Topological Strings on elliptic Calabi-Yau three-folds

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Based on MH, S. Katz and A. Klemm, to appear soon.

Outline

- Introduction and previous works

- Toward solving the compact elliptic Calabi-Yau three-folds (with a main example: elliptic fibration over $\mathbb{CP}^2$)

  1. The involution symmetry

  2. The use of the weak Jacobi forms

  3. Curve counting considerations

- Summary and Conclusion

Introduction and previous works

- Topological strings: A $N = (2,2)$ supersymmetric non-linear sigma model from world sheet Σ to target space X.

$$\Phi_i : \Sigma \to X$$

  Topological string theory is the most interesting and free of world sheet anomaly, when the target space X is a Calabi-Yau 3-fold.

- There are two types of topological twisting: A-model and B-model. We are interested in the topological string partition function

$$Z = \exp(\sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(t_i))$$

  where t_i are Kahler moduli in the case of A-model, and complex structure moduli in the case of B-model.

- The A-model topological string free energy counts holomorphic curves, can be expanded in terms of Gopakumar-Vafa(GV) invariants n_g^β, where β is an element of 2nd integral homology class. It is related to Gromov-Witten invariants by a transformation.

- Mirror symmetry relates topological A-model on manifold X to topological B-model on its mirror manifold. Some very difficult mathematical problems of enumerative geometry can be easily solved by topological B-model methods.

- There are many ways to compute the topological string free energy (e.g. matrix models, topological vertex), with many physical and mathematical applications (e.g. gauge theories, black hole physics).

- Topological strings on non-compact toric Calabi-Yaus are essentially solved to all genera by topological vertex formalism, and also the B-model method using holomorphic anomaly equation.

- A long standing problem: How to solve topological strings on compact Calabi-Yau three-folds? The non-compact Calabi-Yau three-folds are basically described by a Riemann surface, so the geometric structure is simpler than the compact case. Many methods work only on the non-compact case.

- A well known example: the Quintic manifold, a degree 5 hypersurface in $\mathbb{CP}^4$.
  Candelas et al solve the genus zero sector, i.e. counting rational curve, using mirror symmetry and Picard-Fuchs equation.
  The mirror symmetry results are later proven by mathematicians using Kontsevich's localization methods, Givental; Lian, Liu, Yau.

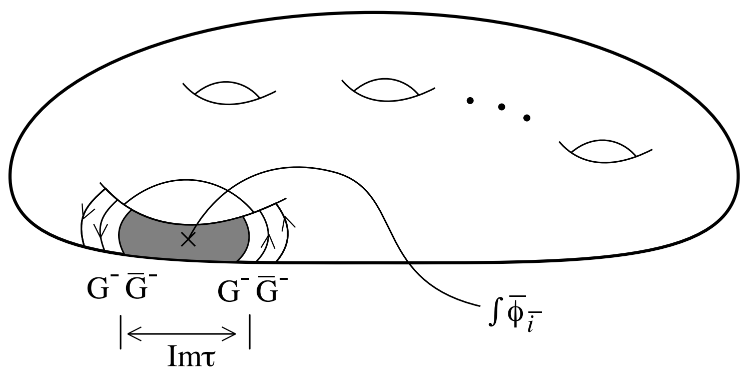
- At higher genus, the only available approach is the BCOV method. One use holomorphic anomaly equation to compute $F^{(g)}$ recursively in genus g. This was done by BCOV (in 1993) up to genus 2.

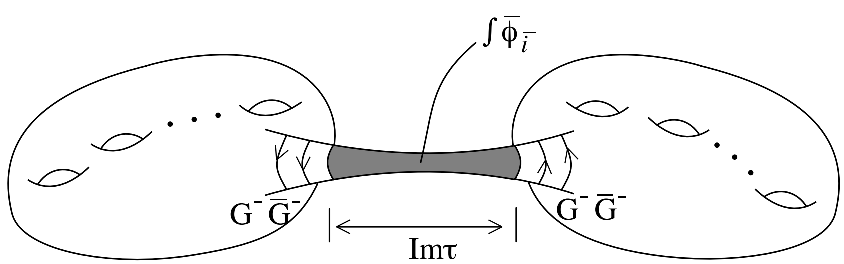
- There are also some mathematical approaches for compact Calabi-Yaus. In particular, Pandharipande's method can in principle compute to all genera. But a collaborator tells me that the method is too complicated that one can not go very far and test the results.

- The holomorphic anomaly comes from the boundaries of the moduli space of genus g Riemann surface. The BCOV holomorphic anomaly equation relates the anholomorphic derivative of the free energy to lower genus data

$$\bar{\partial}_{\bar{k}} F^{(g)} = \frac{1}{2} \bar{C}_{\bar{k}}^{ij} \left(D_i D_j F^{(g-1)} + \sum_{r=1}^{g-1} D_i F^{(r)} D_j F^{(g-r)} \right) .$$

- The degenerations of Riemann surface (c.f. BCOV)





- However, it is difficult to push the BCOV methods to higher genus. Two major difficulties are the the followings.

  1. Holomorphic ambiguity problem. The holomorphic anomaly equation only determine $F^{(g)}$ recursively in terms of lower genus results up to a holomorphic ambiguity, a meromorphic function in the moduli space with a finite number of unknown constants. One need find alternative ways to fix these unknown constants.

  2. Computational complexity in BCOV method: the number of diagrams grows exponentially with genus. A normal laptop can handle the computation only up to about genus 6, even for the simplest one parameter models such as the quintic.

- The calculation was pushed up to genus 3 for the quintic, using further information from the counting of BPS states known as Gopakumar-Vafa invariants. Katz, Klemm, Vafa, hep-th/9910181.

- We made some important progress Huang, Klemm, Quackenbush, hep-th/0612125.

  1. We solve the holomorphic anomaly equation directly without the BCOV Feynman diagrams, by using the idea of formulating topological strings as polynomials Yamaguchi, Yau, hep-th/0406078. The computational complexity of the method grows only polynomially in genus.

  2. We discover boundary conditions at the conifold point of the moduli space, i.e. the "gap" condition c.f. Huang, Klemm, hep-th/0605195, which fix the holomorphic ambiguity to a large extend.

- We are able to solve a class of one-parameter Calabi-Yau models to very high genus, e.g. in principle to genus 51 for the quintic (in practice up to around genus 20). At low genus we have redundant data which provide non-trivial test of the results.

Toward solving the compact elliptic Calabi-Yau three-folds

- In this work, we reconsider the case of the compact elliptic Calabi-Yau three-folds. Some recent works are
  M. Alim and E. Scheidegger, arXiv:1205.1784 [hep-th];
  A. Klemm, J. Manschot and T. Wotschke, arXiv:1205.1795 [hep-th].
  Based on these earlier works, using various conditions, notably the involution symmetry, and the weak Jacobi forms, we make some more progress toward solving the models.

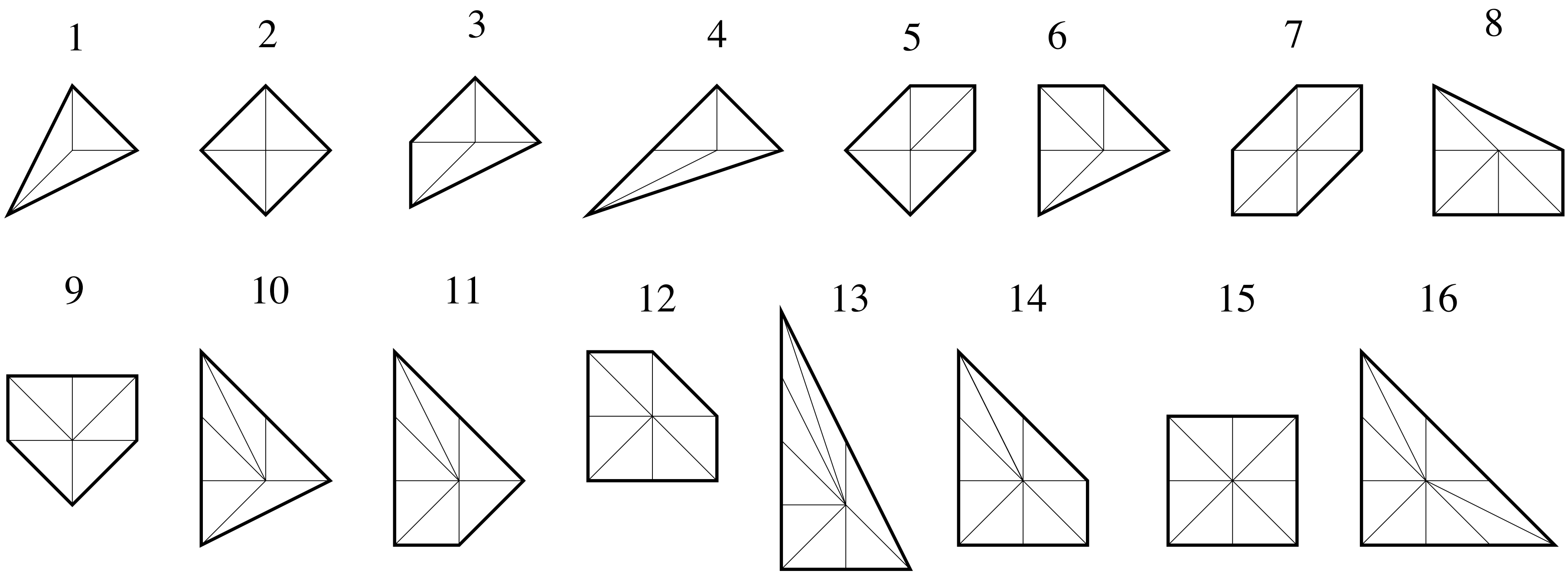
- Some quick introduction of elliptic Calabi-Yau: The construction of mirror pairs of Calabi-Yau n-folds as hypersurfaces in toric ambient spaces $\mathbb{P}^{n+1}_{\Delta}$ follows Batyrev's construction which relies on dual pairs of $n+1$ dimensional reflexive pairs of lattice polyhedra (Δ, Δ^*).

- To give an elliptic fibration structure, we combine a base polyhedron Δ^{B*} and a reflexive fibre polyhedron Δ^{F*} into an $n+1$ dimensional polyhedron Δ^*.

$$
\begin{array}{|cc|cc|}
\multicolumn{2}{c}{\nu_i^* \in \Delta^*} & \multicolumn{2}{c}{\nu_j \in \Delta} \\
 & \nu_i^{F*} & & \nu_j^F \\
\Delta^{B*}_{n-m} & \vdots & s_{ij}\Delta^B_{n-m} & \vdots \\
 & \nu_i^{F*} & & \nu_j^F \\
0\dots0 & & 0\dots0 & \\
\vdots & \Delta^{*F}_{m+1} & \vdots & \Delta^F_{m+1} \\
0\dots0 & & 0\dots0 &
\end{array}\,.
$$

If Δ^{*F} and Δ^{*B} are reflexive, then (Δ, Δ^*) is a reflexive pair. Here $s_{ij} = \langle \nu_i^F, \nu_j^{F*} \rangle + 1 \in \mathbb{N}$, and we indicated the dimensions of some polyhedra by subscripts; elliptic fibrations correspond to $m = 1$.

- For $n = 3$ and $m = 1$ we get many examples by choosing any of the 16 reflexive polyhedra in 2d as Δ^{*F} and Δ^{*B} respectively and specifying in addition $\nu_i^{F*} \in \Delta^{*F}$



- The main example: the elliptic fibration over $\mathbb{P}^2$.

$$\begin{array}{c|cccc|cccc|cc|} & & \nu_i & & & & \bar{\nu}_i^* & & & l^{(E)} & l^{(B)} \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ & 12 & -6 & -1 & -1 & 1 & 0 & -2 & -3 & 0 & 1 \\ & -6 & 12 & -1 & -1 & 0 & 1 & -2 & -3 & 0 & 1 \\ & -6 & -6 & -1 & -1 & -1 & -1 & -2 & -3 & 0 & 1 \\ & 0 & 0 & -1 & -1 & 0 & 0 & -2 & -3 & 1 & -3 \\ & 0 & 0 & 2 & -1 & 0 & 0 & 1 & 0 & 2 & 0 \\ & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 3 & 0 \end{array} .$$

- The geometry has two Kahler parameters from the base and the fiber. The Gopakumar-Vafa(GV) invariants are $n_g^{(d_B,d_E)}$, where d_B and d_E are base and fiber degrees.

- Another example: the elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$. The base has two Kahler parameters. In a limit where one of the base parameters decouple, this reduces to the half K3 geometry and the topological strings compute elliptic genus of E-strings.

- The complex structure parameters of the mirror are denoted z_1, z_2. From standard techniques in mirror symmetry, we can derive the Picard-Fuchs differential equations, and calculate the three-point functions and prepotential.

$$C_{111} = \frac{9}{z_1^3 \Delta_1}, \qquad C_{112} = C_{121} = C_{211} = \frac{3\Delta_3}{z_1^2 z_2 \Delta_1},$$
$$C_{122} = C_{212} = C_{221} = \frac{\Delta_3^2}{z_1 z_2^2 \Delta_1}, \quad C_{222} = \frac{9(\Delta_3^3 + (432 z_1)^3)}{z_2^2 \Delta_1 \Delta_2},$$

where the discriminants are $\Delta_1 = (1 - 432 z_1)^3 - 27 z_2 (432 z_1)^3, \Delta_2 = 1 + 27 z_2$, and for convenience we also define $\Delta_3 = 1 - 432 z_1$.

The involution symmetry

- An involution symmetry for the model has been known in the early days of mirror symmetry.

$$I: \quad (z_1, z_2) \to (x_1, x_2) = \Big(\frac{1}{432} - z_1, -\frac{(432z_1)^3 z_2}{(1-432z_1)^3}\Big).$$

  This is an involution $I^2 = 1$.

- The involution exchanges the two discriminants

$$I(\Delta_1) = (432z_1)^3 \Delta_2, \qquad I(\Delta_2) = \frac{\Delta_1}{(1-432z_1)^3} \ .$$

- It acts on the holomorphic 3-form as $I: \Omega \to i\Omega$. Since Ω defines vacuum line bundle $\mathcal{L}$ and the higher genus amplitudes $\mathcal{F}^{(g)}$ transforms as section $\mathcal{F}^{(g)} \in \mathcal{L}^{2g-2}$ we conclude that

$$I: \quad \mathcal{F}^{(g)} \to (-1)^{g-1} \mathcal{F}^{(g)}.$$

- The three point functions are covariant derivatives of the genus zero prepotential, so transform as a tensor except for a minus sign

$$I: \quad C_{ijk} \to -\frac{\partial z_l}{\partial x_i}\frac{\partial z_m}{\partial x_j}\frac{\partial z_n}{\partial x_k}C_{lmn}.$$

- The BCOV propagators are defined by

$$\partial_{\bar{i}} S = G_{\bar{i}j} S^j, \quad \partial_{\bar{i}} S^j = G_{\bar{i}k} S^{jk}, \qquad \partial_{\bar{i}} S^{ij} = \bar{C}_{\bar{i}}^{ij} ,$$

where $G_{\bar{i}j} = \bar{\partial}_{\bar{i}} \partial_j K$ is the special Kahler metric of the moduli space.

- It is convenient to make a change of variables with the derivative of Kähler potential $K_i = \partial_i K$ by the following (Alim et al, 2007)

$$S^{ij} \to S^{ij}, \quad S^i \to S^i - S^{ij}K_j, \quad S \to S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j.$$

Then the topological string amplitudes $F^{(g)}$ are polynomials of degree $3g-3$ with rational function coefficients, where one assigns degree 1,2,3 respectively to the propagators S^{ij}, S^i, S.

- The propagators do not transform exactly like a tensor under the involution symmetry. Instead, we find

$$I: \quad S^{ij} \to -\frac{\partial x_i}{\partial z_k}\frac{\partial x_j}{\partial z_m}S^{km}, \quad S^i \to -\frac{\partial x_i}{\partial z_k}S^k + f^i, \quad S \to -S + f^0.$$

  Here the minus sign is similar to the three-point functions. We also have the shifts f^i, f^0, which are rational functions of z_1, z_2.

- We integrate the defining equations for the propagators, then make a gauge choice of the holomorphic ambiguities. We determine the shift functions f^i, f^0 such that the involution transformation is equivalent to the original equations. We find the formulas of f^i, f^0 for the gauge choice used in Alim et al 2012.

- We first find a particular holomorphic ambiguity such that the total genus g amplitude transform with $(-1)^{g-1}$ under the involution. Then we only need to consider additional ambiguity of the form

$$I : f^{(g)}(z_1, z_2) \to (-1)^{g-1} f^{(g)}(z_1, z_2).$$

- Overall the number of unknown constants at large genus is reduced to about one quarter of the original one by the involution symmetry.

- Fiber modularity: we can expand for base degree

$$w_0^{2g-2}\mathcal{F}^{(g)} = \sum_{k=0}^{\infty} P_k^{(g)}(q_E) \, (\frac{q_E}{\eta(q_E)^{24}})^{\frac{3k}{2}} q_B^k, \quad g \geq 1.$$

  It was known that the coefficients $P_k^{(g)}(q_E)$ are quasi-modular forms, i.e. polynomials of Eisenstein series E_2, E_4, E_6, of modular weight $18k+2g-2$, and satisfy a modular anomaly equation.

- We find that fiber modularity is basically equivalent to the involution symmetry plus regularity condition at $z_1 \sim \infty$.

- The gap condition at the conifold divisors $\Delta_1\Delta_2 = 0$ greatly reduces the number of unknown constants, to about $\frac{g^2}{54}$ for large genus g, or about $\frac{1}{7}$ of the number after imposing involution symmetry,

- Some remarks: It is sufficient to consider only one point in the conifold divisor. The remaining ambiguity is now a holomorphic function with no pole, transforms with $(-1)^{g-1}$ under involution symmetry.

The use of weak Jacobi forms

- Consider a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$, an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group as

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z), \qquad \forall \quad \begin{pmatrix} a & b \\ a & c \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z})$$

  and under translations of the elliptic parameter as

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z} \ .$$

  Here $k \in \mathbb{Z}$ is called the *weight* and $m \in \mathbb{Z}_{>0}$ is called the *index*.

- Due to the periodicity, the function has a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r} c(n, r) q^n y^r, \qquad \text{where} \quad q = e^{2\pi i \tau}, \quad y = e^{2\pi i z}$$

- A (holomorphic) Jacobi form: $c(n,r)=0$ unless $4mn \geq r^2$.
  (stronger) A cusp Jacobi form: $c(n,r)=0$ unless $4mn > r^2$.
  (weaker) A weak Jacobi form : $c(n,r)=0$ unless $n \geq 0$.

- Some weak Jacobi forms can be constructed by theta functions

$$\begin{aligned}\phi_{-2,1}(\tau,z) &= -\frac{\theta_1(z,\tau)^2}{\eta^6(\tau)},\\ \phi_{0,1}(\tau,z) &= 4[\frac{\theta_2(z,\tau)^2}{\theta_2(0,\tau)^2}+\frac{\theta_3(z,\tau)^2}{\theta_3(0,\tau)^2}+\frac{\theta_4(z,\tau)^2}{\theta_4(0,\tau)^2}].\end{aligned}$$

  Here $\phi_{0,1}(\tau,z)$ is the elliptic genus of K3.

- A. Dabholkar, S. Murthy and D. Zagier, arXiv:1208.4074 [hep-th].
  A weak Jacobi form of given index m and even modular weight k is a polynomial of $E_4(\tau)$, $E_6(\tau)$, $\phi_{0,1}(\tau,z)$, $\phi_{-2,1}(\tau,z)$ whose modular weights and indices are $4,6,0,-2$ and $0,0,1,1$ respectively.

- The even weight weak Jacobi forms have a Taylor expansion in z with coefficients are quasi-modular forms. For example the first coefficients in the expansion of $\phi_{-2,1}(z,\tau)$ and $\phi_{0,1}(z,\tau)$ are

$$\begin{aligned}\phi_{-2,1}(\tau,z) &= -z^2+\frac{E_2 z^4}{12}+\frac{-5E_2^2+E_4}{1440}z^6+\mathcal{O}(z^8),\\ \phi_{0,1}(\tau,z) &= 12-E_2 z^2+\frac{E_2^2+E_4}{24}z^4+\mathcal{O}(z^6).\end{aligned}$$

- They satisfy the modular anomaly equation

$$\left(\partial_{E_2}+\frac{z^2}{12}\right)\phi_{-2,1}(\tau,z)=0, \qquad \left(\partial_{E_2}+\frac{z^2}{12}\right)\phi_{0,1}(\tau,z)=0.$$

- Therefore a weak Jacobi form $\varphi_{k,m}$ of index m satisfies the modular anomaly equation that

$$\left(\partial_{E_2}+\frac{mz^2}{12}\right)\varphi_{k,m}=0\ .$$

- Some recent developments on the elliptic genus of E-strings, which is related to the topological strings on half K3 space.
  Haghighat, Lockhart and Vafa, arXiv:1406.0850 [hep-th];
  Cai, MH and Sun, arXiv:1411.2801 [hep-th];
  J. Kim, S. Kim, Lee, Park and Vafa, arXiv:1411.2324 [hep-th];
  Haghighat, Klemm, Lockhart and Vafa, arXiv:1412.3152 [hep-th].

- The works of Haghighat et al use the M5 brane domain wall blocks from M-strings, and make an ansatz for the M9 brane domain wall blocks. Basic ingredients:
  M-Strings: M2 branes suspended between two M5 branes.
  E-Strings: M2 branes suspended between a M5 brane and a M9 brane.
  Heterotic Strings: M2 branes suspended between two M9 branes.
  We push the calculations to three-strings.

- On the other hand, Kim et al constructs a 2d quiver gauge theory for E-strings, and can in principle compute the elliptic genus of any finite number of E-strings with the beautiful techniques of Jeffrey-Kirwan residues (c.f Kim et al, Benini et al).

- Inspired by these works, we apply the idea to compact elliptic Calabi-Yau manifolds (without refinement). We expand the partition function on base degree

$$Z = Z_0[1 + \sum_{d_B=1}^{\infty} Z_{d_B}(\lambda,\tau) Q_B^{d_B} \frac{Q_E^{\frac{3d_B}{2}}}{\eta^{36d_B}(\tau)}]$$

- We conjecture

$$Z_{d_B}(z,\tau) = \frac{\varphi_{d_B}(z,\tau)}{\prod_{k=1}^{d_B} \phi_{-2,1}(\tau,kz)},$$

where $z = \lambda$ is the genus expansion parameter of topological strings.

- According to the modular anomaly equation, $Z_{d_B}(z,\tau)$ has formally index $\frac{d_B(d_B-3)}{2}$. So the denominator $\varphi_{d_B}(z,\tau)$ is a weak Jacobi form of index $\frac{1}{3}d_B(d_B-1)(d_B+4)$ and weight $16d_B$.

- Castelnuovo's bound: for a given degree (d_B, d_E), the GV invariant $n_g^{(d_B,d_E)}$ vanish at sufficiently large genus g. This fix many coefficients in the ansatz for the weak Jacobi form $\varphi_{d_B}(z,\tau)$ in the denominator.

- The basic constrain is that $\varphi_{-2,1}(\tau, kz)^{-1}$ for $k > 1$ contributes to arbitrarily large genus, and must be cancelled by multi-cover contributions from lower degrees. So the remaining unfixed ansatz is

$$\sum_{k=-1}^{\frac{d_B(d_B-3)}{2}} f_{18d_B+2k}(E_4, E_6)\phi_{-2,1}(z,\tau)^k \phi_{0,1}(z,\tau)^{\frac{d_B(d_B-3)}{2}-k},$$

where $f_{18d_B+2k}(E_4, E_6)$ symbolizes a modular form of weight $18d_B+2k$, which is a polynomials of E_4 and E_6 with many unknown coefficients, so that the total modular weight and index are the same as the more general ansatz. These sub-family of ansatz can not be fixed by vanishing GV conditions.

- This approach can be combined with the B-model holomorphic anomaly approach to compute higher genus topological string amplitudes. Using the involution symmetry and the boundary conditions at the conifold point, we find that the exact formula at base degree d_B can provide sufficient boundary data to fix the B-model formula at genus $9(d_B+1)$, valid for all base and fiber degrees. On the other hand, in order to fix the exact formula at base degree d_B, we need topological free energy of genus no less than $\frac{d_B(d_B-3)}{2}+1$.

- Thus, as long as $9(d_B+1) \geq \frac{(d_B+1)(d_B-2)}{2}+1$, we can repeat this procedure to fix the exact formula with increasing base degrees. In this way we can in principle determine the exact formula up to base degree $d_B = 20$ (for all genera and fiber degrees), and the topological string free energy up to genus 189 (for all base and fiber degrees). In practice we compute up to $d_B = 5$ and genus $g = 8$.

Curve counting considerations

- We make many predictions for the Gopakumar-Vafa invariants. The counting of BPS states can be calculated from the cohomology of the moduli space $\mathcal{M}$ of curves with certain degrees and genus. The algebraic geometric counting is developed in Katz, Klemm, Vafa, 1999.

- The top genus numbers are the easiest to calculate.

$$n_g^{(d_B,d_E)} = (-1)^{\dim(\mathcal{M})}\chi(\mathcal{M}).$$

  There are corrections to the formula below the top genus.

- An example of case $d_B = 1$. The curves of degrees $(d_E, 1)$ has maximal genus d_E, and the moduli space is the product of $\mathbb{P}^2$ and symmetric product of d_E points on a $\mathbb{P}^1$. The dimension and Euler character are $\dim(\mathcal{M}) = d_E + 2, \chi(\mathcal{M}) = 3(d_E + 1)$. So the GV invariants are $n_{d_E}^{(1,d_E)} = (-1)^{d_E} 3(d_E+1)$. Many other checks by S. Katz up to $d_B = 4$.

Summary and Conclusion

- We shall study other examples, e.g. the elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$.

- We make some progress, but still have not completely solve the topological string partition function on the compact Calabi-Yau models. More ideas are needed.

- Explore the connection to CFT's. Localization may help.

Thank You