transforms to

$$I \to \tau(I) \to R\tau(I) \to \tau^{-1}R\tau(I) \to R\tau^{-1}R\tau(I)$$

- Need to study the cycle structure of $R\tau^{-1}R\tau$.

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- In any case the resulting density function is

$$K\big(\lambda_I^{(1)}, \lambda_{R\tau^{-1}R\tau(I)}^{(1)}\big) = \int d\lambda_I^{(0)} d\lambda_{\tau R(I)}^{(n)} \prod_{a=0}^{n-1} d\lambda_{R(I)}^{(a)} \prod_{a=2}^{n} d\lambda_{\tau^{-1}R\tau R(I)}^{(a)}$$

$$\frac{1}{\mathrm{ch}\left(\lambda_I^{(1)} - \lambda_I^{(0)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_I^{(0)} - \lambda_{R(I)}^{(0)}\right)} \prod_{a=0}^{n-2} \frac{1}{\mathrm{ch}\left(\lambda_{R(I)}^{(a)} - \lambda_{R(I)}^{(a+1)}\right)}$$

$$\frac{1}{\mathrm{ch}\left(\lambda_{R(I)}^{(n-1)} - \lambda_{\tau R(I)}^{(n)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_{\tau R(I)}^{(n)} - \lambda_{R\tau R(I)}^{(n)}\right)} \prod_{a=1}^{n-1} \frac{1}{\mathrm{ch}\left(\lambda_{\tau^{-1}R\tau R(I)}^{(a)} - \lambda_{\tau^{-1}R\tau R(I)}^{(a+1)}\right)}$$

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$$\frac{1}{\operatorname{ch}\big(\lambda_I^{(1)} - \lambda_I^{(0)}\big)} \frac{1}{\operatorname{sh}\big(\lambda_I^{(0)} - \lambda_{R(I)}^{(0)}\big)} \prod_{a=0}^{n-2} \frac{1}{\operatorname{ch}\big(\lambda_{R(I)}^{(a)} - \lambda_{R(I)}^{(a+1)}\big)}$$

$$\frac{1}{\operatorname{ch}\big(\lambda_{R(I)}^{(n-1)} - \lambda_{\tau R(I)}^{(n)}\big)} \frac{1}{\operatorname{sh}\big(\lambda_{\tau R(I)}^{(n)} - \lambda_{R\tau R(I)}^{(n)}\big)} \prod_{a=1}^{n-1} \frac{1}{\operatorname{ch}\big(\lambda_{\tau^{-1}R\tau R(I)}^{(a)} - \lambda_{\tau^{-1}R\tau R(I)}^{(a+1)}\big)}$$

- Using the Fourier transform of $1/\operatorname{sh} x$ we get the density operator

$$\hat{K} = \frac{1}{\operatorname{ch}\hat{p}} \frac{i\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}} \frac{1}{\operatorname{ch}\hat{p}} \frac{1}{\operatorname{ch}^{n-3}\hat{p}} \frac{1}{\operatorname{ch}\hat{p}} \frac{-i\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}} \frac{1}{\operatorname{ch}\hat{p}} \frac{1}{\operatorname{ch}^{n-3}\hat{p}}$$

- The D_4-quiver has $n = 2$. Let us in addition include a fundamental field coupled to the central node, masses for the bi-fundamentals and some FI terms

$$Z(N) = \frac{1}{N!^4(2N)!} \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{I=1}^{2N} \frac{e^{2i\pi\left(\zeta_0\lambda_I^{(0)}+\zeta_1\lambda_I^{(1)}+\zeta_2\lambda_I^{(2)}\right)}}{\operatorname{ch}\lambda_I^{(1)}} \prod_{I<J} \operatorname{sh}^2\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)$$

$$\frac{\prod_{i<j}\operatorname{sh}^2\pi\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right)\operatorname{sh}^2\left(\lambda_{N+i}^{(0)} - \lambda_{N+j}^{(0)}\right)\operatorname{sh}^2\left(\lambda_i^{(2)} - \lambda_j^{(2)}\right)\operatorname{sh}^2\left(\lambda_{N+i}^{(2)} - \lambda_{N+j}^{(2)}\right)}{\prod_{i,J}\operatorname{ch}\left(\lambda_i^{(0)} - \lambda_J^{(1)} - m_0\right)\operatorname{ch}\left(\lambda_{N+i}^{(0)} - \lambda_J^{(1)} - m_0\right)\operatorname{ch}\left(\lambda_i^{(2)} - \lambda_J^{(1)} - m_2\right)\operatorname{ch}\left(\lambda_{N+i}^{(2)} - \lambda_J^{(1)} - m_2\right)}$$

- After applying the Cauchy identity we find the density function

$$K\left(\lambda, \lambda'\right) = \int d\lambda_1^{(0)} d\lambda_{N+1}^{(0)} d\lambda^{(1)} d\lambda_1^{(2)} d\lambda_{N+1}^{(2)}$$

$$\frac{1}{\operatorname{ch}\left(\lambda - \lambda_I^{(0)} + m_0\right)} \frac{e^{2\pi i\zeta_0\left(\lambda_1^{(0)}+\lambda_{N+1}^{(0)}\right)}}{\operatorname{sh}\left(\lambda_1^{(0)} - \lambda_{N+1}^{(0)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_{N+1}^{(0)} - \lambda^{(1)} - m_0\right)} \frac{e^{2\pi i\zeta_1\lambda^{(1)}}}{\operatorname{ch}\lambda^{(1)}}$$

$$\frac{1}{\operatorname{ch}\left(\lambda^{(1)} - \lambda_{N+1}^{(2)} + m_2\right)} \frac{e^{2\pi i\zeta_0(\lambda_1^{(2)}+\lambda_{N+1}^{(2)})}}{\operatorname{sh}\left(\lambda_{N+1}^{(2)} - \lambda_1^{(2)}\right)} \frac{1}{\operatorname{ch}\left(\lambda_1^{(2)} - \lambda' - m_2\right)} \frac{e^{2\pi i\zeta_1\lambda'}}{\operatorname{ch}\lambda'}$$

- The D_4-quiver has $n=2$. Let us in addition include a fundamental field coupled to the central node, masses for the bi-fundamentals and some FI terms

$$Z(N)=\frac{1}{N!^4(2N)!}\int\prod_{a=0}^{2}d^{2N}\lambda^{(a)}\prod_{I=1}^{2N}\frac{e^{2i\pi\left(\zeta_0\lambda_I^{(0)}+\zeta_1\lambda_I^{(1)}+\zeta_2\lambda_I^{(2)}\right)}}{\operatorname{ch}\lambda_I^{(1)}}\prod_{I<J}\operatorname{sh}^2\left(\lambda_I^{(1)}-\lambda_J^{(1)}\right)$$

$$\frac{\prod_{i<j}\operatorname{sh}^2\pi\left(\lambda_i^{(0)}-\lambda_j^{(0)}\right)\operatorname{sh}^2\left(\lambda_{N+i}^{(0)}-\lambda_{N+j}^{(0)}\right)\operatorname{sh}^2\left(\lambda_i^{(2)}-\lambda_j^{(2)}\right)\operatorname{sh}^2\left(\lambda_{N+i}^{(2)}-\lambda_{N+j}^{(2)}\right)}{\prod_{i,J}\operatorname{ch}\left(\lambda_i^{(0)}-\lambda_J^{(1)}-m_0\right)\operatorname{ch}\left(\lambda_{N+i}^{(0)}-\lambda_J^{(1)}-m_0\right)\operatorname{ch}\left(\lambda_i^{(2)}-\lambda_J^{(1)}-m_2\right)\operatorname{ch}\left(\lambda_{N+i}^{(2)}-\lambda_J^{(1)}-m_2\right)}$$

- After applying the Cauchy identity we find the density function

$$K\left(\lambda,\lambda'\right)=\int d\lambda_1^{(0)}d\lambda_{N+1}^{(0)}d\lambda^{(1)}d\lambda_1^{(2)}d\lambda_{N+1}^{(2)}$$

$$\frac{1}{\operatorname{ch}\left(\lambda-\lambda_I^{(0)}+m_0\right)}\frac{e^{2\pi i\zeta_0\left(\lambda_1^{(0)}+\lambda_{N+1}^{(0)}\right)}}{\operatorname{sh}\left(\lambda_1^{(0)}-\lambda_{N+1}^{(0)}\right)}\frac{1}{\operatorname{ch}\left(\lambda_{N+1}^{(0)}-\lambda^{(1)}-m_0\right)}\frac{e^{2\pi i\zeta_1\lambda^{(1)}}}{\operatorname{ch}\lambda^{(1)}}$$

$$\frac{1}{\operatorname{ch}\left(\lambda^{(1)}-\lambda_{N+1}^{(2)}+m_2\right)}\frac{e^{2\pi i\zeta_0(\lambda_1^{(2)}+\lambda_{N+1}^{(2)})}}{\operatorname{sh}\left(\lambda_{N+1}^{(2)}-\lambda_1^{(2)}\right)}\frac{1}{\operatorname{ch}\left(\lambda_1^{(2)}-\lambda'-m_2\right)}\frac{e^{2\pi i\zeta_1\lambda'}}{\operatorname{ch}\lambda'}$$

- This can be expressed as the density operator

$$\hat{K}=\frac{e^{-2i\pi m_0\hat{p}}}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_0\hat{q}}\,\frac{\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_0\hat{q}}\,\frac{e^{2i\pi m_0\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2i\pi\zeta_1\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{e^{-2i\pi m_2\hat{p}}}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_2\hat{q}}\,\frac{\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_2\hat{q}}\,\frac{e^{2i\pi m_2\hat{p}}}{\operatorname{ch}\hat{p}}\,\frac{e^{2i\pi\zeta_1\hat{q}}}{\operatorname{ch}\hat{q}}$$

$$=e^{-4i\pi(\zeta_0m_0+\zeta_2m_2)}\,\frac{1}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_0\hat{q}}\,\frac{\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_0\hat{q}}\,\frac{1}{\operatorname{ch}\hat{p}}\,\frac{e^{2i\pi\zeta_1\hat{q}}}{\operatorname{ch}\hat{q}}\,\frac{1}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_2\hat{q}}\,\frac{\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}}\,e^{2i\pi\zeta_2\hat{q}}\,\frac{1}{\operatorname{ch}\hat{p}}\,\frac{e^{2i\pi\zeta_1\hat{q}}}{\operatorname{ch}\hat{q}}$$

Sp-quivers

- The mirror theory is a 2-node linear Sp-quiver with a pair of fundamentals on each node. [Dey,Distler]
- The partition function is

$$Z(N) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} d\lambda_i^{(0)} d\lambda_i^{(1)} \prod_{a=0}^{1} \prod_{i=1}^{N} \frac{\operatorname{sh}^2 2\lambda_i^{(a)}}{\operatorname{ch}^2 \lambda_i^{(a)} \operatorname{ch}\left(\lambda_i^{(a)} - \tilde{\mu}_a\right) \operatorname{ch}\left(\lambda_i^{(a)} + \tilde{\mu}_a\right)}$$

$$\frac{\prod_{a=0}^{1} \prod_{i<j} \operatorname{sh}^2\left(\lambda_i^{(a)} - \lambda_j^{(a)}\right) \operatorname{sh}^2\left(\lambda_i^{(a)} + \lambda_j^{(a)}\right)}{\prod_{i,j} \operatorname{ch}\left(\lambda_i^{(0)} - \lambda_j^{(1)} - \tilde{m}\right) \operatorname{ch}\left(\lambda_i^{(0)} - \lambda_j^{(1)} + \tilde{m}\right) \operatorname{ch}\left(\lambda_i^{(0)} + \lambda_j^{(1)} - \tilde{m}\right) \operatorname{ch}\left(\lambda_i^{(0)} + \lambda_j^{(1)} + \tilde{m}\right)}$$

- Let us define $\lambda_{i+N} = -\lambda_i$

$$Z(N) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} d\lambda_i^{(0)} d\lambda_i^{(1)} \prod_{a=0}^{1} \prod_{i=1}^{N} \frac{\operatorname{sh} 2\lambda_i^{(a)}}{\operatorname{ch}^2 \lambda_i^{(a)}} \frac{1}{\prod_{a=0}^{1} \prod_{I=1}^{2N} \operatorname{ch}\left(\lambda_I^{(a)} - \tilde{\mu}_a\right)}$$

$$\times \frac{\prod_{I<J} \operatorname{sh}\left(\lambda_I^{(0)} - \lambda_J^{(0)}\right) \operatorname{sh}\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)}{\prod_{I,J} \operatorname{ch}\left(\lambda_I^{(0)} - \lambda_J^{(1)} - \tilde{m}\right)}$$

- We can apply the Cauchy identity to the last line, which becomes

$$\sum_{\tau \in S_{2N}} (-1)^\sigma \prod_{I=1}^{2N} \frac{1}{\operatorname{ch}\left(\lambda_I^{(0)} - \lambda_{\tau(I)}^{(1)} - \tilde{m}\right)}$$

- We have a similar issue to before where we have the reflection operator $R(I) = I \pm N$

$$Z(N) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} d\lambda_i^{(0)} d\lambda_i^{(1)} \prod_{a=0}^{1} \prod_{i=1}^{N} \frac{\operatorname{sh} \lambda_i^{(a)}}{\operatorname{ch} \lambda_i^{(a)}} \frac{1}{\prod_{a=0}^{1} \prod_{i=1}^{N} \operatorname{ch}\left(\lambda_i^{(a)} - \tilde{\mu}_a\right) \operatorname{ch}\left(\lambda_{R(i)}^{(a)} - \tilde{\mu}_a\right)}$$
$$\times \sum_{\tau \in S_{2N}} (-1)^{\sigma} \prod_{i=1}^{N} \frac{1}{\operatorname{ch}\left(\lambda_i^{(0)} - \lambda_{\tau(i)}^{(1)} - \tilde{m}\right) \operatorname{ch}\left(\lambda_{R(i)}^{(0)} - \lambda_{\tau R(i)}^{(1)} - \tilde{m}\right)}$$

- The density operator is then

$$\hat{K} = \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q} \operatorname{ch}(\hat{q} - \tilde{\mu}_1) \operatorname{ch}(\hat{q} + \tilde{\mu}_1)} \frac{e^{2i\pi \tilde{m} \hat{p}}}{\operatorname{ch} \hat{p}} \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q} \operatorname{ch}(\hat{q} - \tilde{\mu}_2) \operatorname{ch}(\hat{q} + \tilde{\mu}_2)} \frac{e^{2i\pi \tilde{m} \hat{p}}}{\operatorname{ch} \hat{p}}$$

- Finally we conjugate by

$$\hat{K} \to e^{2i\pi \tilde{\mu}_1 \hat{p}} \, \hat{K} \, e^{-2i\pi \tilde{\mu}_1 \hat{p}}$$

to find

$$\hat{K} = \frac{1}{\operatorname{ch} \hat{q}} e^{2i\pi \tilde{\mu}_1 \hat{p}} \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q}} e^{2i\pi \tilde{\mu}_1 \hat{p}} \frac{1}{\operatorname{ch} \hat{q}} \frac{e^{2i\pi (\tilde{m} - \tilde{\mu}_1 - \tilde{\mu}_2) \hat{p}}}{\operatorname{ch} \hat{p}} \frac{1}{\operatorname{ch} \hat{q}} e^{2i\pi \tilde{\mu}_2 \hat{p}} \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q}} e^{2i\pi \tilde{\mu}_2 \hat{p}} \frac{1}{\operatorname{ch} \hat{q}} \frac{e^{2i\pi (\tilde{m} - \tilde{\mu}_1 - \tilde{\mu}_2) \hat{p}}}{\operatorname{ch} \hat{p}}$$

- We have a similar issue to before where we have the reflection operator $R(I) = I \pm N$

$$Z(N) = \frac{1}{(N!)^2} \int \prod_{i=1}^{N} d\lambda_i^{(0)} d\lambda_i^{(1)} \prod_{a=0}^{1} \prod_{i=1}^{N} \frac{\operatorname{sh} \lambda_i^{(a)}}{\operatorname{ch} \lambda_i^{(a)}} \frac{1}{\prod_{a=0}^{1} \prod_{i=1}^{N} \operatorname{ch}\left(\lambda_i^{(a)} - \tilde{\mu}_a\right) \operatorname{ch}\left(\lambda_{R(i)}^{(a)} - \tilde{\mu}_a\right)}$$

$$\times \sum_{\tau \in S_{2N}} (-1)^\sigma \prod_{i=1}^{N} \frac{1}{\operatorname{ch}\left(\lambda_i^{(0)} - \lambda_{\tau(i)}^{(1)} - \tilde{m}\right) \operatorname{ch}\left(\lambda_{R(i)}^{(0)} - \lambda_{\tau R(i)}^{(1)} - \tilde{m}\right)}$$

- The density operator is then

$$\hat{K} = \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q} \operatorname{ch}(\hat{q} - \tilde{\mu}_1) \operatorname{ch}(\hat{q} + \tilde{\mu}_1)} \frac{e^{2i\pi\tilde{m}\hat{p}}}{\operatorname{ch} \hat{p}} \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q} \operatorname{ch}(\hat{q} - \tilde{\mu}_2) \operatorname{ch}(\hat{q} + \tilde{\mu}_2)} \frac{e^{2i\pi\tilde{m}\hat{p}}}{\operatorname{ch} \hat{p}}$$

- Finally we conjugate by

$$\hat{K} \to e^{2i\pi\tilde{\mu}_1\hat{p}} \hat{K} e^{-2i\pi\tilde{\mu}_1\hat{p}}$$

to find

$$\hat{K} = \frac{1}{\operatorname{ch} \hat{q}} e^{2i\pi\tilde{\mu}_1\hat{p}} \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q}} e^{2i\pi\tilde{\mu}_1\hat{p}} \frac{1}{\operatorname{ch} \hat{q}} \frac{e^{2i\pi(\tilde{m}-\tilde{\mu}_1-\tilde{\mu}_2)\hat{p}}}{\operatorname{ch} \hat{p}} \frac{1}{\operatorname{ch} \hat{q}} e^{2i\pi\tilde{\mu}_2\hat{p}} \frac{\operatorname{sh} \hat{q}}{\operatorname{ch} \hat{q}} e^{2i\pi\tilde{\mu}_2\hat{p}} \frac{1}{\operatorname{ch} \hat{q}} \frac{e^{2i\pi(\tilde{m}-\tilde{\mu}_1-\tilde{\mu}_2)\hat{p}}}{\operatorname{ch} \hat{p}}$$

- A somewhat different formalism gives even and odd operators. [Mezei,Pufu]
- One can calculate from the above density the Airy function expression.

Mirror symmetry

- For the D-quiver, with all masses set to zero we had

$$\hat{K}_D = \frac{1}{\operatorname{ch}\hat{p}}\, e^{2i\pi\zeta_0\hat{q}}\, \frac{\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}}\, e^{2i\pi\zeta_0\hat{q}}\, \frac{1}{\operatorname{ch}\hat{p}}\, \frac{e^{2i\pi\zeta_1\hat{q}}}{\operatorname{ch}\hat{q}}\, \frac{1}{\operatorname{ch}\hat{p}}\, e^{2i\pi\zeta_2\hat{q}}\, \frac{\operatorname{sh}\hat{p}}{\operatorname{ch}\hat{p}}\, e^{2i\pi\zeta_2\hat{q}}\, \frac{1}{\operatorname{ch}\hat{p}}\, \frac{e^{2i\pi\zeta_1\hat{q}}}{\operatorname{ch}\hat{q}}$$

- Compare to the Sp-quiver

$$\hat{K}_{Sp} = \frac{1}{\operatorname{ch}\hat{q}}\, e^{2i\pi\tilde{\mu}_1\hat{p}}\, \frac{\operatorname{sh}\hat{q}}{\operatorname{ch}\hat{q}}\, e^{2i\pi\tilde{\mu}_1\hat{p}}\, \frac{1}{\operatorname{ch}\hat{q}}\, \frac{e^{2i\pi(\tilde{m}-\tilde{\mu}_1-\tilde{\mu}_2)\hat{p}}}{\operatorname{ch}\hat{p}}\, \frac{1}{\operatorname{ch}\hat{q}}\, e^{2i\pi\tilde{\mu}_2\hat{p}}\, \frac{\operatorname{sh}\hat{q}}{\operatorname{ch}\hat{q}}\, e^{2i\pi\tilde{\mu}_2\hat{p}}\, \frac{1}{\operatorname{ch}\hat{q}}\, \frac{e^{2i\pi(\tilde{m}-\tilde{\mu}_1-\tilde{\mu}_2)\hat{p}}}{\operatorname{ch}\hat{p}}$$

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- If we identify

$$\zeta_0 = \tilde{\mu}_1\,, \qquad \zeta_2 = \tilde{\mu}_2 \qquad \zeta_1 = \tilde{m} - \tilde{\mu}_1 - \tilde{\mu}_2\,,$$

  Then the two expressions are identical with

$$\hat{q} \to \hat{p}\,, \qquad \hat{p} \to -\hat{q}$$

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Again, in the Fermi-gas formalism, mirror symmetry is a linear canonical transformation.

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> Again, in the Fermi-gas formalism, mirror symmetry is a linear canonical transformation.

- This mapping of parameters exactly matches what has been previously found via complicated integral identities [Dey,Distler]

Summary

- The Fermi-gas approach allows to solve Many $\mathcal{N} \geq 3$ to all orders in $1/N$.
- Can be generalized to D-quivers and Sp-quivers
- Mirror symmetry is very simple in this setup - just a symplectic transformation on phase space.
- Many more results for D-quivers and Sp-quivers will be in our upcoming paper.

The end