Stringy Differential Geometry and Double Field Theory

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Prologue

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- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
  - Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  - $\nabla_\lambda g_{\mu\nu} = 0,\ \Gamma^\lambda_{[\mu\nu]} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  - Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_{\kappa\lambda\mu\nu} \longrightarrow R$
- On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal footing, as they, or NS-NS sector, form a multiplet of T-duality.
- This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.
- Basically, Riemannian geometry is for *Particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector.

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- In four-dimensional spacetime photon has two physical degrees of freedom, but can be best described by a four component vector.

- Similarly, D-dimensional spacetime may be better understood in terms of **doubled-yet-gauged** $(D + D)$ coordinates.

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Talk based on works with Imtak Jeon & Kanghoon Lee

- Differential geometry with a projection: Application to double field theory
  arXiv:1011.1324 JHEP
- Double field formulation of Yang-Mills theory arXiv:1102.0419 PLB
- Stringy differential geometry, beyond Riemann arXiv:1105.6294 PRD
- Incorporation of fermions into double field theory arXiv:1109.2035 JHEP
- Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity
  arXiv:1112.0069 PRD Rapid Comm.
- Ramond-Ramond Cohomology and O(D,D) T-duality arXiv:1206.3478 JHEP
- **Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N}=2$ $D=10$ Supersymmetric Double Field Theory** arXiv:1210.5078 PLB
- Comments on double field theory and diffeomorphisms arXiv:1304.5946 JHEP
- Covariant action for a string in doubled yet gauged spacetime arXiv:1307.8377 NPB

Parallel works on U-duality

- U-geometry: SL(5) with Yoonji Suh arXiv:1302.1652 JHEP
- M-theory and F-theory from a Duality Manifest Action
  with Chris Blair and Emanuel Malek arXiv:1311.5109 JHEP
- U-gravity: SL(N) with Yoonji Suh arXiv:1402.5027 JHEP

Double Field Theory by Hull & Zwiebach (Hohm), *c.f.* Siegel

- With a "generalized metric" Duff and a redefined dilaton:

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

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$$L_{\rm DFT} = e^{-2d}\left[\mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) + 4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\right]$$

- Spacetime is formally doubled, $y^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, 2, \cdots, D{+}D$.

- T-duality is manifestly realized as usual $\mathbf{O}(D, D)$ rotations Tseytlin, Siegel

$$\mathcal{H}_{AB} \longrightarrow M_A{}^C M_B{}^D \mathcal{H}_{CD}\,, \qquad d \longrightarrow d\,, \qquad M \in \mathbf{O}(D, D)\,.$$

- Yet, DFT (for NS-NS sector) is a D-dimensional theory written in terms of $(D + D)$-dimensional language, i.e. tensors.

- All the fields must live on a D-dimensional null hyperplane or 'section', subject to

$$\partial_A\partial^A = 2\frac{\partial^2}{\partial x^\mu \partial\tilde{x}_\mu} \equiv 0 \quad : \quad \text{section condition}$$

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- Then DFT reduces to the well-known effective action within 'Riemannian' setup:

$$L_{\rm DFT} \Longrightarrow L_{\rm eff.} = \sqrt{-g}e^{-2\phi}\left(R_g + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right).$$

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- On the other hand, in the above formulation of DFT, the diffeomorphism and the B-field gauge symmetry are rather unclear, while $\mathbf{O}(D,D)$ T-duality is manifest.
- The above expression may be analogous to the case of writing the Riemannian scalar curvature, R, in terms of the metric and its derivative.
- It is desirable to explore the underlying differential geometry, beyond Riemann.

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- On the other hand, in the above formulation of DFT, the diffeomorphism and the B-field gauge symmetry are rather unclear, while $\mathbf{O}(D,D)$ T-duality is manifest.
- The above expression may be analogous to the case of writing the Riemannian scalar curvature, R, in terms of the metric and its derivative.
- It is desirable to explore the underlying differential geometry, beyond Riemann.

Double Field Theory by Hull & Zwiebach (Hohm), *c.f.* Siegel

- With a "generalized metric" Duff and a redefined dilaton:

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

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In the remaining of this talk, I will try to explain our proposal for

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  - And their complete covariantization via ‘projection’

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- **Key concepts include**
  - Projector
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  - Semi-covariant curvature
  - And their complete covariantization via 'projection'

c.f. Alternative approaches: Berman-Blair-Malek-Perry, Cederwall, Geissbuhler, Marques et al.

Geometric Constitution of Double Field Theory

- *Notation*

Capital Latin alphabet letters denote the $\mathbf{O}(D,D)$ vector indices, i.e. $A,B,C,\cdots=1,2,\cdots,D{+}D$, which can be freely raised or lowered by the $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right).$$

- *Doubled-yet-gauged spacetime*

The spacetime is formally doubled, being $(D+D)$-dimensional.

However, **the doubled spacetime is gauged** : the coordinate space is equipped with an *equivalence relation*,

$$x^{A} \sim x^{A} + \phi\partial^{A}\varphi\,,$$

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which we call '*coordinate gauge symmetry*'.

Note that ϕ and φ are arbitrary functions in DFT.

Each equivalence class, or gauge orbit, represents a single physical point.

Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.

- *Realization of the coordinate gauge symmetry.*

The equivalence relation is realized in DFT by enforcing that, arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge symmetry shift,

$$\Phi(x+\Delta)=\Phi(x)\,, \qquad \Delta^{A}=\phi\partial^{A}\varphi\,.$$

- *Section condition.*

The invariance under the coordinate gauge symmetry can be shown to be equivalent to the **section condition** ,

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The invariance under the coordinate gauge symmetry can be shown to be equivalent to the **section condition** ,

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Explicitly, acting on arbitrary functions, Φ, Φ', and their products, we have

$$\partial_A \partial^A \Phi = 0 \qquad \text{(weak constraint)}\,,$$

$$\partial_A \Phi \partial^A \Phi' = 0 \qquad \text{(strong constraint)}\,.$$

- ***Diffeomorphism.***

Diffeomorphism symmetry in $\mathbf{O}(D, D)$ DFT is generated by a generalized Lie derivative Siegel, Courant, Grana

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}X_B - \partial_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,,$$

where ω_T denotes the weight.

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where ω_T denotes the weight.

In particular, the generalized Lie derivative of the $\mathbf{O}(D,D)$ invariant metric is trivial,

$$\hat{\mathcal{L}}_X\mathcal{J}_{AB} = 0\,.$$

The commutator is closed by C-bracket Hull-Zwiebach

$$\left[\hat{\mathcal{L}}_X,\hat{\mathcal{L}}_Y\right] = \hat{\mathcal{L}}_{[X,Y]_{\mathrm{C}}}\,,\qquad\qquad [X,Y]^A_{\mathrm{C}} = X^B\partial_B Y^A - Y^B\partial_B X^A + \tfrac{1}{2}Y^B\partial^A X_B - \tfrac{1}{2}X^B\partial^A Y_B\,.$$

- *Dilaton and a pair of two-index projectors.*

The geometric objects in DFT consist of a dilation, d, and a pair of symmetric projection operators,

$$P_{AB} = P_{BA}\,, \qquad \bar{P}_{AB} = \bar{P}_{BA}\,, \qquad P_A{}^B P_B{}^C = P_A{}^C\,, \qquad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C\,.$$

Further, the projectors are orthogonal and complementary,

$$P_A{}^B \bar{P}_B{}^C = 0\,, \qquad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}\,.$$

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Remark: The difference of the two projectors, $P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}$, corresponds to the "generalized metric" which can be also independently defined as a symmetric $\mathbf{O}(D, D)$ element, i.e. $\mathcal{H}_{AB} = \mathcal{H}_{BA}$, $\mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A{}^C$. However, in supersymmetric double field theories it appears that the projectors are more fundamental than the "generalized metric".

- *Integral measure.*

  While the projectors are weightless, the dilation gives rise to the $\mathbf{O}(D,D)$ invariant integral measure with weight one, after exponentiation,

$$e^{-2d}\,.$$

- *Semi-covariant derivative and semi-covariant Riemann curvature.*

We define a semi-covariant derivative,

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,,$$

and

Geometric Constitution of Double Field Theory

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and a semi-covariant Riemann curvature,

$$S_{ABCD} := \tfrac{1}{2} \left(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB} \Gamma_{ECD} \right) .$$

Here R_{ABCD} denotes the ordinary "field strength" of a connection,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED} .$$

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and a semi-covariant "Riemann" curvature,

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$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED} .$$

As I will explain shortly, we may determine the (torsionelss) connection:

$$\begin{aligned}\Gamma_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) ,\end{aligned}$$

which is the DFT generalization of the Christoffel connection.

Geometric Constitution of Double Field Theory

The semi-covariant derivative then obeys the Leibniz rule and annihilates the $\mathbf{O}(D,D)$ invariant constant metric,

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A crucial defining property of the semi-covariant "Riemann" curvature is that, under arbitrary transformation of the connection, it transforms as total derivative,

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB} \,.$$

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Further, the semi-covariant "Riemann" curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature,

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB}\,, \qquad S_{[ABC]D} = 0\,,$$

as well as additional identities concerning the projectors,

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD}{=}0\,, \qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,.$$

It follows that

$$S^{AB}{}_{AB}{=}0\,.$$

- *The uniqueness of the torsionless connection.*

The connection is the unique solution to the following five constraints:

$$\nabla_A P_{BC} = 0\,, \qquad \nabla_A \bar{P}_{BC} = 0\,,$$

$$\nabla_A d = -\tfrac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \tfrac{1}{2} \Gamma^B{}_{BA} = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{ACB} = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} = 0\,,$$

$$\mathcal{P}_{ABC}{}^{DEF} \Gamma_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF} \Gamma_{DEF} = 0\,.$$

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$$\mathcal{P}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,.$$

- The first two relations are the compatibility conditions with all the geometric objects , or NS-NS sector, in DFT.
- The third constraint is the compatibility condition with the $\mathbf{O}(D,D)$ invariant constant metric, *i.e.* $\nabla_A \mathcal{J}_{BC} = 0$.

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- The next cyclic property makes the semi-covariant derivative compatible with the generalized Lie derivative as well as with the C-bracket,

$$\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla)\,, \qquad [X,Y]_{\rm C}(\partial) = [X,Y]_{\rm C}(\nabla)\,.$$

- The last formulae are projection conditions which we impose intentionally in order to ensure the uniqueness.

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- The last formulae are projection conditions which we impose intentionally in order to ensure the uniqueness.

- *Six-index projection operators.*

The six-index projection operators are explicitly,

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D} ,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D} ,$$

which satisfy the 'projection' properties,

$$\mathcal{P}_{ABC}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{ABC}{}^{GHI} , \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{ABC}{}^{GHI} .$$

Further, they are symmetric and traceless,

$$\mathcal{P}_{ABCDEF} = \mathcal{P}_{DEFABC} , \qquad \mathcal{P}_{ABCDEF} = \mathcal{P}_{A[BC]D[EF]} , \qquad P^{AB} \mathcal{P}_{ABCDEF} = 0 ,$$

$$\bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{DEFABC} , \qquad \bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{A[BC]D[EF]} , \qquad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0 .$$

Crucially, the projection operator dictates the anomalous terms in the diffeomorphic transformations of the semi-covariant derivative and the semi-covariant Riemann curvature,

$$(\delta_X-\hat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n}=\sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E X_F\, T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

$$(\delta_X-\hat{\mathcal{L}}_X)S_{ABCD}=2\nabla_{[A}\Big((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F X_G\Big)+2\nabla_{[C}\Big((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F X_G\Big).$$

- *Complete covariantizations.*

Both the semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized, through appropriate contractions with the projectors:

$$
\begin{array}{lll}
P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1\cdots B_n}\,, & \bar{P}_{C}{}^{D}P_{A_1}{}^{B_1}\cdots P_{A_n}{}^{B_n}\nabla_{D}T_{B_1\cdots B_n}\,, & \\
P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1\cdots D_n}\,, & \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1\cdots D_n} & \text{(divergences)}\,,\\
P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1\cdots D_n}\,, & \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1\cdots D_n} & \text{(Laplacians)}\,,
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P^{AB} \bar{P}_{C_1}{}^{D_1} \cdots \bar{P}_{C_n}{}^{D_n} \nabla_A T_{BD_1 \cdots D_n} \,, & \bar{P}^{AB} P_{C_1}{}^{D_1} \cdots P_{C_n}{}^{D_n} \nabla_A T_{BD_1 \cdots D_n} & \text{(divergences)} \,, \\
P^{AB} \bar{P}_{C_1}{}^{D_1} \cdots \bar{P}_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 \cdots D_n} \,, & \bar{P}^{AB} P_{C_1}{}^{D_1} \cdots P_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 \cdots D_n} & \text{(Laplacians)} \,,
\end{array}
$$

and

$$
\begin{array}{ll}
P_A{}^C \bar{P}_B{}^D S_{CED}{}^E & \text{(“Ricci” curvature)} \,, \\
(P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD} & \text{(scalar curvature)} \,.
\end{array}
$$

- *Action.*

The action of $\mathbf{O}(D, D)$ DFT is given by the fully covariant scalar curvature,

$$\int_{\Sigma_D} e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\,,$$

where the integral is taken over a section, Σ_D.

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The dilation and the projector equations of motion correspond to the vanishing of the scalar curvature and the "Ricci" curvature respectively.

Note: It is precisely the above expression that allows the '1.5 formalism' to work in the full order supersymmetric extensions of $\mathcal{N} = 1, 2,\ D = 10$ Jeon-Lee-JHP

- *Section.*

Up to $\mathbf{O}(D,D)$ duality rotations, the solution to the section condition is unique. It is a D-dimensional section, Σ_D, characterized by the independence of the dual coordinates, i.e.

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0\,,$$

while the whole doubled coordinates are given by

$$x^A = (\tilde{x}_\mu, x^\nu)\,,$$

where μ,ν are now D-dimensional indices.

- *Riemannian reduction.*

  To perform the Riemannian reduction to the D-dimensional section, Σ_D, we parametrize the dilation and the projectors in terms of D-dimensional Riemannian metric, $g_{\mu\nu}$, ordinary dilaton, ϕ, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

$$P_{AB} - \bar{P}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{|g|}e^{-2\phi}\,.$$

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  The DFT scalar curvature then reduces upon the section to

$$(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\Big|_{\Sigma_D} = R_g + 4\Delta\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} ,$$

  where as usual, $H_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]}$.

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  DFT-diffeomorphim $\Rightarrow$ D-dimensional diffeomorphism plus B-field gauge symmetry.

- *Riemannian reduction.*

To perform the Riemannian reduction to the D-dimensional section, Σ_D, we parametrize the dilation and the projectors in terms of D-dimensional Riemannian metric, $g_{\mu\nu}$, ordinary dilaton, ϕ, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

$$P_{AB} - \bar{P}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{|g|}e^{-2\phi}\,.$$

The DFT scalar curvature then reduces upon the section to

$$(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\Big|_{\Sigma_D} = R_g + 4\Delta\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\,,$$

where as usual, $H_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]}$.

DFT-diffeomorphim $\Rightarrow$ D-dimensional diffeomorphism plus B-field gauge symmetry.

Up to field redefinitions, the above is the most general parametrization of the "generalized metric", $\mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}$, when its upper left $D \times D$ block is non-degenerate.

- *Non-Riemannian backgrounds.*

When the upper left $D\times D$ block of $\mathcal{H}_{AB}=(P-\bar{P})_{AB}$ is degenerate – where g^{-1} might be positioned – the Riemannian metric ceases to exist upon the section, Σ_D.

Nevertheless, DFT and a doubled sigma model –which I will discuss later– have no problem with describing such a non-Riemannian background.

An extreme example of such a non-Riemannian background is the flat background where

$$\mathcal{H}_{AB}=(P-\bar{P})_{AB}=\mathcal{J}_{AB}\,.$$

This is a vacuum solution to the bosonic $\mathbf{O}(D,D)$ DFT and the corresponding doubled sigma model reduces to a certain 'chiral' sigma model.

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Allowing non-Riemannian backgrounds, DFT is NOT a mere reformulation of SUGRA. It describes a new class of string theory backgrounds. *c.f.* Gomis-Ooguri

SUSY

Based on the differential geometry I just described,

incorporating fermions and the R-R sector (*i.e.* vielbein formalism),

it is possible to construct the maximally supersymmetric double field theory

to the full order (*i.e.* quartic order) in fermions.

$\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory

Symmetries of $\mathcal{N} = 2$ $D = 10$ SDFT

- **O**(D, D) **T-duality**

- **Gauge symmetries**

  1. **DFT-diffeomorphism (generalized Lie derivative)**

  2. **A *pair* of local Lorentz symmetries, Spin**$(1, D-1)_L \times$ **Spin**$(D-1, 1)_R$

  3. **local** $\mathcal{N} = 2$ **SUSY with** 32 **supercharges.**

- **All the bosonic symmetries will be realized manifestly and simultaneously.**

- **The theory is chiral with respect to both Local Lorentz groups.**

- **Consequently, there is no distinction of IIA and IIB** $\Longrightarrow$ **Unificaiton of IIA and IIB**

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  - NS-NS sector $\begin{cases} \text{DFT-dilaton:} & d \\ \text{DFT-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - R-R potential: $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions**
  - DFT-dilatinos: $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
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Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(D,D)$ **& DFT-diffeom. vector**	$\mathcal{J}_{AB}$
$p, q, \cdots$	**Spin**$(1, D-1)_L$ **vector**	$\eta_{pq} = \textbf{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	**Spin**$(1, D-1)_L$ **spinor**	$C_{+\alpha\beta}\,, \quad (\gamma^p)^T = C_+\gamma^p C_+^{-1}$
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R-R potential and Fermions carry NOT $(D+D)$-dimensional BUT undoubled D-dimensional indices.

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***A priori*, O(D,D) rotates only the O(D,D) vector indices (capital Roman), and the R-R sector and all the fermions are O(D,D) T-duality singlet.**

The usual IIA $\Leftrightarrow$ IIB exchange will follow only after fixing a gauge.

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All the fields are required to satisfy the section condition,

$$\partial_A \partial^A \equiv 0\,.$$

- The DFT-dilaton gives rise to a scalar density with weight one,

$$e^{-2d}\,.$$

- The DFT-vielbeins satisfy the **four defining properties:**

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}}=0\,,\qquad V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}\,.$$

- For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,

$$\gamma^{(D+1)}\psi_{\bar{p}}=\mathbf{c}\,\psi_{\bar{p}}\,,\qquad \gamma^{(D+1)}\rho=-\mathbf{c}\,\rho\,,$$

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where $\mathbf{c}$ and $\mathbf{c}'$ are arbitrary independent two sign factors, $\mathbf{c}^2=\mathbf{c}'^2=1$.

- Lastly for the R-R sector, we set the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, to be in the **bi-fundamental** spinorial representation of $\mathbf{Spin}(1,D{-}1)_{L}\times\mathbf{Spin}(D{-}1,1)_{R}$. It possesses the chirality,

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- *A priori* all the possible four different sign choices are equivalent up to $\mathbf{Pin}(1, D{-}1)_L \times \mathbf{Pin}(D{-}1, 1)_R$ rotations.
- That is to say, $\mathcal{N} = 2$ $D = 10$ SDFT is chiral with respect to both $\mathbf{Pin}(1, D{-}1)_L$ and $\mathbf{Pin}(D{-}1, 1)_R$, and the theory is unique, unlike IIA/IIB SUGRAs.
- Hence, without loss of generality, we may safely set

$$\mathbf{c} \equiv \mathbf{c}' \equiv +1\,.$$

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- The DFT-vielbeins generate **a pair of two-index projectors,**

$$P_{AB} := V_A{}^p V_{Bp}\,, \qquad P_A{}^B P_B{}^C = P_A{}^C\,, \qquad \bar{P}_{AB} := \bar{V}_A{}^{\bar{p}} \bar{V}_{B\bar{p}}\,, \qquad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C\,,$$

which are symmetric, orthogonal and complementary to each other,

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- It follows

$$P_A{}^B V_{Bp} = V_{Ap}\,, \qquad \bar{P}_A{}^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}\,, \qquad \bar{P}_A{}^B V_{Bp} = 0\,, \qquad P_A{}^B \bar{V}_{B\bar{p}} = 0\,.$$

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- Surely we also get the six-index projectors

$$\mathcal{P}_{CAB}{}^{DEF} := P_{C}{}^{D}P_{[A}{}^{[E}P_{B]}{}^{F]} + \tfrac{2}{D-1}P_{C[A}P_{B]}{}^{[E}P^{F]D}\,, \qquad \mathcal{P}_{CAB}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI}\,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_{C}{}^{D}\bar{P}_{[A}{}^{[E}\bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1}\bar{P}_{C[A}\bar{P}_{B]}{}^{[E}\bar{P}^{F]D}\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{CAB}{}^{GHI}\,,$$

which are symmetric and traceless,

$$\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}\,, \qquad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}\,,$$

$$\mathcal{P}^{A}{}_{ABDEF} = 0\,, \quad P^{AB}\mathcal{P}_{ABCDEF} = 0\,, \qquad \bar{\mathcal{P}}^{A}{}_{ABDEF} = 0\,, \quad \bar{P}^{AB}\bar{\mathcal{P}}_{ABCDEF} = 0\,,$$

and play crucial roles in the construction of the completely covariant derivatives and curvatures.

- Having all the ‘right’ field-variables prepared, we now discuss their derivatives or

  ‘semi-covariant derivatives’ .

Semi-covariant derivatives

- For each gauge symmetry we assign a corresponding connection,
  - Γ_A for the DFT-diffeomorphism (generalized Lie derivative),
  - Φ_A for the 'unbarred' local Lorentz symmetry, $\mathbf{Spin}(1, D-1)_L$,
  - $\bar{\Phi}_A$ for the 'barred' local Lorentz symmetry, $\mathbf{Spin}(D-1, 1)_R$.
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$$\nabla_A = \partial_A + \Gamma_A \,, \qquad\qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A \,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,.$$

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- ***By definition,*** **the master derivative annihilates all the 'constants',**

$$\mathcal{D}_A\mathcal{J}_{BC}=\nabla_A\mathcal{J}_{BC}=\Gamma_{AB}{}^{D}\mathcal{J}_{DC}+\Gamma_{AC}{}^{D}\mathcal{J}_{BD}=0\,,$$

$$\mathcal{D}_A\eta_{pq}=D_A\eta_{pq}=\Phi_{Ap}{}^{r}\eta_{rq}+\Phi_{Aq}{}^{r}\eta_{pr}=0\,,$$

$$\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=D_A\bar{\eta}_{\bar{p}\bar{q}}=\bar{\Phi}_{A\bar{p}}{}^{\bar{r}}\bar{\eta}_{\bar{r}\bar{q}}+\bar{\Phi}_{A\bar{q}}{}^{\bar{r}}\bar{\eta}_{\bar{p}\bar{r}}=0\,,$$

$$\mathcal{D}_AC_{+\alpha\beta}=D_AC_{+\alpha\beta}=\Phi_{A\alpha}{}^{\delta}C_{+\delta\beta}+\Phi_{A\beta}{}^{\delta}C_{+\alpha\delta}=0\,,$$

$$\mathcal{D}_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=D_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=\bar{\Phi}_{A\bar{\alpha}}{}^{\bar{\delta}}\bar{C}_{+\bar{\delta}\bar{\beta}}+\bar{\Phi}_{A\bar{\beta}}{}^{\bar{\delta}}\bar{C}_{+\bar{\alpha}\bar{\delta}}=0\,,$$

including the gamma matrices,

$$\mathcal{D}_A(\gamma^p)^\alpha{}_\beta=D_A(\gamma^p)^\alpha{}_\beta=\Phi_A{}^p{}_q(\gamma^q)^\alpha{}_\beta+\Phi_A{}^\alpha{}_\delta(\gamma^p)^\delta{}_\beta-(\gamma^p)^\alpha{}_\delta\Phi_A{}^\delta{}_\beta=0\,,$$

$$\mathcal{D}_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=D_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=\bar{\Phi}_A{}^{\bar{p}}{}_{\bar{q}}(\bar{\gamma}^{\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}+\bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\delta}}(\bar{\gamma}^{\bar{p}})^{\bar{\delta}}{}_{\bar{\beta}}-(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\delta}}\bar{\Phi}_A{}^{\bar{\delta}}{}_{\bar{\beta}}=0\,.$$

- It follows then that the connections are all anti-symmetric,

$$\Gamma_{ABC} = -\Gamma_{ACB}\,,$$

$$\Phi_{Apq} = -\Phi_{Aqp}\,, \qquad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = -\bar{\Phi}_{A\bar{q}\bar{p}}\,, \qquad \bar{\Phi}_{A\bar{\alpha}\bar{\beta}} = -\bar{\Phi}_{A\bar{\beta}\bar{\alpha}}\,,$$

and as usual,

$$\Phi_A{}^{\alpha}{}_{\beta} = \tfrac{1}{4}\Phi_{Apq}(\gamma^{pq})^{\alpha}{}_{\beta}\,, \qquad \bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\beta}} = \tfrac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}(\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}\,.$$

- ***Further,*** **the master derivative is <u>compatible with the whole NS-NS sector,</u>**

$$\mathcal{D}_A d = \nabla_A d := -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA} = 0\,,$$

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0\,,$$

$$\mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

- It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$

and the connections are related to each other,

$$\Gamma_{ABC} = V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}\,,$$

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- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \,,$$

$$\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq} \,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} \,.$$

Here Γ^0_{CAB} is the torsionless DFT-Christoffel connection which we fixed earlier,

$$\begin{aligned}\Gamma^0_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \,,\end{aligned}$$

and, with the corresponding derivative, $\nabla^0_A = \partial_A + \Gamma^0_A$,

$$\Phi^0_{Apq} = V^B{}_p \nabla^0_A V_{Bq} = V^B{}_p \partial_A V_{Bq} + \Gamma^0_{ABC} V^B{}_p V^C{}_q \,,$$

$$\bar{\Phi}^0_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla^0_A \bar{V}_{B\bar{q}} = \bar{V}^B{}_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma^0_{ABC} \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}} \,.$$

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- Further, the extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_{A} d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 \,, \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 \,.$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_{A} \,, \qquad \bar{\psi}_{\bar{p}}\gamma_{A}\psi_{\bar{q}} \,, \qquad \bar{\rho}\gamma_{Apq}\rho \,, \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} \,,$$

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Semi-covariant curvature

- The usual curvatures for the three connections,

$$R_{CDAB}=\partial_{A}\Gamma_{BCD}-\partial_{B}\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}\,,$$

$$F_{ABpq}=\partial_{A}\Phi_{Bpq}-\partial_{B}\Phi_{Apq}+\Phi_{Apr}\Phi_{B}{}^{r}{}_{q}-\Phi_{Bpr}\Phi_{A}{}^{r}{}_{q}\,,$$

$$\bar{F}_{AB\bar{p}\bar{q}}=\partial_{A}\bar{\Phi}_{B\bar{p}\bar{q}}-\partial_{B}\bar{\Phi}_{A\bar{p}\bar{q}}+\bar{\Phi}_{A\bar{p}\bar{r}}\bar{\Phi}_{B}{}^{\bar{r}}{}_{\bar{q}}-\bar{\Phi}_{B\bar{p}\bar{r}}\bar{\Phi}_{A}{}^{\bar{r}}{}_{\bar{q}}\,,$$

  are, from $[\mathcal{D}_{A},\mathcal{D}_{B}]V_{Cp}=0$ and $[\mathcal{D}_{A},\mathcal{D}_{B}]\bar{V}_{C\bar{p}}=0$, related to each other,

$$R_{ABCD}=F_{CDpq}V_{A}{}^{p}V_{B}{}^{q}+\bar{F}_{CD\bar{p}\bar{q}}\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B}{}^{\bar{q}}\,.$$

- However, the crucial object in DFT is

$$S_{ABCD}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^{E}{}_{AB}\Gamma_{ECD}\right),$$

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$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right),$$

  which we named the semi-covariant "Riemann" curvature.

Properties of the semi-covariant curvature

- Precisely the same symmetric property as the ordinary Riemann curvature,

$$S_{ABCD}=\tfrac{1}{2}\left(S_{[AB][CD]}+S_{[CD][AB]}\right)\,,$$

$$S^{0}_{[ABC]D}=0\,.$$

- Projection property,

$$P_{I}{}^{A}\bar{P}_{J}{}^{B}P_{K}{}^{C}\bar{P}_{L}{}^{D}S_{ABCD}\equiv 0\,.$$

- Under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S_{ABCD}=\mathcal{D}_{[A}\delta\Gamma_{B]CD}+\mathcal{D}_{[C}\delta\Gamma_{D]AB}-\tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^{E}{}_{CD}-\tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^{E}{}_{AB}\,,$$

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$$\delta S_{ABCD} = \mathcal{D}_{[A}\delta\Gamma_{B]CD} + \mathcal{D}_{[C}\delta\Gamma_{D]AB} - \tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^{E}{}_{CD} - \tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^{E}{}_{AB}\,,$$

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Properties of the semi-covariant curvature

- Precisely the same symmetric property as the ordinary Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}\left(S_{[AB][CD]} + S_{[CD][AB]}\right)\,,$$

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- In general, as discussed earlier in this talk, under DFT-diffeomorphisms the variation of the semi-covariant derivative contains an *anomalous part* dictated by the six-index projectors,

$$\delta_X \left(\nabla_C T_{A_1\cdots A_n}\right) \equiv \hat{\mathcal{L}}_X \left(\nabla_C T_{A_1\cdots A_n}\right) + \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}\,T_{\cdots B\cdots}\,,$$

and hence,

$$\delta_X \;\neq\; \hat{\mathcal{L}}_X\,.$$

- However, the characteristic property of our master semi-covariant derivative is that, **contracted with the projectors, vielbeins as well as gamma matrices, it can generate various fully covariant quantities,** as listed below.

‘Semi-covariance’

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Projector-aided, fully covariant derivatives

- **For O(D, D) tensors:** *we recall*

$$P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,$$

$$\bar{P}_{C}{}^{D}P_{A_1}{}^{B_1}P_{A_2}{}^{B_2}\cdots P_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,$$

$$\left.\begin{array}{l} P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\,,\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\end{array}\right\}\quad\text{Divergences}\,,$$

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- **For Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$ tensors:**

$$
\begin{array}{ll}
\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, & \mathcal{D}_{\bar{p}} T_{q_1 q_2\cdots q_n}\,, \\
\mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, & \mathcal{D}^{\bar{p}} T_{\bar{p}q_1 q_2\cdots q_n}\,, \\
\mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, & \mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}} T_{q_1 q_2\cdots q_n}\,,
\end{array}
$$

where we set

$$
\mathcal{D}_p := V^A{}_p\mathcal{D}_A\,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}}\mathcal{D}_A\,.
$$

These are the pull-back of the previous results using the DFT-vielbeins.

Projector-aided, fully covariant derivatives

- **Dirac operators for fermions,** $\rho^{\alpha},\ \psi_{\bar{p}}^{\alpha},\ \rho'^{\bar{\alpha}},\ \psi_{p}'^{\bar{\alpha}}$:

$$\gamma^{p}\mathcal{D}_{p}\rho=\gamma^{A}\mathcal{D}_{A}\rho\,,\qquad\qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}}=\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,,\qquad\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}=\mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\psi}^{A}\gamma_{p}(\mathcal{D}_{A}\psi_{\bar{q}}-\tfrac{1}{2}\mathcal{D}_{\bar{q}}\psi_{A})\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'=\bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p}=\bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,,\qquad\qquad \mathcal{D}_{p}\psi'^{p}=\mathcal{D}_{A}\psi'^{A}\,,$$

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Projector-aided, fully covariant derivatives

- **For Spin$(1, D-1)_L \times$ Spin$(D-1, 1)_R$ bi-fundamental spinorial fields,** $\mathcal{T}^{\alpha}{}_{\bar{\beta}}$ **:**

$$\mathcal{D}_{+}\mathcal{T} := \gamma^{A}\mathcal{D}_{A}\mathcal{T} + \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{T}\bar{\gamma}^{A}\,,$$

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- Especially for the torsionless case, the corresponding operators are **nilpotent**

$$(\mathcal{D}^{0}_{+})^{2}\mathcal{T} \equiv 0\,, \qquad\qquad (\mathcal{D}^{0}_{-})^{2}\mathcal{T} \equiv 0\,,$$

  and hence, they define **O(D, D) covariant cohomology.**

- **The field strength of the R-R potential,** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$**, is then defined by**

$$\mathcal{F} := \mathcal{D}^{0}_{+}\mathcal{C}\,.$$

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized

$$\delta\mathcal{C} = \mathcal{D}^{0}_{+}\Delta \qquad \Longrightarrow \qquad \delta\mathcal{F} = \mathcal{D}^{0}_{+}(\delta\mathcal{C}) = (\mathcal{D}^{0}_{+})^{2}\Delta \equiv 0\,.$$

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- **Scalar curvature:**

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \,.$$

- **"Ricci" curvature:**

$$S_{p\bar{q}} + \tfrac{1}{2}\mathcal{D}_{\bar{r}}\bar{\Delta}_{p\bar{q}}{}^{\bar{r}} + \tfrac{1}{2}\mathcal{D}_{r}\Delta_{\bar{q}p}{}^{r} \,,$$

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Combining all the results above, we are now ready to spell

- $\mathcal{N}=2$ $D=10$ **Supersymmetric Double Field Theory**

- **Lagrangian** :

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_{+}^{-1}\mathcal{F}^{T}C_{+}$.

- As they are contracted with the DFT-vielbeins properly, **every term in the Lagrangian is fully covariant.**

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- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\Gamma_{ABC}=\Gamma^{0}_{ABC}+i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho-2i\bar{\rho}\gamma_{BC}\psi_{A}-i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}}+4i\bar{\psi}_{B}\gamma_{A}\psi_{C}$$
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which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold: $\mathcal{D}^{\star}_{A}$ for the unprimed fermions and $\mathcal{D}'^{\star}_{A}$ for the primed fermions, set by

$$\Gamma^{\star}_{ABC}=\Gamma_{ABC}-i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho+i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_{A}+i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}}-2i\bar{\psi}_{B}\gamma_{A}\psi_{C}+i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A}\,,$$

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- **The $\mathcal{N}=2$ supersymmetry transformation rules are**

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,$$

$$\delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,$$

$$\delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^{q}(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon\mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon\rho = -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,$$

$$\delta_\varepsilon\rho' = -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,$$

$$\delta_\varepsilon\psi_{\bar{p}} = \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,$$

$$\delta_\varepsilon\psi'_p = \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,,$$

where

$$\hat{\Gamma}_{ABC} = \Gamma_{ABC} - i\tfrac{17}{48}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{1}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 3i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,,$$

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- The Lagrangian is **pseudo :** It is necessary to impose a **self-duality** of the R-R field strength by hand,

$$\tilde{\mathcal{F}}_{-} := \left(1-\gamma^{(D+1)}\right)\left(\mathcal{F}-i\tfrac{1}{2}\rho\bar{\rho}'+i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{q}}\right)\equiv 0\,.$$

- **Lagrangian** :

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

- The Lagrangian is **pseudo** **:** It is necessary to impose a **self-duality** of the R-R field strength by hand,

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- Under the $\mathcal{N}=2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

$$\delta_{\varepsilon}\mathcal{L}_{\rm Type\,II} \simeq -\tfrac{1}{8}e^{-2d}\,\bar{V}^{A}{}_{\bar{q}}\delta_{\varepsilon}V_{Ap}{\rm Tr}\left(\gamma^{p}\tilde{\mathcal{F}}_{-}\bar{\gamma}^{\bar{q}}\overline{\tilde{\mathcal{F}}_{-}}\right)\,,$$

where

$$\tilde{\mathcal{F}}_{-} := \left(1-\gamma^{(D+1)}\right)\left(\mathcal{F}-i\tfrac{1}{2}\rho\bar{\rho}^{\prime}+i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}^{\prime}_{p}\bar{\gamma}^{\bar{q}}\right)\,.$$

This verifies, *to the full order in fermions*, **the supersymmetric invariance of the action, modulo the self-duality.**

- For a **nontrivial consistency check,** the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_{\varepsilon}\tilde{\mathcal{F}}_{-} = -i\left(\tilde{\mathcal{D}}_{\bar{p}}\rho+\gamma^{p}\tilde{\mathcal{D}}_{p}\psi_{\bar{p}}-\gamma^{p}\mathcal{F}\bar{\gamma}_{\bar{p}}\psi^{\prime}{}_{p}\right)\bar{\varepsilon}^{\prime}\bar{\gamma}^{\bar{p}}-i\gamma^{p}\varepsilon\left(\tilde{\mathcal{D}}^{\prime}_{p}\bar{\rho}^{\prime}+\tilde{\mathcal{D}}^{\prime}_{\bar{p}}\bar{\psi}^{\prime}_{p}\bar{\gamma}^{\bar{p}}-\bar{\psi}_{\bar{p}}\gamma_{p}\mathcal{F}\bar{\gamma}^{\bar{p}}\right)\,.$$

$\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

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Equations of Motion for Bosons

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'{}_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^{q}\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

This is DFT-generalization of Einstein equation.

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$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

Namely, the on-shell Lagrangian vanishes!

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$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

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Equations of Motion for Fermions

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$$\gamma^{p}\tilde{\mathcal{D}}_{p}\rho - \tilde{\mathcal{D}}_{\bar{p}}\psi^{\bar{p}} - \mathcal{F}\rho' = 0\,, \qquad \bar{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\rho' - \tilde{\mathcal{D}}_{p}\psi'^{p} - \bar{\mathcal{F}}\rho = 0\,.$$

- Gravitinos,

$$\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^{p}\tilde{\mathcal{D}}_{p}\psi_{\bar{p}} - \gamma^{p}\mathcal{F}\bar{\gamma}_{\bar{p}}\psi'_{p} = 0\,, \qquad \tilde{\mathcal{D}}_{p}\rho' + \bar{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\psi'_{p} - \bar{\gamma}^{\bar{p}}\bar{\mathcal{F}}\gamma_{p}\psi_{\bar{p}} = 0\,.$$

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Truncation to $\mathcal{N} = 1$ $D = 10$ SDFT [1112.0069]

- Turning off the primed fermions and the R-R sector truncates the $\mathcal{N} = 2$ $D = 10$ SDFT to $\mathcal{N} = 1$ $D = 10$ SDFT,

$$\mathcal{L}_{\mathcal{N}=1} = e^{-2d}\Big[\tfrac{1}{8}\left(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD}\right)S_{ACBD} + i\tfrac{1}{2}\bar{\rho}\gamma^{A}\mathcal{D}^{\star}_{A}\rho - i\bar{\psi}^{A}\mathcal{D}^{\star}_{A}\rho - i\tfrac{1}{2}\bar{\psi}^{B}\gamma^{A}\mathcal{D}^{\star}_{A}\psi_{B}\Big]\,.$$

- $\mathcal{N} = 1$ **Local SUSY:**

$$\begin{aligned}
\delta_{\varepsilon} d &= -i\tfrac{1}{2}\bar{\varepsilon}\rho\,,\\
\delta_{\varepsilon} V_{Ap} &= -i\bar{\varepsilon}\gamma_{p}\psi_{A}\,,\\
\delta_{\varepsilon} \bar{V}_{A\bar{p}} &= i\bar{\varepsilon}\gamma_{A}\psi_{\bar{p}}\,,\\
\delta_{\varepsilon}\rho &= -\gamma^{A}\hat{\mathcal{D}}_{A}\varepsilon\,,\\
\delta_{\varepsilon}\psi_{\bar{p}} &= \bar{V}^{A}{}_{\bar{p}}\hat{\mathcal{D}}_{A}\varepsilon - i\tfrac{1}{4}(\bar{\rho}\psi_{\bar{p}})\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\rho)\psi_{\bar{p}}\,.
\end{aligned}$$

$\mathcal{N}=1$ SUSY Algebra [1112.0069]

- Commutator of supersymmetry reads

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\equiv\hat{\mathcal{L}}_{X_3}+\delta_{\varepsilon_3}+\delta_{\mathbf{so}(1,9)_L}+\delta_{\mathbf{so}(9,1)_R}+\delta_{\mathrm{trivial}}\,.$$

where

$$X_3^A=i\bar{\varepsilon}_1\gamma^A\varepsilon_2\,,\qquad \varepsilon_3=i\tfrac{1}{2}\left[(\bar{\varepsilon}_1\gamma^p\varepsilon_2)\gamma_p\rho+(\bar{\rho}\varepsilon_2)\varepsilon_1-(\bar{\rho}\varepsilon_1)\varepsilon_2\right]\,,\qquad \text{etc.}$$

and $\delta_{\mathrm{trivial}}$ corresponds to the fermionic equations of motion.

Now I am going to sketch

- the parametrization of the DFT-field-variables in terms of Riemannian ones,
- the diagonal gauge of Spin$(1, D{-}1)_L \times$ Spin$(D{-}1, 1)_R$,
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- Nevertheless, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian ("metric-less") string theory backgrounds. *c.f.* Gomis-Ooguri

- Note also 'global' aspects of interest in DFT:
  - T-fold Hull
  - "non-geometry" Berman-Cederwall-Perry, Papadopoulos
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Parametrization: Reduction to Generalized Geometry

- Recall the defining algebraic properties of the DFT-vielbeins,

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}}=0\,,\qquad V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}\,.$$

- We may parametrize the solution in terms of Riemannian variables.
- Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

$$V_{Ap}=\frac{1}{\sqrt{2}}\begin{pmatrix}(e^{-1})_{p}{}^{\mu}\\ \\ (B+e)_{\nu p}\end{pmatrix},\qquad\qquad \bar{V}_{A\bar{p}}=\frac{1}{\sqrt{2}}\begin{pmatrix}(\bar{e}^{-1})_{\bar{p}}{}^{\mu}\\ \\ (B+\bar{e})_{\nu\bar{p}}\end{pmatrix}.$$

Here $e_{\mu}{}^{p}$ and $\bar{e}_{\nu}{}^{\bar{p}}$ are two copies of the D-dimensional vielbeins, or zehnbeins, corresponding to the same spacetime metric,

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Parametrization: Reduction to Generalized Geometry

- Instead, we may choose an alternative parametrization,

$$V_A{}^p = \frac{1}{\sqrt{2}}\begin{pmatrix} (\beta+\tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^p{}_\nu \end{pmatrix}, \qquad \bar{V}_A{}^{\bar{p}} = \frac{1}{\sqrt{2}}\begin{pmatrix} (\beta+\bar{\tilde{e}})^{\mu p} \\ (\bar{\tilde{e}}^{-1})^p{}_\nu \end{pmatrix},$$

  where $\beta^{\mu p} = \beta^{\mu\nu}(\tilde{e}^{-1})^p{}_\nu$, $\beta^{\mu\bar{p}} = \beta^{\mu\nu}(\bar{\tilde{e}}^{-1})^p{}_\nu$, and $\tilde{e}^\mu{}_p$, $\bar{\tilde{e}}^\mu{}_{\bar{p}}$ correspond to a pair of T-dual vielbeins for winding modes,

$$\tilde{e}^\mu{}_p\tilde{e}^\nu{}_q\eta^{pq} = -\bar{\tilde{e}}^\mu{}_{\bar{p}}\bar{\tilde{e}}^\nu{}_{\bar{q}}\eta^{\bar{p}\bar{q}} = (g - Bg^{-1}B)^{-1\,\mu\nu}\,.$$

- Note that in the T-dual winding mode sector, the D-dimensional curved spacetime indices are all upside-down: $\tilde{x}_\mu$, $\tilde{e}^\mu{}_p$, $\bar{\tilde{e}}^\mu{}_{\bar{p}}$, $\beta^{\mu\nu}$ (*cf.* x^μ, $e_\mu{}^p$, $\bar{e}_\mu{}^{\bar{p}}$, $B_{\mu\nu}$).

Parametrization: Reduction to Generalized Geometry

- Two parametrizations:

$$V_{Ap}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}(e^{-1})_{p}{}^{\mu}\\ \\ (B+e)_{\nu p}\end{array}\right)\,,\qquad\qquad \bar{V}_{A\bar{p}}=\frac{1}{\sqrt{2}}\left(\begin{array}{c}(\bar{e}^{-1})_{\bar{p}}{}^{\mu}\\ \\ (B+\bar{e})_{\nu\bar{p}}\end{array}\right)$$

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- Yet if we consider dimensional reductions from D to lower dimensions, there is no longer preferred parametrization.

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- This reduces (S)DFT to Generalized Geometry

  Hitchin; Grana, Minasian, Petrini, Waldram

- For example, the $\mathbf{O}(D,D)$ covariant Dirac operators become

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\begin{aligned}
&\sqrt{2}\gamma^A\mathcal{D}_A\rho \equiv \gamma^m\left(\partial_m\rho + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\rho + \tfrac{1}{24}H_{mnp}\gamma^{np}\rho - \partial_m\phi\rho\right),\\
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Diagonal gauge fixing and Reduction to SUGRA

- Setting the diagonal gauge,

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

with $\eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}}$, $\bar{\gamma}^{\bar{p}} = \gamma^{(D+1)}\gamma^p$, $\bar{\gamma}^{(D+1)} = -\gamma^{(D+1)}$, breaks the local Lorentz symmetry,

$$\mathbf{Spin}(1, D{-}1)_L \times \mathbf{Spin}(D{-}1, 1)_R \quad \Longrightarrow \quad \mathbf{Spin}(1, D{-}1)_D\,.$$

- And it reduces SDFT to SUGRA:

$\mathcal{N} = 2$ $D = 10$ **SDFT** $\quad\Longrightarrow\quad$ $10D$ **Type II democratic SUGRA**

Bergshoeff, *et al.*; Coimbra, Strickland-Constable, Waldram

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- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \textstyle\sum_p' \tfrac{1}{p!}\, \mathcal{C}_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p}$$

  and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \textstyle\sum_p' \tfrac{1}{(p+1)!}\, \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}}$$

  where $\sum_p'$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1 a_2 \cdots a_p} = p\left(D_{[a_1}\mathcal{C}_{a_2 \cdots a_p]} - \partial_{[a_1}\phi\, \mathcal{C}_{a_2 \cdots a_p]}\right) + \tfrac{p!}{3!(p-3)!}\, H_{[a_1 a_2 a_3}\mathcal{C}_{a_4 \cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to a 'twisted K-theory' exterior derivative and its dual, after the diagonal gauge fixing,

$$\mathcal{D}^0_+ \quad\Longrightarrow\quad \mathrm{d} + (H - \mathrm{d}\phi)\wedge$$

$$\mathcal{D}^0_- \quad\Longrightarrow\quad *\,[\,\mathrm{d} + (H - \mathrm{d}\phi)\wedge\ \,]\,*$$

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \sum_p' \tfrac{1}{p!}\, \mathcal{C}_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p}$$

and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \sum_p' \tfrac{1}{(p+1)!}\, \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}}$$

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- **In this way,** **ordinary SUGRA $\equiv$ gauge-fixed SDFT,**

$$\textbf{Spin}(1,D{-}1)_L\times\textbf{Spin}(D{-}1,1)_R\quad\Longrightarrow\quad\textbf{Spin}(1,D{-}1)_D\,.$$

Modifying $\mathbf{O}(D, D)$ transformation rule

- The diagonal gauge, $e_{\mu}{}^{p} \equiv \bar{e}_{\mu}{}^{\bar{p}}$, is incompatible with the vectorial $\mathbf{O}(D, D)$ transformation rule of the DFT-vielbein.

- In order to preserve the diagonal gauge, it is necessary to modify the $\mathbf{O}(D, D)$ transformation rule.

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Modifying $\mathbf{O}(D, D)$ transformation rule

- The $\mathbf{O}(D, D)$ rotation must accompany a compensating $\mathbf{Pin}(D{-}1, 1)_R$ local Lorentz rotation, $\bar{L}_{\bar{q}}{}^{\bar{p}}$, $S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}}$ which we can construct explicitly as below.

$$\bar{L} = \bar{e}^{-1}\left[\mathbf{a}^{t} - (g+B)\mathbf{b}^{t}\right]\left[\mathbf{a}^{t} + (g-B)\mathbf{b}^{t}\right]^{-1}\bar{e}\,, \qquad \bar{\gamma}^{\bar{q}}\bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1}\bar{\gamma}^{\bar{p}}S_{\bar{L}}\,,$$

where $\mathbf{a}$ and $\mathbf{b}$ are parameters of a given $\mathbf{O}(D, D)$ group element,

$$M_{A}{}^{B} = \begin{pmatrix} \mathbf{a}^{\mu}{}_{\nu} & \mathbf{b}^{\mu\sigma} \\ \mathbf{c}_{\rho\nu} & \mathbf{d}_{\rho}{}^{\sigma} \end{pmatrix}.$$

Modified O(D,D) Transformation Rule After The Diagonal Gauge Fixing

$$
\begin{array}{ccc}
d & \longrightarrow & d \\
V_{A}{}^{p} & \longrightarrow & M_{A}{}^{B}\,V_{B}{}^{p} \\
\bar{V}_{A}{}^{\bar{p}} & \longrightarrow & M_{A}{}^{B}\,\bar{V}_{B}{}^{\bar{q}}\,\bar{L}_{\bar{q}}{}^{\bar{p}} \\
\mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\alpha}{}_{\bar{\alpha}} & \longrightarrow & \mathcal{C}^{\alpha}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\alpha}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}} \\
\rho^{\alpha} & \longrightarrow & \rho^{\alpha} \\
\rho'^{\bar{\alpha}} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\rho'^{\bar{\beta}} \\
\psi^{\alpha}_{\bar{p}} & \longrightarrow & (\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}}\,\psi^{\alpha}_{\bar{q}} \\
\psi'^{\bar{\alpha}}_{p} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\psi'^{\bar{\beta}}_{p}
\end{array}
$$

- **All the barred indices are now to be rotated.** Consistent with Hassan
- **The R-R sector can be also mapped to O(D,D) spinors.**

Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach

Flipping the chirality: IIA $\Leftrightarrow$ IIB

- **If and only if $\det(\bar{L}) = -1$, the modified $\mathbf{O}(D,D)$ rotation flips the chirality of the theory, since**

$$\bar{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L})\, S_{\bar{L}} \bar{\gamma}^{(D+1)}\,.$$

- Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $\mathbf{O}(D,D)$ T-duality.

- However, since $\bar{L}$ explicitly depends on the parametrization of V_{Ap} and $\bar{V}_{A\bar{p}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $\mathbf{O}(D,D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.

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Worldsheet Perspective

String propagates in doubled-yet-gauged spacetime

- The section condition is equivalent to the 'coordinate gauge symmetry', 1304.5946

$$x^M \sim x^M + \varphi\partial^M\varphi' .$$

A 'physical point' is one-to-one identified with a 'gauge orbit' in coordinate space.

- The coordinate gauge symmetry can be concretely realized on worldsheet, 1307.8377

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{2}\sqrt{-h}h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

where

$$D_iX^M = \partial_iX^M - \mathcal{A}_i^M\,, \qquad \mathcal{A}_i^M\partial_M \equiv 0\,.$$

- The Lagrangian is quite symmetric thanks to the auxiliary gauge field, $\mathcal{A}_i^M$:
  - String worldsheet diffeomorphisms plus Weyl symmetry (as usual)
  - $\mathbf{O}(D,D)$ T-duality
  - Target spacetime diffeomorphisms
  - The coordinate gauge symmetry

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

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- For example, under target spacetime 'finite' diffeomorphism *à la* Zwiebach-Hohm

$$L_M{}^N := \partial_A X'^B\,, \qquad \bar{L} := \mathcal{J} L^t \mathcal{J}^{-1}\,,$$

$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, \qquad \bar{F} := \mathcal{J} F^t \mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,$$

each field transforms as

$$\begin{array}{lll} X^M & \longrightarrow & X'^M(X)\,, \\ \mathcal{H}_{MN}(X) & \longrightarrow & \mathcal{H}'_{MN}(X') = \bar{F}_M{}^K \bar{F}_N{}^L \mathcal{H}_{KL}(X)\,, \\ \mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N (L-F)_N{}^M \qquad : \quad \mathcal{A}'^M \partial'_M \equiv 0\,, \\ DX^M & \longrightarrow & D'X'^M = DX^N F_N{}^M\,, \end{array}$$

such that the worldsheet action remains invariant, up to total derivatives.

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- The Equation Of Motion for X^{L} can be conveniently organized in terms of our DFT-Christoffel connection:

$$\frac{1}{\sqrt{-h}}\partial_{i}\left(\sqrt{-h}D^{i}X^{M}\mathcal{H}_{ML}+\epsilon^{ij}\partial_{i}\mathcal{A}_{jL}\right)-2\Gamma_{LMN}\,(PD_{i}X)^{M}(\bar{P}D^{i}X)^{N}=0\,,$$

which is comparable to the *geodesic motion* of a point particle, $\ddot{Y}^{\lambda}+\Gamma^{\lambda}_{\mu\nu}\dot{Y}^{\mu}\dot{Y}^{\nu}=0.$

- The EOM of $\mathcal{A}_{i}^{M}$ implies *a priori*,

$$\delta\mathcal{A}_{iM}\left(\mathcal{H}^{M}{}_{N}D^{i}X^{N}+\tfrac{1}{\sqrt{-h}}\epsilon^{ij}D_{j}X^{M}\right)=0\,.$$

Especially, for the case of the 'non-degenerate' Riemannian background, a complete self-duality follows

$$\mathcal{H}^{M}{}_{N}D^{i}X^{N}+\tfrac{1}{\sqrt{-h}}\epsilon^{ij}D_{j}X^{M}=0\,.$$

- Finally, the EOM of h_{ij} gives the Virasoro constraints,

$$\left(D_{i}X^{M}D_{j}X^{N}-\tfrac{1}{2}h_{ij}D_{k}X^{M}D^{k}X^{N}\right)\mathcal{H}_{MN}=0\,.$$

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- After parametrization, $X^M = (\tilde{Y}_\mu, Y^\nu)$, $\mathcal{H}_{MN}(G,B)$, and integrating out $\mathcal{A}_i^M$, it can produce either the standard string action for the 'non-degenerate' Riemannian case,

$$\frac{1}{4\pi\alpha'}\mathcal{L} \equiv \frac{1}{2\pi\alpha'}\Big[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y) + \tfrac{1}{2}\epsilon^{ij}\partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \tfrac{1}{2}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu\Big] ,$$

with the bonus of the topological term introduced by Giveon-Rocek, Hull

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or chiral actions for 'degenerate' non-Riemannian cases, e.g. for $\mathcal{H}_{AB} = \mathcal{J}_{AB}$,

$$\frac{1}{4\pi\alpha'}\mathcal{L} \equiv \frac{1}{4\pi\alpha'}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu , \qquad \partial_i Y^\mu + \frac{1}{\sqrt{-h}}\epsilon_i{}^j \partial_j Y^\mu = 0 .$$

c.f. Gomis-Ooguri

U-duality

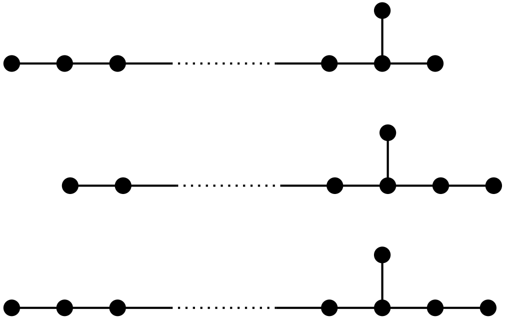
Parallel to the stringy differential geometry for O(D, D) T-duality,

it is possible to construct M-theoretic differential geometry for each U-duality group.

	$A_{N-1} \equiv \mathbf{sl}(N)$
	$D_{N-1} \equiv \mathbf{so}(N{-}1, N{-}1)$
	E_{N-1}
	E_N

Table: Dynkin diagrams for A_{N-1}, D_{N-1}, E_{N-1} and E_N

- E_{11} : conjectured to be the ultimate duality group. West
- E_{10} : Damour, Nicolai, Henneaux and further E_n $(n \leq 8)$ "Exceptional Field Theory"
- D_{10} : Double Field Theory
- A_{10} : **U-gravity**



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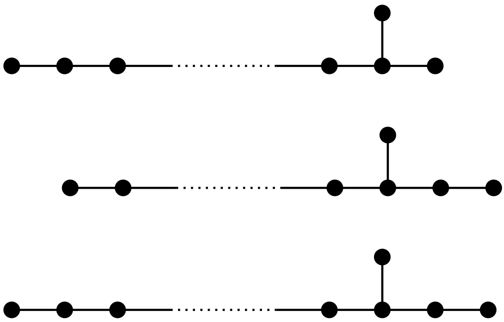

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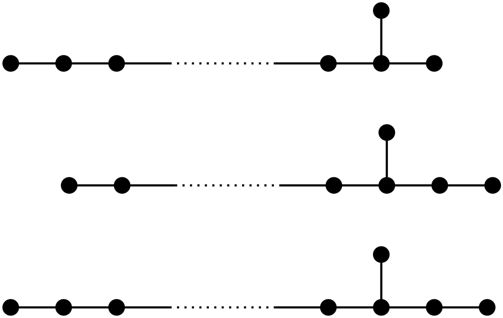

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Geometric Constitution of U-gravity

- *Notation.*

  Small Latin alphabet letters denote the $\mathbf{SL}(N)$ vector indices, i.e. $a, b, c, \cdots = 1, 2, \cdots, N$.

Geometric Constitution of U-gravity

- *Extended-yet-gauged spacetime.*

  - The spacetime is formally extended, being $\frac{1}{2}N(N-1)$-dimensional. The coordinates carry a pair of anti-symmetric $\mathbf{SL}(N)$ vector indices,

    $$x^{ab} = -x^{ba} = x^{[ab]}\,,$$

    and hence so does the derivative,

    $$\partial_{ab} = -\partial_{ba} = \partial_{[ab]} = \frac{\partial}{\partial x^{ab}}\,, \qquad \partial_{ab}x^{cd} = \delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c\,.$$

  - However, *the spacetime is gauged*: the coordinate space is equipped with an equivalence relation ('Coordinate Gauge Symmetry'),

    $$x^{ab} \sim x^{ab} + \frac{1}{(N-4)!}\epsilon^{abc_1\cdots c_{N-4}de}\phi_{c_1\cdots c_{N-4}}\partial_{de}\varphi\,,$$

    where $\phi_{c_1\cdots c_{N-4}}$ and φ are arbitrary functions in U-gravity.

  - Each equivalence class, or gauge orbit defined by the equivalence relation represents a single physical point, and diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.

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- *Realization of the coordinate gauge symmetry.*

The equivalence relation is realized in U-gravity by enforcing that, arbitrary functions and their arbitrary derivatives are invariant under the coordinate gauge symmetry shift,

$$\Phi(x+\Delta)=\Phi(x)\,,\qquad\qquad \Delta^{ab}=\tfrac{1}{(N-4)!}\epsilon^{abc_{1}\cdots c_{N-4}de}\phi_{c_{1}\cdots c_{N-4}}\partial_{de}\varphi\,.$$

- *Section condition.*
  - The invariance under the coordinate gauge symmetry is, in fact, equivalent to a section condition, *c.f.* Berman-Perry for $N=5$

$$\partial_{[ab}\partial_{cd]}\equiv 0\,.$$

  - Acting on arbitrary functions, Φ, Φ', and their products, the section condition leads to

$$\partial_{[ab}\partial_{cd]}\Phi=\partial_{[ab}\partial_{c]d}\Phi=0\qquad\text{(weak constraint)}\,,$$

$$\partial_{[ab}\Phi\partial_{cd]}\Phi'=\tfrac{1}{2}\partial_{[ab}\Phi\partial_{c]d}\Phi'-\tfrac{1}{2}\partial_{d[a}\Phi\partial_{bc]}\Phi'=0\qquad\text{(strong constraint)}\,.$$

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$$\partial_{[ab}\Phi\partial_{cd]}\Phi' = \tfrac{1}{2}\partial_{[ab}\Phi\partial_{c]d}\Phi' - \tfrac{1}{2}\partial_{d[a}\Phi\partial_{bc]}\Phi' = 0 \qquad \text{(strong constraint)}\,.$$

- *Diffeomorphism.*

U-gravity diffeomorphism is generated by a generalized Lie derivative,

c.f. Berman-Perry for $N=5$

$$
\begin{aligned}
\hat{\mathcal{L}}_X T^{a_1 a_2\cdots a_p}{}_{b_1 b_2\cdots b_q} := \;& \tfrac{1}{2}X^{cd}\partial_{cd}T^{a_1 a_2\cdots a_p}{}_{b_1 b_2\cdots b_q} \\
&+\tfrac{1}{2}(\tfrac{1}{2}p-\tfrac{1}{2}q+\omega)\partial_{cd}X^{cd}\,T^{a_1 a_2\cdots a_p}{}_{b_1 b_2\cdots b_q} \\
&-\textstyle\sum_{i=1}^{p} T^{a_1\cdots c\cdots a_p}{}_{b_1 b_2\cdots b_q}\partial_{cd}X^{a_i d} \\
&+\textstyle\sum_{j=1}^{q}\partial_{b_j d}X^{cd}\,T^{a_1 a_2\cdots a_p}{}_{b_1\cdots c\cdots b_q}\,.
\end{aligned}
$$

Here we let the tensor density, $T^{a_1 a_2\cdots a_p}{}_{b_1 b_2\cdots b_q}$, carry the 'total' weight, $\frac{1}{2}p-\frac{1}{2}q+\omega$, such that each upper or lower index contributes to the total weight by $+\frac{1}{2}$ or $-\frac{1}{2}$ respectively, while ω corresponds to a possible 'extra' weight.

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Note

$$\hat{\mathcal{L}}_X\delta^a{}_b=0\,,$$

and the commutator,

$$\left[\hat{\mathcal{L}}_X,\hat{\mathcal{L}}_Y\right]=\hat{\mathcal{L}}_{[X,Y]_{\mathrm{G}}}\,,\qquad [X,Y]^{ab}_{\mathrm{G}}=\tfrac{1}{2}X^{cd}\partial_{cd}Y^{ab}-\tfrac{3}{2}X^{[ab}\partial_{cd}Y^{cd]}-(X\leftrightarrow Y)\,.$$

- *U-metric.*

  The only geometric object in $\mathbf{SL}(N)$ U-gravity is a metric, or U-metric, which is a generic non-degenerate $N \times N$ symmetric matrix, obeying surely the section condition,

$$M_{ab} = M_{ba} = M_{(ab)} \,.$$

  Like in Riemannian geometry, the U-metric with its inverse, M^{ab}, may freely lower or raise the positions of the N-dimensional $\mathbf{SL}(N)$ vector indices.

- *Integral measure.*

  While the U-metric has no extra weight, its determinant, $M \equiv \det(M_{ab})$, acquires an extra weight, $\omega = 4 - N$. The duality invariant integral measure is then

$$|M|^{\frac{1}{4-N}} \,.$$

Geometric Constitution of U-gravity

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- *Semi-covariant derivative and semi-covariant Riemann curvature.*

We define a semi-covariant derivative,

$$\nabla_{cd} T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} := \partial_{cd} T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} + \tfrac{1}{2}(\tfrac{1}{2}p - \tfrac{1}{2}q + \omega)\Gamma_{cde}{}^{e} T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q}$$
$$- \textstyle\sum_{i=1}^{p} T^{a_1 \cdots e \cdots a_p}{}_{b_1 b_2 \cdots b_q}\Gamma_{cde}{}^{a_i} + \sum_{j=1}^{q} \Gamma_{cdb_j}{}^{e} T^{a_1 a_2 \cdots a_p}{}_{b_1 \cdots e \cdots b_q}\,,$$

and a semi-covariant Riemann curvature,

$$\begin{aligned} S_{abcd} :=\ & 3\partial_{[ab}\Gamma_{e][cd]}{}^{e} + 3\partial_{[cd}\Gamma_{e][ab]}{}^{e} + \tfrac{1}{4}\Gamma_{abe}{}^{e}\Gamma_{cdf}{}^{f} + \tfrac{1}{2}\Gamma_{abe}{}^{f}\Gamma_{cdf}{}^{e} \\ & + \Gamma_{ab[c}{}^{e}\Gamma_{d]ef}{}^{f} + \Gamma_{cd[a}{}^{e}\Gamma_{b]ef}{}^{f} + \Gamma_{ea[c}{}^{f}\Gamma_{d]fb}{}^{e} - \Gamma_{eb[c}{}^{f}\Gamma_{d]fa}{}^{e}\,. \end{aligned}$$

- *Semi-covariant derivative and semi-covariant Riemann curvature.*

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$$\nabla_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}:=\partial_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}+\tfrac{1}{2}(\tfrac{1}{2}p-\tfrac{1}{2}q+\omega)\Gamma_{cde}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}$$
$$-\textstyle\sum_{i=1}^{p}T^{a_1\cdots e\cdots a_p}{}_{b_1b_2\cdots b_q}\Gamma_{cde}{}^{a_i}+\sum_{j=1}^{q}\Gamma_{cdb_j}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1\cdots e\cdots b_q}\,,$$

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The semi-covariant derivative obeys the Leibniz rule and annihilates the Kronecker delta symbol,

$$\nabla_{cd}\delta^{a}{}_{b}=0\,.$$

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$$\nabla_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} := \partial_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} + \tfrac{1}{2}(\tfrac{1}{2}p - \tfrac{1}{2}q + \omega)\Gamma_{cde}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}$$
$$-\textstyle\sum_{i=1}^{p} T^{a_1\cdots e\cdots a_p}{}_{b_1b_2\cdots b_q}\Gamma_{cde}{}^{a_i} + \textstyle\sum_{j=1}^{q}\Gamma_{cdb_j}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1\cdots e\cdots b_q}\,,$$

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A crucial defining property of the semi-covariant Riemann curvature is that, under arbitrary transformation of the connection it transforms as total derivative,

$$\delta S_{abcd} = 3\nabla_{[ab}\delta\Gamma_{e][cd]}{}^{e} + 3\nabla_{[cd}\delta\Gamma_{e][ab]}{}^{e}\,.$$

- *Semi-covariant derivative and semi-covariant Riemann curvature.*

We define a semi-covariant derivative,

$$\nabla_{cd} T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} := \partial_{cd} T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} + \tfrac{1}{2}(\tfrac{1}{2}p - \tfrac{1}{2}q + \omega)\Gamma_{cde}{}^{e} T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q}$$
$$- \sum_{i=1}^{p} T^{a_1 \cdots e \cdots a_p}{}_{b_1 b_2 \cdots b_q} \Gamma_{cde}{}^{a_i} + \sum_{j=1}^{q} \Gamma_{cdb_j}{}^{e} T^{a_1 a_2 \cdots a_p}{}_{b_1 \cdots e \cdots b_q} \,,$$

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Further, the semi-covariant Riemann curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature, including the Bianchi identity,

$$S_{abcd} = S_{[ab][cd]} = S_{cdab} \,, \qquad S_{[abc]d} = 0 \,.$$

- *Connection.*

$$
\begin{aligned}
\Gamma_{abcd} \quad &= A_{abcd} + \tfrac{1}{2}(A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \\
&\quad + \tfrac{1}{N-2}\left(M_{ac}A^{e}{}_{(bd)e} - M_{ad}A^{e}{}_{(bc)e} + M_{bd}A^{e}{}_{(ac)e} - M_{bc}A^{e}{}_{(ad)e}\right)\,,
\end{aligned}
$$

where

$$
A_{abcd} := -\tfrac{1}{2}\partial_{ab}M_{cd} + \tfrac{1}{2(N-4)}M_{cd}\partial_{ab}\ln|M|\,.
$$

This connection is the unique solution to the following five constraints:

$$
\Gamma_{abcd} + \Gamma_{abdc} = 2A_{abcd}\,, \tag{1}
$$

$$
\Gamma_{abc}{}^{d} + \Gamma_{bac}{}^{d} = 0\,, \tag{2}
$$

$$
\Gamma_{abc}{}^{d} + \Gamma_{bca}{}^{d} + \Gamma_{cab}{}^{d} = 0\,, \tag{3}
$$

$$
\Gamma_{cab}{}^{c} + \Gamma_{cba}{}^{c} = 0\,, \tag{4}
$$

$$
\mathcal{P}_{abcd}{}^{efgh}\Gamma_{efgh} = 0\,. \tag{5}
$$

- *Connection.*

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Eq.(1) is equivalent to the U-metric compatibility condition,

$$\nabla_{ab}M_{cd} = 0 \, .$$

- *Connection.*

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Eq.(2) is natural from the skew-symmetric nature of the coordinates, $x^{(ab)} = 0$ and hence $\partial_{(ab)} = \nabla_{(ab)} = 0$.

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$$\Gamma_{abcd} = A_{abcd} + \tfrac{1}{2}(A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) + \tfrac{1}{N-2}\left(M_{ac}A^{e}{}_{(bd)e} - M_{ad}A^{e}{}_{(bc)e} + M_{bd}A^{e}{}_{(ac)e} - M_{bc}A^{e}{}_{(ad)e}\right),$$

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Eqs.(3,4) make the semi-covariant derivative compatible with the generalized Lie derivative and the generalized bracket: $\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla)$, $[X,Y]_{\mathrm{G}}(\partial) = [X,Y]_{\mathrm{G}}(\nabla)$.

- *Connection.*

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Eq.(5) is a projection condition which ensures the uniqueness.

- *Projection operator.*

The above eight-index projection operator is explicitly,

$$\begin{aligned}\mathcal{P}_{abcd}{}^{klmn} = \quad & \tfrac{1}{2}\delta_{[a}{}^{[k}\delta_{b]}{}^{l]}\delta_{[c}{}^{[m}\delta_{d]}{}^{n]} + \tfrac{1}{2}\delta_{[c}{}^{[k}\delta_{d]}{}^{l]}\delta_{[a}{}^{[m}\delta_{b]}{}^{n]} + \tfrac{1}{2}M_{c[a}\delta_{b]}{}^{m}M^{n[k}\delta_{d}{}^{l]} - \tfrac{1}{2}M_{c[a}\delta_{b]}{}^{[k}M^{l]n}\delta_{d}{}^{m} \\ & + \tfrac{1}{N-2}\left(\delta_{[a}{}^{n}M_{b][c}M^{m[k}\delta_{d]}{}^{l]} + \delta_{[c}{}^{n}M_{d][a}M^{m[k}\delta_{b]}{}^{l]} - M_{c[a}M_{b]d}M^{m[k}M^{l]n}\right) ,\end{aligned}$$

which satisfies

$$\mathcal{P}_{abcd}{}^{pqrs}\mathcal{P}_{pqrs}{}^{klmn} = \mathcal{P}_{abcd}{}^{klmn} , \qquad \mathcal{P}_{abs}{}^{sklmn} = 0 ,$$

$$\mathcal{P}_{abcd}{}^{klmn} = \mathcal{P}_{[ab]cd}{}^{[kl]mn} , \qquad \mathcal{P}_{ab[cd]}{}^{klmn} = \mathcal{P}_{cd[ab]}{}^{klmn} .$$

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Crucially, the projection operator dictates the anomalous terms under diffeomorphism:

$$(\delta_X - \hat{\mathcal{L}}_X)(\nabla_{ab}T^{c_1\cdots c_p}{}_{d_1 d_2\cdots d_q}) = -\textstyle\sum_{i=1}^{p} T^{c_1\cdots e\cdots c_p}{}_{d_1\cdots d_q}\Omega_{abe}{}^{c_i} + \sum_{j=1}^{q}\Omega_{abd_j}{}^{e}T^{c_1\cdots c_p}{}_{d_1\cdots e\cdots d_q}\,,$$

$$(\delta_X - \hat{\mathcal{L}}_X)S_{abcd} = 2\nabla_{e[a}\Omega_{b][cd]}{}^{e} + 2\nabla_{e[c}\Omega_{d][ab]}{}^{e}\,,$$

where

$$\Omega_{abcd} = \mathcal{P}_{abcd}{}^{klm}{}_{n}\partial_{kl}\partial_{me}X^{ne}\,.$$

- *Complete covariantizations.*

The semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized by (anti-)symmetrizing or contracting the $\mathbf{SL}(N)$ vector indices properly,

$$\nabla_{[ab}T_{c_1c_2\cdots c_q]}\,,\qquad \nabla_{ab}T^{a}\,,\qquad \nabla^{a}{}_{b}T_{[ca]}+\nabla^{a}{}_{c}T_{[ba]}\,,\qquad \nabla^{a}{}_{b}T_{(ca)}-\nabla^{a}{}_{c}T_{(ba)}\,,$$

$$\nabla_{ab}T^{[abc_1c_2\cdots c_q]}\quad(\text{divergence})\,,\qquad \nabla_{ab}\nabla^{[ab}T^{c_1c_2\cdots c_q]}\quad(\text{Laplacian})\,,$$

and

$$S_{ab}:=S_{acb}{}^{c}=S_{ba}\qquad(\text{“Ricci” curvature})\,,$$

$$S:=M^{ab}S_{ab}=S_{ab}{}^{ab}\qquad(\text{scalar curvature})\,.$$

Geometric Constitution of U-gravity

- *Action.*

  The action of $\mathbf{SL}(N)$ U-gravity is given by the fully covariant scalar curvature,

$$\int_{\Sigma} M^{\frac{1}{4-N}}\, S\,,$$

  where the integral is taken over a section, Σ.

- *The Einstein equation of motion.*

  The equation of motion corresponds to the vanishing of the 'Einstein' tensor,

$$S_{ab} + \tfrac{1}{2(N-4)} M_{ab} S = 0\,.$$

  Diffeomorphism symmetry of the action implies a conservation relation,

$$\nabla^{c}{}_{[a} S_{b]c} + \tfrac{3}{8} \nabla_{ab} S = 0\,.$$

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- *Two inequivalent sections.*

Up to $\mathbf{SL}(N)$ rotations, there exist two inequivalent solutions to the section condition :

1. Σ_{N-1} is an $(N-1)$-dimensional section given by

$$\partial_{\alpha\beta}=0\,,\qquad \partial_{\alpha N}\neq 0\,,$$

where $\alpha,\beta=1,2,\cdots,N-1$.

2. Σ_3 is a three-dimensional section characterized by

$$\partial_{\mu i}=0\,,\qquad \partial_{ij}=0\,,\qquad \partial_{\mu\nu}\neq 0\,,$$

where $\mu,\nu=1,2,3$ and $i,j=4,5,\cdots,N$. We may further dualize

$$\tilde{x}_{\mu}\equiv\tfrac{1}{2}\varepsilon_{\mu\nu\rho}x^{\nu\rho}\,,\qquad \tilde{\partial}^{\mu}\tilde{x}_{\nu}=\delta^{\mu}{}_{\nu}\,.$$

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For a triplet of arbitrary functions, we note

$$\partial_{[ab}\Phi\partial_{c][d}\Phi'\partial_{ef]}\Phi''=0\quad\text{on}\quad\Sigma_{N-1}\,,\qquad\qquad \partial_{[ab}\Phi\partial_{c][d}\Phi'\partial_{ef]}\Phi''\neq 0\quad\text{on}\quad\Sigma_{3}\,.$$

Since this is an $\mathbf{SL}(N)$ covariant statement, the two sections are inequivalent.

- *Two inequivalent sections.*

Up to **SL**(N) rotations, there exist two inequivalent solutions to the section condition:

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where $\mu,\nu=1,2,3$ and $i,j=4,5,\cdots,N$. We may further dualize

$$\tilde{x}_{\mu}\equiv\tfrac{1}{2}\varepsilon_{\mu\nu\rho}x^{\nu\rho}\,,\qquad\qquad \tilde{\partial}^{\mu}\tilde{x}_{\nu}=\delta^{\mu}{}_{\nu}\,.$$

Note: in the case of **SL**(5)*, they correspond to* $\mathcal{M}$*-theory and type IIB theory respectively (with the compactification on seven-manifold).* *Blair-Malek-JHP*

- *Riemannian reductions.*

1. Reduction to Σ_{N-1} through $(N-1)$-dimensional Riemannian metric, $g_{\alpha\beta}$, a vector, v^α, and a scalar, ϕ,

$$M_{ab}=\left(\begin{array}{cc}\frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|}\left(-e^{\phi}+v^2\right)\end{array}\right), \qquad |M|^{\frac{1}{4-N}}=e^{\frac{1}{4-N}\phi}\sqrt{|g|}\,.$$

The U-gravity scalar curvature reduces upon the section, Σ_{N-1}, to

$$S|_{\Sigma_{N-1}}=2e^{-\phi}\left[R_g-\tfrac{(N-3)(3N-8)}{4(N-4)^2}\partial_\alpha\phi\partial^\alpha\phi+\tfrac{N-2}{N-4}\Delta\phi+\tfrac{1}{2}e^{-\phi}\left(\bigtriangledown_\alpha v^\alpha\right)^2\right].$$

The vector field can be dualized to an $(N-2)$-form potential.

2. Reduction to Σ_3, employing 'dual' upside-down notations,

$$M^{ab}=\left(\begin{array}{cc}\frac{\tilde{g}^{\mu\nu}}{\sqrt{|\tilde{g}|}} & -\tilde{v}^{j\mu} \\ -\tilde{v}^{i\nu} & \sqrt{|\tilde{g}|}(e^{-\tilde{\phi}}\tilde{\mathcal{M}}^{ij}+\tilde{v}^{i\lambda}\tilde{v}^{j}{}_{\lambda})\end{array}\right), \qquad |M|^{\frac{1}{4-N}}=e^{\frac{N-3}{4-N}\tilde{\phi}}\sqrt{|\tilde{g}|}\,.$$

The U-gravity scalar curvature reduces upon the section, Σ_3, to

$$S|_{\Sigma_3}=-2R_{\tilde{g}}+\tfrac{(N-3)(3N-8)}{2(N-4)^2}\tilde{\partial}^\mu\tilde{\phi}\tilde{\partial}_\mu\tilde{\phi}-\tfrac{4(N-3)}{N-4}\tilde{\Delta}\tilde{\phi}-\tfrac{1}{2}\tilde{\partial}^\mu\tilde{\mathcal{M}}_{ij}\tilde{\partial}_\mu\tilde{\mathcal{M}}^{ij}+e^{\tilde{\phi}}\tilde{\mathcal{M}}_{ij}\tilde{\bigtriangledown}^\mu\tilde{v}^i{}_\mu\tilde{\bigtriangledown}^\nu\tilde{v}^j{}_\nu\,,$$

which manifests $\mathbf{SL}(N-3)$ S-duality.

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- *Non-Riemannian backgrounds.*

When the upper left $(N-1)\times(N-1)$ block of the U-metric is degenerate – where $\frac{g_{\alpha\beta}}{\sqrt{|g|}}$ might have been positioned – the Riemannian metric ceases to exist upon Σ_{N-1}. Nevertheless, $\mathbf{SL}(N)$ U-gravity has no problem with describing such a non-Riemannian background, as long as the whole $N\times N$ U-metric is non-degenerate.

Similarly upon Σ_3, U-gravity may allow the upper left 3×3 block of the inverse of the U-metric to be degenerate.

Conclusion

Summary

- Riemannian geometry is for *particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- Novel differential geometic ingredients:
  - ▷ Spacetime being extended-yet-gauged (section condition)
  - ▷ Semi-covariant derivative and semi-covariant curvature
  - ▷ Complete covariantizations of them through 'projection'.

- $\mathcal{N}=2$ $D=10$ SDFT has been constructed to the full order in fermions. The theory unifies IIA and IIB SUGRAs, and allows non-Riemannian 'metric-less' backgrounds.

- Precisely parallel formulation for $\mathbf{SL}(N)$ U-duality under the name, **U-gravity.**

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Outlook

- Further study and classification of the non-Riemannian, 'metric-less' backgrounds.
- Quantization of the string action on doubled-yet-gauged spacetime.
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