Surface operators in 4d N=2 gauge theories

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Surface operators

* 2-dimensional (= co-dimension 2) defects in 4D field theories

* defined via

  - singular behavior of fields near the surface

  - 2D degrees of freedom localized on the surface

1/2-BPS surface operators in **4D** $\mathcal{N} = 2$ **SUSY** gauge theories

$$4D\ \mathcal{N} = 2 \quad \Longrightarrow \quad 2D\ \mathcal{N} = (2,2)$$

have been studied extensively using localization technique.

Labels of surface operators

Surface operators supporting 2D gauge theory

- **discrete labels** . . . choice of gauge group
- **continuous labels** . . . vortex counting (FI-theta) parameters

Labels of surface operators

Surface operators providing singular boundary condition

* $U(N)$ gauge field near $w=0$ in $\mathbb{C}^2(z,w)$,

$$A \sim \mathrm{diag}(\underbrace{\alpha_1, \cdots, \alpha_1}_{n_1 \text{ times}}, \underbrace{\alpha_2, \cdots, \alpha_2}_{n_2 \text{ times}}, \cdots, \underbrace{\alpha_M, \cdots, \alpha_M}_{n_M \text{ times}}) d\theta$$

$(\theta = \mathrm{arg} w)$

- **discrete labels** . . . Levi subgroup $\mathbb{L} = U(n_1) \times \cdots \times U(n_M)$
- **continuous labels** . . . $(\alpha_1, \cdots, \alpha_M)$ $(2\pi > \alpha_1 > \cdots > \alpha_M \geq 0)$

Class S theories

$\boxed{N \text{ M5-branes wrapped on } M_4 \times \Sigma}$

When $M_4 = \mathbb{R}^4_{\epsilon_1,\epsilon_2}$, A_{N-1} W-algebra

$M_4 = S^4_b$, A_{N-1} Toda CFT arises on Σ .

Surface operators on $C_2 \subset M_4$ in class S theories :

1. **Co-dimension 4 defects** in (2,0) theory

   M2-branes ending at $C_2 \times \{\mathrm{pt}\} \subset M_4 \times \Sigma$

   . . . Toda degenerate insertion at a point in Σ .

2. **Co-dimension 2 defects** in (2,0) theory

   M5-branes intersecting at $C_2 \times \Sigma \subset M_4 \times \Sigma$

   . . . change of CFT on Σ .

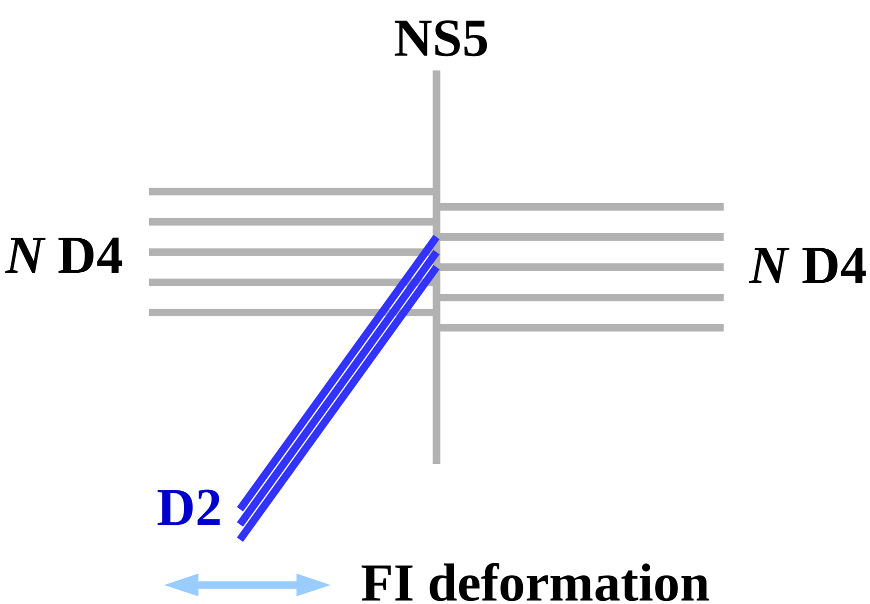
Surface operators
made of co-dimension 4 defects

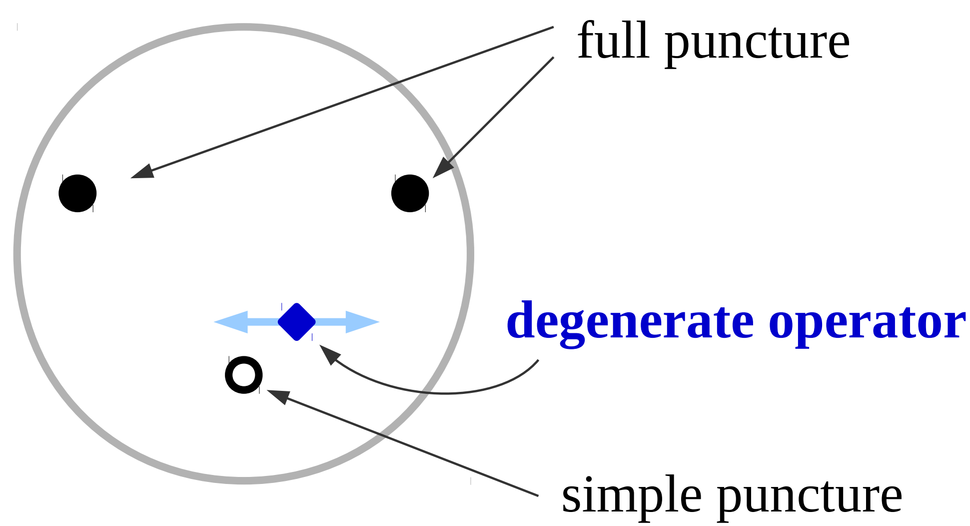
Surface operators from co-dimension 4 defects

Gomis-Le Floch '14

Identified the worldvolume theory of the surface operators put in the 4D theory of free N^2 hypermultiplets.

Obtained the expected Toda degenerate correlators.





Degenerate Toda representations

A_{N-1} Toda representations are labeled by N-component momentum

$$\vec{a} = (a_1, \cdots, a_N); \quad \sum_i a_i = 0.$$

Fully degenerate representations have discrete, pure imaginary momentum

$$\vec{a} = ib(\vec{\rho} + \vec{\lambda}_1) + ib^{-1}(\vec{\rho} + \vec{\lambda}_2)$$

$\vec{\rho}$: $SU(N)$ Weyl vector

$\vec{\lambda}_1, \vec{\lambda}_2$: highest weights of $SU(N)$ reps

They correspond to two intersecting surface operators

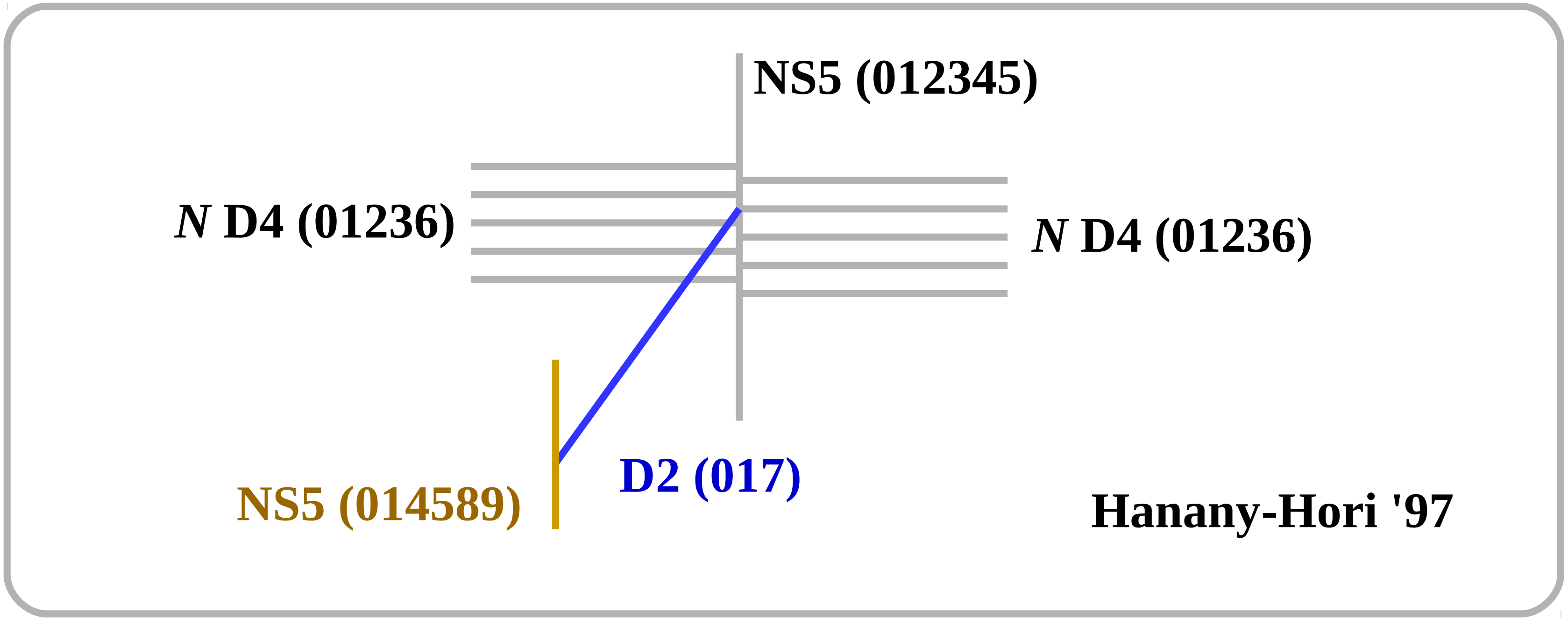
$$\mathcal{O}_{\vec{\lambda}_1}(x_3 = x_4 = 0), \quad \mathcal{O}_{\vec{\lambda}_2}(x_1 = x_2 = 0).$$

Surface operators are labeled by SU(N) representations.

Surface operator: fundamental rep

If it is near a simple puncture, the 2D theory is

$\mathcal{N} = (2,2)$ SQED with N electrons and N positrons.



D4 positions gauge the global symmetry of the 2D theory

$$SU(N) \times SU(N) \times U(1)\,.$$

Partition function on $S^2 \subset S^4$

*** 2D *N*-flavor SQED : (Coulomb branch localization)**

$$Z_{2\mathrm{D}} = \sum_{h \in \mathbb{Z}/2} \int \frac{d\sigma}{2\pi} z^{i\sigma+h} \bar{z}^{i\sigma-h} \prod_{s=1}^{N} \frac{\Gamma(-h-i\sigma+im_s)}{\Gamma(1-h+i\sigma-im_s)} \cdot \frac{\Gamma(h+i\sigma-i\tilde{m}_s)}{\Gamma(1+h-i\sigma+i\tilde{m}_s)}$$

$$(z = e^{-2\pi\zeta+i\theta})$$

*** 4D hypermultiplets** : $$Z_{4\mathrm{D}} = \prod_{s,t=1}^{N} \Upsilon^{-1}\left(\tfrac{Q}{2} - im_s^{(4)} + i\tilde{m}_t^{(4)}\right)$$

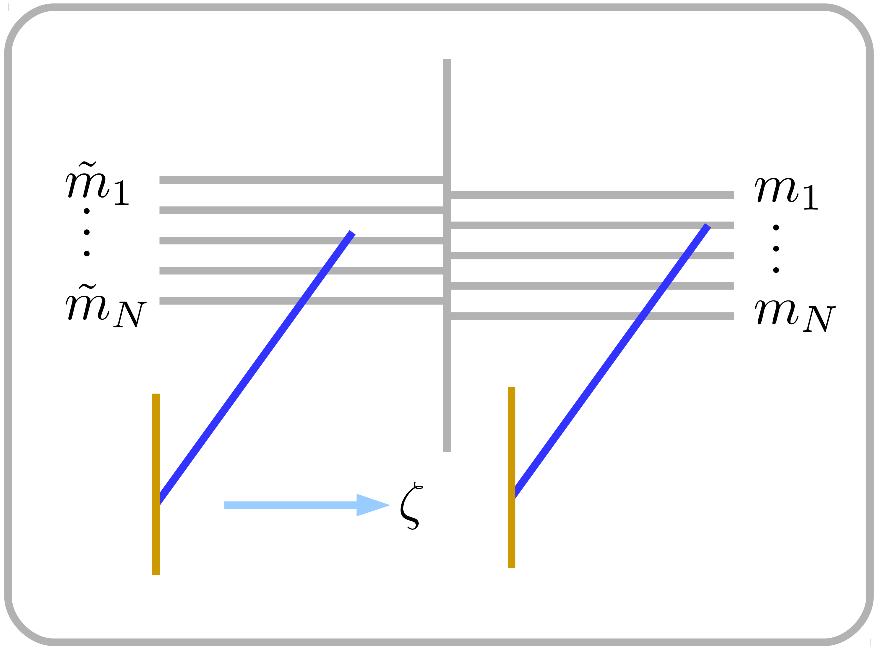
$$\Upsilon(x) \equiv \prod_{m,n\geq 0} (x+mb+nb^{-1})(x-Q-mb-nb^{-1})$$

* It turns out $m_s = bm_s^{(4)} - \frac{i}{4}(1-b^2), \quad \tilde{m}_s = b\tilde{m}_s^{(4)} + \frac{i}{4}(1-b^2)$.

Partition function on $S^2 \subset S^4$

* contour integral = sum of residues over poles

$$\zeta > 0 \quad \Rightarrow \quad \sigma = m_s - i|h| - i\mathbb{Z}_{\geq 0}$$
$$\zeta < 0 \quad \Rightarrow \quad \sigma = \tilde{m}_s + i|h| + i\mathbb{Z}_{\geq 0}$$
$$(s = 1, \cdots, N)$$



Higgs branch formula for $(\zeta > 0)$

$$Z_{\text{2D}} = \sum_{t=1}^{N} (x\bar{x})^{im_t} \frac{\prod_{s\neq t}^{N} \gamma(im_t - im_s)}{\prod_{s=1}^{N} \gamma(1 - im_t + i\tilde{m}_s)} F_t(m, \tilde{m}|x) F_t(m, \tilde{m}|\bar{x}),$$

$$x \equiv z(-1)^N, \quad F_t(m, \tilde{m}|x) = \sum_{n\geq 0} \prod_{s=1}^{N} \frac{(im_t - i\tilde{m}_s)_n}{(1 - im_s + im_t)_n} x^n \,.$$

Comparison

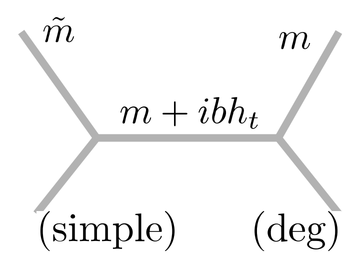
Partition function on $S^2 \subset S^4$

$$Z \;=\; Z_{\text{4D}} \cdot \sum_{t=1}^{N} \frac{\prod_{s \neq t}^{N} \gamma(im_t - im_s)}{\prod_{s=1}^{N} \gamma(1 - im_t + i\tilde{m}_s)} (x\bar{x})^{im_t} F_t(m, \tilde{m}|x) F_t(m, \tilde{m}|\bar{x}),$$

Toda correlator

$$\left\langle V^{\text{full}}_{\tilde{m}}(\infty) V^{\text{simple}}(1) V^{\text{deg}}(x) V^{\text{full}}_{m}(0) \right\rangle$$

$$= \sum_{t=1}^{N} C(\tilde{m}, \text{simple}, m + ibh_t) \cdot C(m + ibh_t, \text{deg}, m) \cdot \left| \quad \right|^2$$



$h_1, \cdots, h_N$: fundamental weights of SU(N)

Comparison

N Higgs branch vacua = N D4-branes the D2 can end on

= N intermediate channels

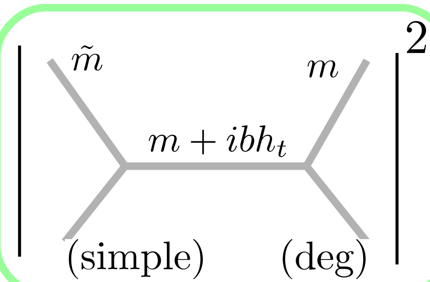
Partition function on $S^2 \subset S^4$

$$Z = Z_{\text{4D}} \cdot \sum_{t=1}^{N} \frac{\prod_{s\neq t}^{N} \gamma(im_t - im_s)}{\prod_{s=1}^{N} \gamma(1 - im_t + i\tilde{m}_s)} (x\bar{x})^{im_t} F_t(m, \tilde{m}|x) F_t(m, \tilde{m}|\bar{x}),$$

Toda correlator

$$\langle V^{\text{full}}_{\tilde{m}}(\infty) V^{\text{simple}}(1) V^{\text{deg}}(x) V^{\text{full}}_{m}(0) \rangle$$

$$= \sum_{t=1}^{N} C(\tilde{m}, \text{simple}, m + ibh_t) \cdot C(m + ibh_t, \text{deg}, m) \cdot$$

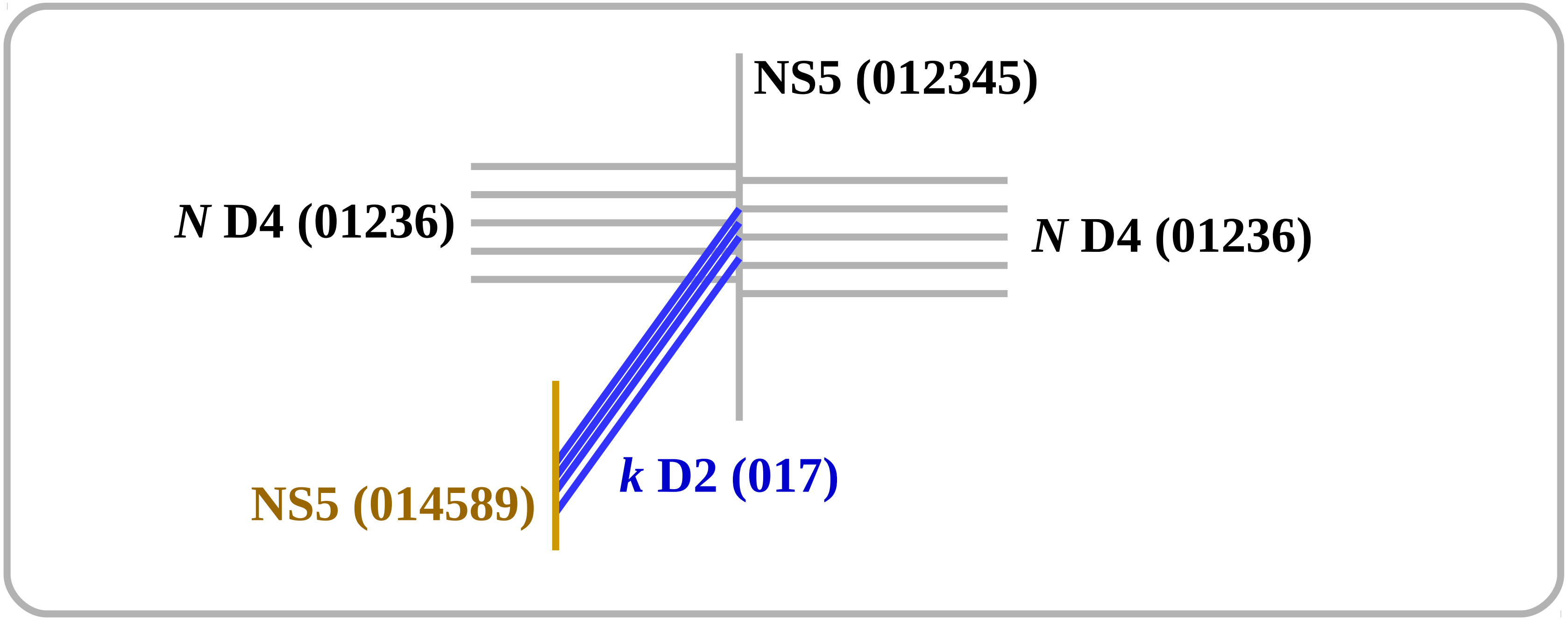


$h_1, \cdots, h_N$: fundamental weights of SU(N)

Surface operator: *k*-th anti-sym rep

If it is near a simple puncture, the 2D theory is

$\mathcal{N} = (2,2)\ U(k)$ **SQCD** with ***N*** fundamentals and ***N*** anti-fundamentals.



D4 positions gauge the global symmetry of the 2D theory

$$SU(N) \times SU(N) \times U(1)\,.$$

Partition function on $S^2 \subset S^4$

*** 2D *N*-flavor SU(*k*) SQCD : Coulomb branch localization**

$$Z_{\text{SQCD}} = \frac{1}{k!}\sum_{h\in(\mathbb{Z}/2)^k}\int\frac{d^k\sigma}{(2\pi)^k}\, z^{\text{Tr}(i\sigma+h)}\,\bar{z}^{\text{Tr}(i\sigma-h)}\prod_{i<j}\left[(\sigma_i-\sigma_j)^2+(h_i-h_j)^2\right]$$

$$\cdot\prod_{j=1}^{k}\prod_{s=1}^{N}\frac{\Gamma(-h_j-i\sigma_j+im_s)}{\Gamma(1-h_j+i\sigma_j-im_s)}\,\frac{\Gamma(h_j+i\sigma_j-i\tilde{m}_s)}{\Gamma(1+h_j-i\sigma_j+i\tilde{m}_s)}$$

$$= \frac{1}{k!}\prod_{i<j}^{k}\left[-(z_i\partial_{z_i}-z_j\partial_{z_j})(\bar{z}_i\partial_{\bar{z}_i}-\bar{z}_j\partial_{\bar{z}_j})\right]\prod_{j=1}^{k}Z^{(\text{SQED})}(m,\tilde{m},z_i,\bar{z}_i)\Big|_{z_i=z}$$

= . . . , integration contours can be closed

Higgs branch formula for $(\zeta > 0)$

$$Z \;=\; \sum_{\{t\}} (x\bar{x})^{i\sum_i m_{t_i}} \frac{\prod_{u\in\{t\}}\prod_{s\in\overline{\{t\}}}\gamma(im_s - im_u)}{\prod_{u\in\{t\}}\prod_s \gamma(1-im_u+i\tilde{m}_s)} F_{\{t\}}(m,\tilde{m}|x)F_{\{t\}}(m,\tilde{m}|\bar{x}),$$

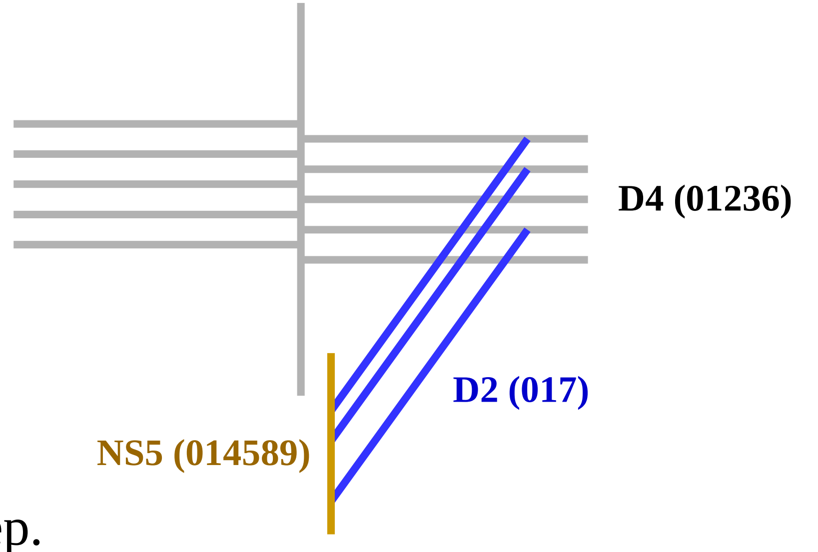
$$F_{\{t\}}(m,\tilde{m}|x) \equiv \sum_{k_1,\cdots,k_N\geq 0} x^{\sum k_i} \prod_{i<j} \frac{k_i - k_j + im_{t_i} - im_{t_j}}{im_{t_i} - im_{t_j}} \prod_{i=1}^{N} \frac{\prod_{u=1}^{N_a}(im_{t_i} - i\tilde{m}_u)_{k_i}}{\prod_{u=1}^{N_f}(1-im_u+im_{t_i})_{k_i}}$$

Higgs branch vacua are labeled by

$\{t\} = \{t_1,\cdots,t_k\} \subset \{1,\cdots,N\}$

k D2-branes end on ***N*** D4-branes

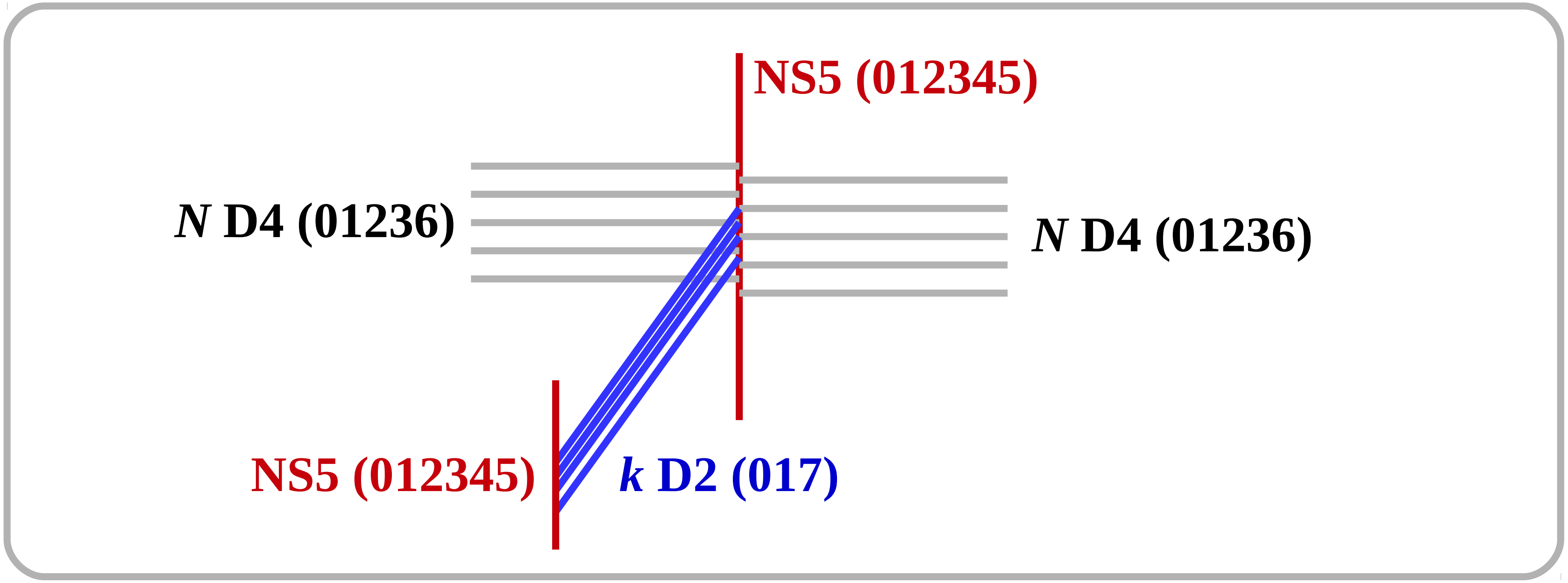
in a **fermionic** manner



${}_N C_k$ = dimension of k-th anti-symmetric rep.

Surface operator: *k*-th sym rep

$\mathcal{N} = (2,2)\ U(k)$ **SQCD** with ***N*** **fund**, ***N*** **anti-fund** and **1 adjoint.**



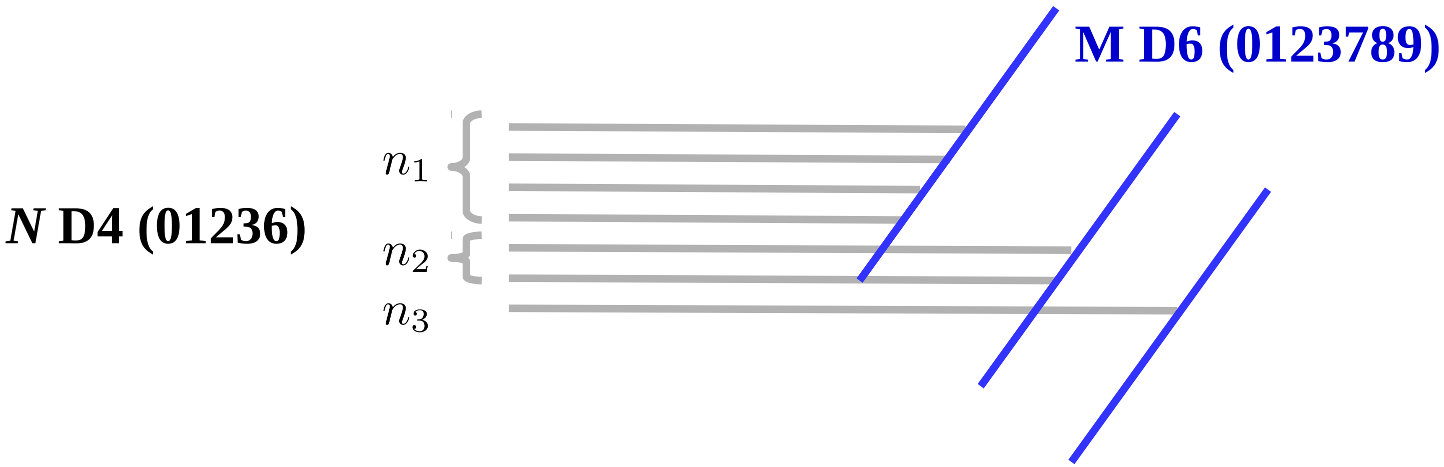
One can try **(Gomis-Le Floch '14)**

* construct more general reps

* match the dualities of gauge theory with symmetry of Toda correlators

Surface operators
made of co-dimension 2 defects

Co-dimension 2 defects of (2,0) theory



In IIA description of class S-theories,

Some co-dimension 2 defects are described by

D4-branes ending on ***M* D6-branes.**

Co-dimension 2 defects in (2,0) theory of type A_{N-1}
are labeled by partition of *N*.

$$N \;=\; n_1 + n_2 + \cdots + n_M$$

Surface operator from co-dimension 2 defects

$SU(N)$ gauge field near the surface operator at $w = 0$

$$A \sim \mathrm{diag}(\underbrace{\alpha_1, \cdots, \alpha_1}_{n_1 \text{ times}}, \underbrace{\alpha_2, \cdots, \alpha_2}_{n_2 \text{ times}}, \cdots, \underbrace{\alpha_M, \cdots, \alpha_M}_{n_M \text{ times}}) d\theta$$

$$\Big(\alpha_1 > \alpha_2 > \cdots > \alpha_M > \alpha_1 - 2\pi \,, \ \theta = \arg w \Big)$$

- **discrete labels** $\ldots$ Levi subgroup $\mathbb{L} = S[U(n_1) \times \cdots \times U(n_M)]$
- **continuous labels** $\cdots$ $(\alpha_1, \cdots, \alpha_M)$

Gauge field configurations are labeled by instanton number k

as well as magnetic fluxes

$$m^I = \int_{w=0} \mathrm{tr}\left(\frac{F}{2\pi}\right)_{I\text{-th block}} \quad \Big(\sum_I m^I = 0\Big)$$

Parabolic subalgebra

. . . a subalgebra $\mathbb{P} \subset SL(N)$ spanned by elements x satisfying

$$[\alpha, x] = \lambda x, \quad \lambda \geq 0$$

$$\Big(\alpha \equiv \mathrm{diag}(\underbrace{\alpha_1, \cdots, \alpha_1}_{n_1 \ \text{times}}, \underbrace{\alpha_2, \cdots, \alpha_2}_{n_2 \ \text{times}}, \ \cdots \ , \underbrace{\alpha_M, \cdots, \alpha_M}_{n_M \ \text{times}})\Big)$$

$$x = \begin{pmatrix} (*)_{n_1 \times n_1} & (*)_{n_1 \times n_2} & \cdots & (*)_{n_1 \times n_M} \\ 0 & (*)_{n_2 \times n_2} & \cdots & (*)_{n_2 \times n_M} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & (*)_{n_M \times n_M} \end{pmatrix}$$

Note : $\mathbb{P}$ depends on the ordering of $(n_1, \cdots, n_M)$ while $\mathbb{L}$ does not.

Partial flag manifold

$$F(k_1, \cdots, k_M) \;\equiv\; \text{space of different nested embeddings}$$
$$\mathbb{C}^{k_1} \subset \mathbb{C}^{k_2} \subset \cdots \subset \mathbb{C}^{k_M} \;\; (k_1 < k_2 < \cdots < k_M)$$

* **coset space description**

$$F(k_1, \cdots, k_M) \;=\; SU(N)/\mathbb{L} \;=\; SL(N)/\mathbb{P}, \quad (N \equiv k_M)$$

$$\mathbb{L} = S[U(k_1) \times U(k_2 - k_1) \times \cdots \times U(k_M - k_{M-1})]$$

* Different $\mathbb{P}$ for the same $\mathbb{L}$ corresponds to different complex structure.

* **GLSM description**

$$F(k_1, \cdots, k_M) \;=\; \left(\mathbb{C}^{k_1 k_2} \times \mathbb{C}^{k_2 k_3} \times \cdots \times \mathbb{C}^{k_{M-1} k_M}\right) // \prod_{i=1}^{M-1} GL(k_i)$$

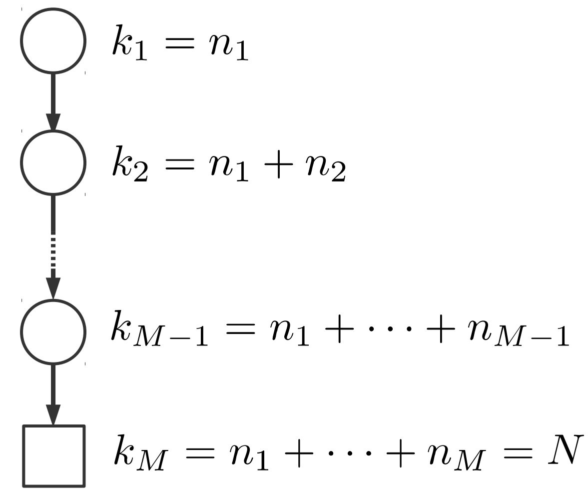
2D field theory description

Gukov-Witten '06

Surface operators of type $N = n_1 + \cdots + n_M$

in $\mathcal{N} = 2$ pure SYM

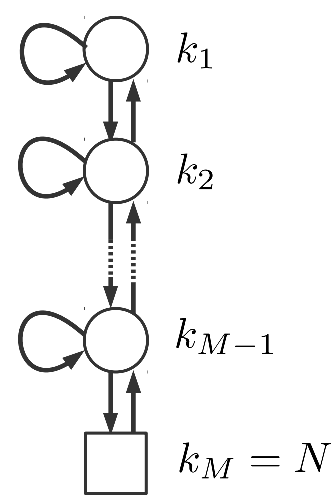
$\boxed{\mathcal{N} = (2,2) \text{ GLSM for } SL(N)/\mathbb{P}}$



Surface operators of the same type

in $\mathcal{N} = 4$ SYM

$\boxed{\mathcal{N} = (4,4) \text{ GLSM for } \mathrm{T}^*[SL(N)/\mathbb{P}]}$



Partition function for *ramified instantons*

calculable by using an **equivalence :**

moduli space of U(N) instantons

with **surface operator** labeled by

$$N = n_1 + \cdots + n_M$$

# of instantons k

magnetic fluxes $m^I \quad \left(\sum_{I=1}^{M} m^I = 0\right)$

moduli space of U(N) instantons

on **orbifold** $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$, acting by

$$\mathrm{diag}(\underbrace{\omega, \cdots \omega}_{n_1}, \cdots, \underbrace{\omega^M, \cdots, \omega^M}_{n_M})$$

on fundamental representation.

# of fractional instantons

$$k_I \quad (I = 1, \cdots, M)$$

$$k_M = k, \quad k_{I+1} = k_I + m^I$$

ADHM construction of moduli space

RECAP : moduli space of ordinary *k*-instantons of U(*N*)

$$\mathcal{M}_{N,k} = \frac{\left\{ A_{(k,k)},\ B_{(k,k)},\ P_{(k,N)},\ Q_{(N,k)} \,\middle|\, [A,B] + PQ = 0 \right\}}{GL(k)}$$

Infinitesimal action of $g = \mathrm{diag}(a_1, \cdots, a_N) \in U(N)$

and $\epsilon_1 J_{12} + \epsilon_2 J_{34} \in SO(4)$

$$\delta(A,B,P,Q) \ = \ \Big(\epsilon_1 A\ ,\ \epsilon_2 B\ ,\ -Pg\ ,\ gQ + (\epsilon_1 + \epsilon_2) Q\Big)$$

The point (A,B,P,Q) is fixed under this action if

$$^{\exists}\phi \text{ s.t. } \delta(A,B,P,Q) = \Big([\phi, A]\, ,\, [\phi, B]\, ,\, \phi P\, ,\, -Q\phi\Big)$$

ADHM construction of moduli space

RECAP : moduli space of ordinary *k*-instantons of U(*N*)

$$\mathcal{M}_{N,k} = \frac{\left\{ A_{(k,k)},\ B_{(k,k)},\ P_{(k,N)},\ Q_{(N,k)} \,\middle|\, [A,B] + PQ = 0 \right\}}{GL(k)}$$

Each fixed point is labeled by *k* boxes forming *N* Young diagrams.

The fixed points are the representatives of equivariant cohomology

$$H_{a,\epsilon_1,\epsilon_2}(\mathcal{M}_{N,k})$$

AGT : the symmetry W_N acts on the space $\bigoplus_k H_{a,\epsilon_1,\epsilon_2}(\mathcal{M}_{N,k})$

Instanton partition function

Simplest case : N=4 SYM

$$Z_{\rm inst} = \sum_{\vec{Y}} q^{|Y_1|+\cdots+|Y_N|} = \prod_{n\geq 1} (1-q^n)^{-N}$$

Adding a column of height n *to any one of*

the diagrams $Y_1, \cdots, Y_N$ *costs* q^n .

Orbifolding

For instantons on orbifold, we apply the **projection**

$$\begin{aligned} [\phi^\circ, A] &= 0, \\ [\phi^\circ, B] &= \omega B, \\ \phi^\circ P - P g^\circ &= 0, \\ g^\circ Q - Q\phi^\circ &= \omega Q \end{aligned}$$

$$g^\circ = \mathrm{diag}\Big(\underbrace{\omega, \cdots, \omega}_{n_1}, \cdots, \underbrace{\omega^M, \cdots, \omega^M}_{n_M}\Big)$$

$$\phi^\circ = \mathrm{diag}\Big(\underbrace{\omega, \cdots, \omega}_{k_1}, \cdots, \underbrace{\omega^M, \cdots, \omega^M}_{k_M}\Big) \qquad (\omega = e^{\frac{2\pi i}{M}})$$

$$A = \begin{pmatrix} A^{(1)}_{(k_1\times k_1)} & & & \\ & A^{(2)}_{(k_2\times k_2)} & & \\ & & \ddots & \\ & & & A^{(M)}_{(k_M\times k_M)} \end{pmatrix} \qquad P = \begin{pmatrix} P^{(1)}_{(k_1\times n_1)} & & & \\ & P^{(2)}_{(k_2\times n_2)} & & \\ & & \ddots & \\ & & & P^{(M)}_{(k_M\times n_M)} \end{pmatrix}$$

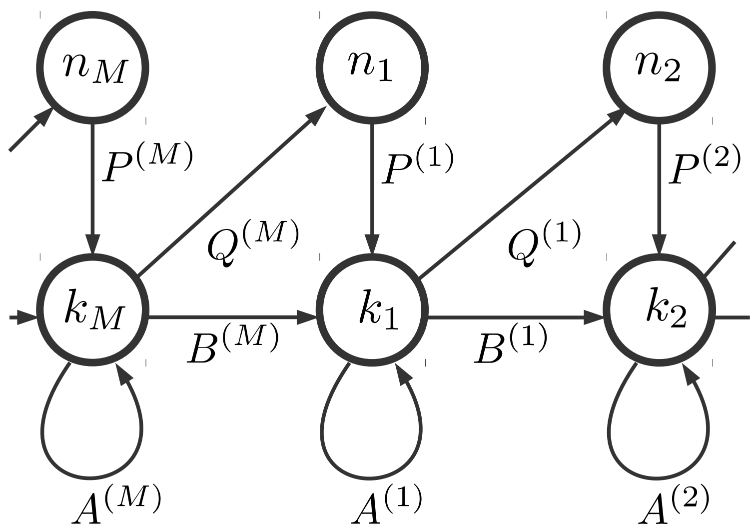
$$B = \begin{pmatrix} & & & B^{(M)}_{(k_1\times k_M)} \\ B^{(1)}_{(k_2\times k_1)} & & & \\ & \ddots & & \\ & & B^{(M)}_{(k_M\times k_{M-1})} & \end{pmatrix} \qquad Q = \begin{pmatrix} & & & Q^{(M)}_{(n_1\times k_M)} \\ Q^{(1)}_{(n_2\times k_1)} & & & \\ & \ddots & & \\ & & Q^{(M)}_{(n_M\times k_{M-1})} & \end{pmatrix}$$

Orbifolding

Moduli space :

$$\mathcal{M}_{N,\vec{k}} = \left\{ \begin{array}{l} A^{(I)}_{(k_I \times k_I)},\ B^{(I)}_{(k_{I+1} \times k_I)},\ P^{(I)}_{(k_I \times n_I)},\ Q^{(I)}_{(n_{I+1} \times k_I)} \\ \text{s.t.}\ \ A^{(I+1)} B^{(I)} - B^{(I)} A^{(I)} + P^{(I+1)} Q^{(I)} = 0 \end{array} \right\} \Big/ \prod GL(k_I)$$

Chain-saw quiver :



Fixed points and Young diagrams

On $\mathcal{M}_{N,\vec{k}}$, we consider the action of

$$g = \mathrm{diag}(a_1, \cdots, a_N) + \frac{\epsilon_2}{M}\Big(\underbrace{1, \cdots, 1}_{n_1}, \cdots, \underbrace{M, \cdots, M}_{n_M}\Big) \in U(N)$$

and $\epsilon_1 J_{12} + \dfrac{\epsilon_2}{M} J_{34} \in SO(4)$

The fixed points are labeled by k boxes forming N diagrams.

$k = (k_1 + k_2 + \cdots + k_M)$ ... k_I boxes are assigned the $\mathbb{Z}_M$ charge I ,

$N = (n_1 + n_2 + \cdots + n_M)$ ... N diagrams form M groups,

$$Y_{s,I} = \begin{array}{|c|c|c|c|} \cline{1-1} I+2 \\ \cline{1-3} I+1 & I+1 & I+1 \\ \hline I & I & I & I \\ \hline \end{array} \qquad \begin{pmatrix} s = 1, \cdots, n_I \\ I = 1, \cdots, M \end{pmatrix}$$

The boxes in the j-th row of $Y_{s,I}$ carry $\mathbb{Z}_M$ charge $I + j - 1$.

Ramified instanton partition function

Simplest case : N=4 SYM

$$Z_{\text{inst}} = \sum_{\vec{Y}} \prod_I q_I^{k_I} = \prod_{I=1}^{M} \prod_{J=1}^{\infty} \Big(1 - \prod_{s=I}^{I+J-1} q_s\Big)^{-n_I}$$

Adding a column of height J *to any one of*

the diagrams $Y_{1,I}, \cdots, Y_{n_I,I}$ *costs* $q_I q_{I+1} \cdots q_{I+J-1} = \prod_{s=I}^{I+J-1} q_s$.

$$\begin{array}{|c|}\hline I+2 \\ \hline I+1 \\ \hline I \\ \hline \end{array}$$

Kanno-Tachikawa '11

compared this with the character of general W algebra.

General W algebra

$W(\widehat{SU}(N),[n_I])$ . . . Drinfeld-Sokolov reduction of current algebra $\widehat{SU}(N)$

according to the decomposition $N=(n_1+\cdots+n_M)$

Important properties :

$(N\times N)$ matrix is decomposed into $M\times M$ blocks.

(I,J)-th block is of size $(n_I\times n_J)$. It gives rise to currents

$$U^I{}_{J,s}(z)=\sum_n U^I{}_{J,s;n}z^{-n-s},$$

$$\text{spin } s=\frac{1}{2}|n_I-n_J|+1,\cdots,\frac{1}{2}|n_I+n_J|$$

Classification of mode operators $U^I{}_{J,s;n}$

- **creation operators** : $n < 0$ or $(n = 0, I < J)$
- **annihilation operators** : $n > 0$ or $(n = 0, I > J)$
- **Cartan generators** : $n = 0$ and $I = J$

In particular, the Cartan generator $U^I{}_{I,s=1;0} \equiv T_I$ satisfies

$$[T_I, U^J{}_{K,s;n}] = (\delta_{IJ} - \delta_{IK}) U^J{}_{K,s;n} \, .$$

We also have, as usual,

$$[L_0, U^J{}_{K,s;n}] = -n U^J{}_{K,s;n} \, .$$

Character

. . . Trace of the operator $\mathcal{O} \equiv q^{L_0} \prod_{I=1}^{M} y_I^{T_I}$ over a Verma module.

Now observe

$$\mathcal{O}\, U^I{}_{J,s;-n} \mathcal{O}^{-1} \;=\; (q^n y_I / y_J) \cdot U^I{}_{J,s;-n}$$

$$\boxed{q_1 \equiv y_1/y_2 \,,\; q_2 \equiv y_2/y_3, \;, \;\cdots \;, \; q_M \equiv q y_M / q_1}$$

$$= \; (q_I q_{I+1} \cdots q_{I+\ell-1}) \cdot U^I{}_{J,s;-n}$$

($\ell \equiv J - I + Mn > 0$ for creation operators)

Creation operator corresponds to addition of a column!

Character formula : $\mathrm{Tr}\mathcal{O} = \prod_{I=1}^{M} \prod_{J=1}^{\infty} \left(1 - \prod_{s=I}^{I+J-1} q_s\right)^{-\min(n_I, n_J)}$

Other interesting topics

- Co-dimension 2 defects in field theories in other dimensions.
  (next talk)

- Significance in **geometric Langlands program**

- **Nekrasov-Shatashvili** limit :
  Ramified instanton partition function as eigenfunctions
  of integrable Hamiltonians

- **Ribault-Teschner correspondence :**
  WZW correlator = Toda correlator with degenerate operator insertions.

Fundamental problems

- Why is imposing singular boundary condition the same as introducing 2D localized degree of freedom ?

- How the instantons with surface defects are related to the instantons in orbifolds?

- What happens at intersection of surface defects?