Stringy Differential Geometry and Double Field Theory

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Prologue
In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.

- Diffeomorphism: $\partial_\mu \rightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$

- $\nabla_\lambda g_{\mu\nu} = 0, \quad \Gamma^\lambda_{[\mu\nu]} = 0 \quad \rightarrow \quad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

- Curvature: $[\nabla_\mu, \nabla_\nu] \rightarrow R_{\kappa\lambda\mu\nu} \rightarrow \mathcal{R}$

On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ on an equal footing, as they, or NS-NS sector, form a multiplet of T-duality.

This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.

Basically, Riemannian geometry is for Particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector.
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My talk today aims to introduce such a **Stringy Geometry** which is defined in **doubled-yet-gauged** spacetime.

In four-dimensional spacetime photon has two physical degrees of freedom, but can be best described by a four component vector.

Similarly, $D$-dimensional spacetime may be better understood in terms of **doubled-yet-gauged** $(D+D)$ coordinates.
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Talk based on works with Imtak Jeon & Kanghoon Lee

- Differential geometry with a projection: Application to double field theory
  arXiv:1011.1324 JHEP

- Double field formulation of Yang-Mills theory
  arXiv:1102.0419 PLB

- Stringy differential geometry, beyond Riemann
  arXiv:1105.6294 PRD

- Incorporation of fermions into double field theory
  arXiv:1109.2035 JHEP

- Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity

- Ramond-Ramond Cohomology and O(D,D) T-duality
  arXiv:1206.3478 JHEP

- Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory
  arXiv:1210.5078 PLB

- Comments on double field theory and diffeomorphisms
  arXiv:1304.5946 JHEP

- Covariant action for a string in doubled yet gauged spacetime
  arXiv:1307.8377 NPB
Parallel works on U-duality


- **M-theory and F-theory from a Duality Manifest Action**
  with Chris Blair and Emanuel Malek arXiv:1311.5109 JHEP

With a “generalized metric” Duff and a redefined dilaton:

\[
\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \quad e^{-2d} = \sqrt{-g}e^{-2\phi}
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DFT Lagrangian constructed by Hull & Zwiebach (Hohm) reads

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L_{\text{DFT}} = e^{-2d} \left[ \mathcal{H}^{AB} \left( 4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \partial_A \mathcal{H}^{CD} \partial_C \mathcal{H}_{BD} \right) + 4\partial_A \mathcal{H}^{AB} \partial_B d - \partial_A \partial_B \mathcal{H}^{AB} \right]
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Spacetime is formally doubled, \( y^A = (\tilde{x}_\mu, x^\nu) \), \( A = 1, 2, \ldots, D+D \).

T-duality is manifestly realized as usual \( O(D, D) \) rotations Tseytlin, Siegel

\[
\mathcal{H}_{AB} \rightarrow M_A^C M_B^D \mathcal{H}_{CD}, \quad d \rightarrow d, \quad M \in O(D, D).
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Yet, DFT (for NS-NS sector) is a \( D \)-dimensional theory written in terms of \( (D + D) \)-dimensional language, i.e. tensors.

All the fields must live on a \( D \)-dimensional null hyperplane or ‘section’, subject to

\[
\partial_A \partial^A = 2 \frac{\partial^2}{\partial x^\mu \partial \tilde{x}_\mu} \equiv 0 : \text{section condition}
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Double Field Theory by Hull & Zwiebach (Hohm), c.f. Siegel

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Up to O(D, D) rotation, we may fix the section, or choose to set

\[ \frac{\partial}{\partial \tilde{x}_\mu} \equiv 0. \]

Then DFT reduces to the well-known effective action within ‘Riemannian’ setup:

\[ L_{\text{DFT}} \rightarrow L_{\text{eff.}} = \sqrt{-g}e^{-2\phi} \left( R_g + 4(\partial \phi)^2 - \frac{1}{12} H^2 \right). \]

where the diffeomorphism and the B-field gauge symmetry are ‘tamed’ under our control,

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On the other hand, in the above formulation of DFT, the diffeomorphism and the B-field gauge symmetry are rather unclear, while O(D, D) T-duality is manifest.

The above expression may be analogous to the case of writing the Riemannian scalar curvature, \( R \), in terms of the metric and its derivative.

It is desirable to explore the underlying differential geometry, beyond Riemann.
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It is desirable to explore the underlying differential geometry, beyond Riemann.
With a “generalized metric” Duff and a redefined dilaton:

\[ H_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1} \end{pmatrix}, \quad e^{-2d} = \sqrt{-g}e^{-2\phi} \]

DFT Lagrangian constructed by Hull & Zwiebach (Hohm) reads

\[ L_{DFT} = e^{-2d} \left[ H^{AB} \left( 4\partial_A \partial_B d - 4\partial_A d \partial_B d + \frac{1}{8} \partial_A H^{CD} \partial_B H_{CD} - \frac{1}{2} \partial_A H^{CD} \partial_C H_{BD} \right) + 4\partial_A H^{AB} \partial_B d - \partial_A \partial_B H^{AB} \right] \]

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The above expression may be analogous to the case of writing the Riemannian scalar curvature, $R$, in terms of the metric and its derivative.

It is desirable to explore the underlying differential geometry, beyond Riemann.
In the remaining of this talk, I will try to explain our proposal for

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- Key concepts include
  - Projector
  - Semi-covariant derivative
  - Semi-covariant curvature
  - And their complete covariantization via ‘projection’
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**the Stringy Differential Geometry of DFT**

- **Key concepts include**
  - Projector
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  - Semi-covariant curvature
  - And their complete covariantization via ‘projection’

*c.f. Alternative approaches: Berman-Blair-Malek-Perry, Cederwall, Geissbuhler, Marques et al.*
Geometric Constitution of Double Field Theory
**Notation**

Capital Latin alphabet letters denote the $O(D, D)$ vector indices, i.e. $A, B, C, \cdots = 1, 2, \cdots, D+D$, which can be freely raised or lowered by the $O(D, D)$ invariant constant metric,

\[ J_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]
Doubled-yet-gauged spacetime

The spacetime is formally doubled, being \((D+D)\)-dimensional. However, **the doubled spacetime is gauged**: the coordinate space is equipped with an equivalence relation,

\[
x^A \sim x^A + \phi \partial^A \varphi,
\]

which we call ‘coordinate gauge symmetry’.

Note that \(\phi\) and \(\varphi\) are arbitrary functions in DFT.
Doubled-yet-gauged spacetime

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which we call ‘coordinate gauge symmetry’.

Note that \(\phi\) and \(\varphi\) are arbitrary functions in DFT.

Each equivalence class, or gauge orbit, represents a single physical point.

Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.
**Realization of the coordinate gauge symmetry.**

The equivalence relation is realized in DFT by enforcing that, arbitrary functions and their arbitrary derivatives, denoted here collectively by $\Phi$, are invariant under the coordinate gauge symmetry shift,

$$\Phi(x + \Delta) = \Phi(x), \quad \Delta^A = \phi \partial^A \phi.$$
**Section condition.**

The invariance under the coordinate gauge symmetry can be shown to be equivalent to the section condition,

\[ \partial_A \partial^A \equiv 0. \]
The invariance under the coordinate gauge symmetry can be shown to be equivalent to the section condition, 
\[ \partial_A \partial^A \equiv 0. \]

Explicitly, acting on arbitrary functions, \( \Phi \), \( \Phi' \), and their products, we have
\[ \partial_A \partial^A \Phi = 0 \quad \text{(weak constraint)}, \]
\[ \partial_A \Phi \partial^A \Phi' = 0 \quad \text{(strong constraint)}. \]
Diffeomorphism.

Diffeomorphism symmetry in $O(D, D)$ DFT is generated by a generalized Lie derivative

\[ \mathcal{L}_X T_{A_1 \ldots A_n} := X^B \partial_B T_{A_1 \ldots A_n} + \omega_T \partial_B X^B T_{A_1 \ldots A_n} + \sum_{i=1}^{n} (\partial_A X^B - \partial_B X^A) T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n}, \]

where $\omega_T$ denotes the weight.
Diffeomorphism symmetry in $O(D, D)$ DFT is generated by a generalized Lie derivative.

Siegel, Courant, Grana

$$\hat{L}_X T_{A_1 \cdots A_n} := X^B \partial_B T_{A_1 \cdots A_n} + \omega_T \partial_B X^B T_{A_1 \cdots A_n} + \sum_{i=1}^{n} (\partial_{A_i} X_B - \partial_B X_{A_i}) T_{A_1 \cdots A_{i-1}} B A_{i+1} \cdots A_n,$$

where $\omega_T$ denotes the weight.

In particular, the generalized Lie derivative of the $O(D, D)$ invariant metric is trivial,

$$\hat{L}_X J_{AB} = 0.$$

The commutator is closed by C-bracket Hull-Zwiebach

$$[\hat{L}_X, \hat{L}_Y] = \hat{L}_{[X, Y]_C}, \quad [X, Y]_C^A = X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B.$$
Dilaton and a pair of two-index projectors.

The geometric objects in DFT consist of a dilation, \( d \), and a pair of symmetric projection operators,

\[
P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_A^B P_B^C = P_A^C, \quad \bar{P}_A^B \bar{P}_B^C = \bar{P}_A^C.
\]

Further, the projectors are orthogonal and complementary,

\[
P_A^B \bar{P}_B^C = 0, \quad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}.
\]
Dilaton and a pair of two-index projectors.

The geometric objects in DFT consist of a dilation, $d$, and a pair of symmetric projection operators,

$$ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_{A}^{B} P_{B}^{C} = P_{A}^{C}, \quad \bar{P}_{A}^{B} \bar{P}_{B}^{C} = \bar{P}_{A}^{C}. $$

Further, the projectors are orthogonal and complementary,

$$ P_{A}^{B} \bar{P}_{B}^{C} = 0, \quad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}. $$

Remark: The difference of the two projectors, $P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}$, corresponds to the “generalized metric” which can be also independently defined as a symmetric $O(D, D)$ element, i.e. $\mathcal{H}_{AB} = \mathcal{H}_{BA}, \mathcal{H}_{A}^{B} \mathcal{H}_{B}^{C} = \delta_{A}^{C}$. However, in supersymmetric double field theories it appears that the projectors are more fundamental than the “generalized metric”.
While the projectors are weightless, the dilation gives rise to the $O(D, D)$ invariant integral measure with weight one, after exponentiation,
\[ e^{-2d} . \]
Semi-covariant derivative and semi-covariant Riemann curvature.

We define a semi-covariant derivative,

$$\nabla_C T_{A_1 A_2 \ldots A_n} := \partial_C T_{A_1 A_2 \ldots A_n} - \omega^T_G B T_{A_1 A_2 \ldots A_n} + \sum_{i=1}^n \Gamma^B_C A_i B T_{A_1 \ldots A_i-1 A_i+1 \ldots A_n},$$

and


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\[ \nabla C T_{A_1 A_2 \cdots A_n} := \partial C T_{A_1 A_2 \cdots A_n} - \omega T \Gamma^B_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma^B_{CA_i} T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} , \]

and a semi-covariant Riemann curvature,

\[ S_{ABCD} := \frac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma^{E}_{AB} \Gamma_{ECD} \right) . \]

Here \( R_{ABCD} \) denotes the ordinary “field strength” of a connection,

\[ R_{CDAB} = \partial A \Gamma_{BCD} - \partial B \Gamma_{ACD} + \Gamma_{AC}^{E} \Gamma_{BED} - \Gamma_{BC}^{E} \Gamma_{AED} . \]
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and a semi-covariant “Riemann” curvature,

\[ S_{ABCD} := \frac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma^E_{AB} \Gamma_{ECD} \right). \]

Here \( R_{ABCD} \) denotes the ordinary “field strength” of a connection,

\[ R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}. \]

As I will explain shortly, we may determine the (torsionless) connection:

\[ \Gamma_{CAB} = 2 \left( P \partial_C P \bar{P} \right)_{[AB]} + 2 \left( \bar{P}_A^D P_B^E - P_A^D P_B^E \right) \partial_D P_{EC} \]

\[ - \frac{4}{D-1} \left( \bar{P}_{C[A} \bar{P}_B^D + P_{C[A} P_B^D \right) \left( \partial_D d + \left( P \partial^E P \bar{P} \right)_{[ED]} \right), \]

which is the DFT generalization of the Christoffel connection.
The semi-covariant derivative then obeys the Leibniz rule and annihilates the $O(D, D)$ invariant constant metric,

$$\nabla_A \mathcal{J}_{BC} = 0.$$
The semi-covariant derivative then obeys the Leibniz rule and annihilates the $O(D, D)$ invariant constant metric,

$$\nabla_A J_{BC} = 0.$$ 

A crucial defining property of the semi-covariant “Riemann” curvature is that, under arbitrary transformation of the connection, it transforms as total derivative,

$$\delta S_{ABCD} = \nabla_{[A} \delta \Gamma_{B]CD} + \nabla_{[C} \delta \Gamma_{D]AB}.$$
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A crucial defining property of the semi-covariant “Riemann” curvature is that, under arbitrary transformation of the connection, it transforms as total derivative,

$$\delta S_{ABCD} = \nabla_{[A} \delta \Gamma_{B]} CD + \nabla_{[C} \delta \Gamma_{D]} AB.$$  

Further, the semi-covariant “Riemann” curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature,

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB}, \quad S_{[ABC]D} = 0,$$

as well as additional identities concerning the projectors,

$$P_I^A P_J^B \bar{P}_K^C \bar{P}_L^D S_{ABCD} = 0, \quad P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} = 0.$$  

It follows that

$$S^{AB}_{AB} = 0.$$
**The uniqueness of the torsionless connection.**

The connection is the unique solution to the following five constraints:

\[
\begin{align*}
\nabla_A P_{BC} &= 0, & \nabla_A \bar{P}_{BC} &= 0, \\
\nabla_A d &= -\frac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \frac{1}{2} \Gamma^B_{BA} = 0, \\
\Gamma_{ABC} + \Gamma_{ACB} &= 0, \\
\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} &= 0, \\
\mathcal{P}^{ABC}_{DEF} \Gamma_{DEF} &= 0, & \bar{\mathcal{P}}^{ABC}_{DEF} \Gamma_{DEF} &= 0.
\end{align*}
\]
The uniqueness of the torsionless connection.

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\[ \Gamma_{ABC} + \Gamma_{ACB} = 0, \]
\[ \Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} = 0, \]
\[ P_{ABC}^{DEF} \Gamma_{DEF} = 0, \quad \bar{P}_{ABC}^{DEF} \Gamma_{DEF} = 0. \]

The first two relations are the compatibility conditions with all the geometric objects, or NS-NS sector, in DFT.

The third constraint is the compatibility condition with the \( \mathcal{O}(D,D) \) invariant constant metric, \( i.e. \nabla_A \mathcal{J}_{BC} = 0. \)
**The uniqueness of the torsionless connection.**

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\mathcal{P}_{ABC}^{DEF} \Gamma_{DEF} = 0, \quad \bar{\mathcal{P}}_{ABC}^{DEF} \Gamma_{DEF} = 0.
\]

The next cyclic property makes the semi-covariant derivative compatible with the generalized Lie derivative as well as with the C-bracket,

\[
\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla), \quad [X, Y]_C(\partial) = [X, Y]_C(\nabla).
\]

The last formulae are projection conditions which we impose intentionally in order to ensure the uniqueness.
The uniqueness of the torsionless connection.

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\]

The last formulae are projection conditions which we impose intentionally in order to ensure the uniqueness.
Six-index projection operators.

The six-index projection operators are explicitly,

\[
P_{CABDEF} := P_C^D P_{[A}^{[E} P_{B]}^{F]} + \frac{2}{D-1} P_{[A} P_{B]}^{[E} P^{F]} D^, \]

\[
\bar{P}_{CABDEF} := \bar{P}_C^D \bar{P}_{[A}^{[E} \bar{P}_{B]}^{F]} + \frac{2}{D-1} \bar{P}_{[A} \bar{P}_{B]}^{[E} \bar{P}^{F]} D^, \]

which satisfy the ‘projection’ properties,

\[
P_{ABC}^{D} \bar{P}_{DEF}^{GHI} = P_{ABC}^{GHI}, \quad \bar{P}_{ABC}^{D} \bar{P}_{DEF}^{GHI} = \bar{P}_{ABC}^{GHI}. \]

Further, they are symmetric and traceless,

\[
P_{ABCDEF} = P_{DEFABC}, \quad P_{ABCDEF} = P_{A[BC]D[EF]}, \quad P^{AB} P_{ABCDEF} = 0, \]

\[
\bar{P}_{ABCDEF} = \bar{P}_{DEFABC}, \quad \bar{P}_{ABCDEF} = \bar{P}_{A[BC]D[EF]}, \quad \bar{P}^{AB} \bar{P}_{ABCDEF} = 0. \]
Crucially, the projection operator dictates the anomalous terms in the diffeomorphic transformations of the semi-covariant derivative and the semi-covariant Riemann curvature,

\[
(\delta_X - \hat{L}_X) \nabla_C T_{A_1 \ldots A_n} = \sum_{i=1}^{n} 2(\mathcal{P} + \mathcal{P})_{CA_i}^{BDEF} \partial_D \partial_E X_F T_{A_1 \ldots A_{i-1} B A_{i+1} \ldots A_n},
\]

\[
(\delta_X - \hat{L}_X) S_{ABCD} = 2\nabla_{[A} \left( (\mathcal{P} + \mathcal{P})_{B][CD}^{EFG} \partial_E \partial_F X_G \right) + 2\nabla_{[C} \left( (\mathcal{P} + \mathcal{P})_{D][AB}^{EFG} \partial_E \partial_F X_G \right).
\]
**Complete covariantizations.**

Both the semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized, through appropriate contractions with the projectors:

\[
P_C^D \bar{P}_{A_1}^{B_1} \cdots \bar{P}_{A_n}^{B_n} \nabla^D T_{B_1 \cdots B_n}, \quad \bar{P}_C^D P_{A_1}^{B_1} \cdots P_{A_n}^{B_n} \nabla^D T_{B_1 \cdots B_n},
\]

\[
P^{AB} \bar{P}_{C_1}^{D_1} \cdots \bar{P}_{C_n}^{D_n} \nabla^A T_{B D_1 \cdots D_n}, \quad \bar{P}^{AB} P_{C_1}^{D_1} \cdots P_{C_n}^{D_n} \nabla^A T_{B D_1 \cdots D_n} \quad \text{(divergences)},
\]

\[
P^{AB} \bar{P}_{C_1}^{D_1} \cdots \bar{P}_{C_n}^{D_n} \nabla^A \nabla^B T_{D_1 \cdots D_n}, \quad \bar{P}^{AB} P_{C_1}^{D_1} \cdots P_{C_n}^{D_n} \nabla^A \nabla^B T_{D_1 \cdots D_n} \quad \text{(Laplacians)},
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and
**Complete covariantizations.**

Both the semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized, through appropriate contractions with the projectors:

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\]

\[
P^{AB} \tilde{P}_{C_1}^{D_1} \ldots \tilde{P}_{C_n}^{D_n} \nabla_A T_{B D_1} \ldots D_n,
\]

\[
P^{AB} \tilde{P}_{C_1}^{D_1} \ldots \tilde{P}_{C_n}^{D_n} \nabla_A \nabla_B T_{D_1} \ldots D_n,
\]

and

\[
P_A^C \tilde{P}_B^D S_{C E D}^E
\]

\[
(P^{A C} P^{B D} - \tilde{P}^{A C} \tilde{P}^{B D}) S_{A B C D}
\]

(“Ricci” curvature),

(divergences),

(Laplacians),

and

(scalar curvature).
**Action.**

The action of $O(D, D)$ DFT is given by the fully covariant scalar curvature,

$$
\int_{\Sigma_D} e^{-2d} (P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD},
$$

where the integral is taken over a section, $\Sigma_D$. 
**Action.**

The action of $O(D, D)$ DFT is given by the fully covariant scalar curvature,

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where the integral is taken over a section, $\Sigma_D$.

The dilation and the projector equations of motion correspond to the vanishing of the scalar curvature and the “Ricci” curvature respectively.
Action.

The action of $O(D, D)$ DFT is given by the fully covariant scalar curvature,

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$$

where the integral is taken over a section, $\Sigma_D$.

The dilation and the projector equations of motion correspond to the vanishing of the scalar curvature and the "Ricci" curvature respectively.

Note: It is precisely the above expression that allows the ‘1.5 formalism’ to work in the full order supersymmetric extensions of $\mathcal{N} = 1, 2, D = 10$ Jeon-Lee-JHP
**Section.**

Up to $O(D,D)$ duality rotations, the solution to the section condition is unique. It is a $D$-dimensional section, $\Sigma_D$, characterized by the independence of the dual coordinates, i.e.

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0,$$

while the whole doubled coordinates are given by

$$x^A = (\tilde{x}_\mu, x^\nu),$$

where $\mu, \nu$ are now $D$-dimensional indices.
Riemannian reduction.

To perform the Riemannian reduction to the $D$-dimensional section, $\Sigma_D$, we parametrize the dilation and the projectors in terms of $D$-dimensional Riemannian metric, $g_{\mu\nu}$, ordinary dilaton, $\phi$, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

$$ P_{AB} - \bar{P}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \quad e^{-2d} = \sqrt{|g|}e^{-2\phi}. $$
**Riemannian reduction.**

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The DFT scalar curvature then reduces upon the section to

$$ (P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD} \bigg|_{\Sigma_D} = R_g + 4\Delta \phi - 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda \mu \nu} H^{\lambda \mu \nu}, $$

where as usual, $H_{\lambda \mu \nu} = 3 \partial_{[\lambda} B_{\mu \nu]}$. 

Jeong-Hyuck Park

Stringy Differential Geometry and Supersymmetric Double Field Theory
**Riemannian reduction.**

To perform the Riemannian reduction to the $D$-dimensional section, $\Sigma_D$, we parametrize the dilation and the projectors in terms of $D$-dimensional Riemannian metric, $g_{\mu\nu}$, ordinary dilaton, $\phi$, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

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where as usual, $H_{\lambda\mu\nu} = 3\partial_{[\lambda} B_{\mu\nu]}$.

DFT-diffeomorphism $\Rightarrow$ $D$-dimensional diffeomorphism plus $B$-field gauge symmetry.
**Riemannian reduction.**

To perform the Riemannian reduction to the $D$-dimensional section, $\Sigma_D$, we parametrize the dilation and the projectors in terms of $D$-dimensional Riemannian metric, $g_{\mu\nu}$, ordinary dilaton, $\phi$, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

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The DFT scalar curvature then reduces upon the section to

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where as usual, $H_{\lambda\mu\nu} = 3\partial_{[\lambda} B_{\mu\nu]}$.

DFT-diffeomorphim $\Rightarrow$ $D$-dimensional diffeomorphism plus $B$-field gauge symmetry.

Up to field redefinitions, the above is the most general parametrization of the "generalized metric", $\mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}$, when its upper left $D \times D$ block is non-degenerate.
Non-Riemannian backgrounds.

When the upper left $D \times D$ block of $\mathcal{H}_{AB} = (P - \bar{P})_{AB}$ is degenerate – where $g^{-1}$ might be positioned – the Riemannian metric ceases to exist upon the section, $\Sigma_D$.

Nevertheless, DFT and a doubled sigma model – which I will discuss later – have no problem with describing such a non-Riemannian background.

An extreme example of such a non-Riemannian background is the flat background where

$$\mathcal{H}_{AB} = (P - \bar{P})_{AB} = \mathcal{J}_{AB}.$$

This is a vacuum solution to the bosonic $O(D, D)$ DFT and the corresponding doubled sigma model reduces to a certain ‘chiral’ sigma model.
**Non-Riemannian backgrounds.**

When the upper left $D \times D$ block of $\mathcal{H}_{AB} = (P - \bar{P})_{AB}$ is degenerate – where $g^{-1}$ might be positioned – the Riemannian metric ceases to exist upon the section, $\Sigma_D$.

Nevertheless, DFT and a doubled sigma model – which I will discuss later – have no problem with describing such a non-Riemannian background.

An extreme example of such a non-Riemannian background is the flat background where

$$\mathcal{H}_{AB} = (P - \bar{P})_{AB} = \mathcal{J}_{AB}. $$

This is a vacuum solution to the bosonic DFT and the corresponding doubled sigma model reduces to a certain ‘chiral’ sigma model.

Allowing non-Riemannian backgrounds, DFT is NOT a mere reformulation of SUGRA. It describes a new class of string theory backgrounds.  

*c.f.* Gomis-Ooguri
SUSY
Based on the differential geometry I just described,

incorporating fermions and the R-R sector (i.e. vielbein formalism),

it is possible to construct the maximally supersymmetric double field theory

to the full order (i.e. quartic order) in fermions.
$\mathcal{N} = 2 \ D = 10$ Supersymmetric Double Field Theory
Symmetries of $\mathcal{N} = 2 \ D = 10$ SDFT

- **O($D, D$) T-duality**

- **Gauge symmetries**
  1. DFT-diffeomorphism (generalized Lie derivative)
  2. A *pair* of local Lorentz symmetries, $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$
  3. *Local* $\mathcal{N} = 2$ SUSY with 32 supercharges.

- All the bosonic symmetries will be realized manifestly and simultaneously.
- The theory is chiral with respect to both Local Lorentz groups.
- Consequently, there is no distinction of IIA and IIB $\implies$ Unification of IIA and IIB
- While the theory is unique, it contains type IIA and IIB SUGRA backgrounds as different kind of solutions.
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For this, it is crucial to have the right field variables.

We shall postulate $O(D, D)$ covariant genuine DFT-field-variables, and NOT employ Riemannian variables such as metric, $B$-field, R-R $p$-forms.
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Field contents of $\mathcal{N} = 2$ $D = 10$ SDFT

**Bosons**

- NS-NS sector
  - DFT-dilaton: $d$
  - DFT-vielbeins: $V_{Ap}$, $\bar{V}_{A\bar{p}}$
- R-R potential: $C^{\alpha \bar{\alpha}}$

**Fermions**

- DFT-dilatinos: $\rho^\alpha$, $\rho'^{\bar{\alpha}}$
- Gravitinos: $\psi_\alpha^p$, $\psi'^{\bar{\alpha}}_p$
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Jeong-Hyuck Park

Stringy Differential Geometry and Supersymmetric Double Field Theory
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  - **NS-NS sector**
    - DFT-dilaton: $d$
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  - **R-R potential:** $C^{\alpha\bar{\alpha}}$

- **Fermions**
  - DFT-dilatinos: $\rho^{\alpha}$, $\rho'^{\bar{\alpha}}$
  - Gravitinos: $\psi_{\alpha}^{\bar{p}}$, $\psi'^{\bar{\alpha}}_{\bar{p}}$

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  - DFT-dilatinos: $\rho^{\alpha}, \rho^{\bar{\alpha}}$
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R-R potential and Fermions carry NOT $(D + D)$-dimensional
BUT undoubled $D$-dimensional indices.
Field contents of $\mathcal{N} = 2 \ D = 10$ SDFT

- **Bosons**
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*A priori*, $O(D, D)$ rotates only the $O(D, D)$ vector indices (capital Roman), and the R-R sector and all the fermions are $O(D, D)$ T-duality singlet.

The usual IIA $\Leftrightarrow$ IIB exchange will follow only after fixing a gauge.
Field contents of $\mathcal{N} = 2 \ D = 10$ SDFT

- **Bosons**
  - NS-NS sector
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  - DFT-dilatinos: $\rho^{\alpha}$, $\rho^{\prime\bar{\alpha}}$
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All the fields are required to satisfy the section condition,

$$\partial_{A}\partial^{A} \equiv 0.$$
The DFT-dilaton gives rise to a scalar density with weight one,
\[ e^{-2d}. \]

The DFT-vielbeins satisfy the four defining properties:
\[
V_A p V^A q = \eta_{pq}, \quad \bar{V}_{\bar{A}} \bar{V}^A q = \bar{\eta}_{\bar{p} \bar{q}}, \quad V_A \bar{V}^A q = 0, \quad V_A V_B p + \bar{V}_{\bar{A}} \bar{V}_{\bar{B}} \bar{p} = J_{AB}.
\]

For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,
\[
\gamma^{(D+1)} \psi_{\bar{p}} = c \psi_{\bar{p}}, \quad \gamma^{(D+1)} \rho = -c \rho,
\]
\[
\bar{\gamma}^{(D+1)} \psi'_{p} = c' \psi'_{p}, \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho',
\]
where \( c \) and \( c' \) are arbitrary independent two sign factors, \( c^2 = c'^2 = 1 \).

Lastly for the R-R sector, we set the R-R potential, \( C^{\alpha}_{\bar{\alpha}} \), to be in the bi-fundamental spinorial representation of \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \). It possesses the chirality,
\[
\gamma^{(D+1)} \bar{C} \bar{\gamma}^{(D+1)} = cc' C.
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Spin$(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ chiralities:

\[ \gamma^{(D+1)} \psi p = c \psi p, \quad \gamma^{(D+1)} \rho = -c \rho, \]
\[ \bar{\gamma}^{(D+1)} \psi' p = c' \psi' p, \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho', \]
\[ \gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C. \]

A priori all the possible four different sign choices are equivalent up to Pin$(1, D-1)_L \times \text{Pin}(D-1, 1)_R$ rotations.

That is to say, $\mathcal{N} = 2$ $D = 10$ SDFT is chiral with respect to both Pin$(1, D-1)_L$ and Pin$(D-1, 1)_R$, and the theory is unique, unlike IIA/IIB SUGRAs.

Hence, without loss of generality, we may safely set $c \equiv c' \equiv +1$.

Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
\textbf{Spin}(1, D-1)_L \times \textbf{Spin}(D-1, 1)_R \text{ chiralities:}

\begin{align*}
\gamma^{(D+1)} \psi^p &= c \psi^p, & \gamma^{(D+1)} \rho &= -c \rho, \\
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\gamma^{(D+1)} C \bar{\gamma}^{(D+1)} &= cc'.
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\textbf{Spin}(1, D−1)_{L} \times \textbf{Spin}(D−1, 1)_{R} \textit{chiralities:}

\[ \gamma^{(D+1)} \psi_{\bar{p}} = c \psi_{\bar{p}} , \quad \gamma^{(D+1)} \rho = -c \rho , \]
\[ \bar{\gamma}^{(D+1)} \psi'_{\bar{p}} = c' \psi'_{\bar{p}} , \quad \bar{\gamma}^{(D+1)} \rho' = -c' \rho' , \]
\[ \gamma^{(D+1)} C \bar{\gamma}^{(D+1)} = cc' C . \]

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Later we shall see that while the theory is unique, it contains type IIA and IIB supergravity backgrounds as different kind of solutions.
The DFT-vielbeins generate **a pair of two-index projectors**,\[
\begin{align*}
P_{AB} &:= V_A^p V_B^p, \\
P_A^B P_B^C &= P_A^C, \\
\bar{P}_{AB} &:= \bar{V}_{A}^{\bar{p}} \bar{V}_{B}^{\bar{p}}, \\
\bar{P}_A^B \bar{P}_B^C &= \bar{P}_A^C,
\end{align*}
\]
which are symmetric, orthogonal and complementary to each other,
\[
\begin{align*}
P_{AB} &= P_{BA}, \\
\bar{P}_{AB} &= \bar{P}_{BA}, \\
P_A^B \bar{P}_B^C &= 0, \\
P_A^B + \bar{P}_A^B &= \delta_A^B.
\end{align*}
\]

It follows
\[
\begin{align*}
P_A^B V_{Bp} &= V_{Ap}, \\
\bar{P}_A^B \bar{V}_{B\bar{p}} &= \bar{V}_{A\bar{p}}, \\
\bar{P}_A^B V_{Bp} &= 0, \\
P_A^B \bar{V}_{B\bar{p}} &= 0.
\end{align*}
\]

Note also
\[
\mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}.
\]

However, our emphasis lies on the ‘projectors’ rather than the “generalized metric".
The DFT-vielbeins generate a pair of two-index projectors,

\[ P_{AB} := V_A^p V_{Bp}, \quad P_A^B P_B^C = P_A^C, \quad \bar{P}_{AB} := \bar{V}_A^{\bar{p}} \bar{V}_{B\bar{p}}, \quad \bar{P}_A^B \bar{P}_B^C = \bar{P}_A^C, \]

which are symmetric, orthogonal and complementary to each other,

\[ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_A^B \bar{P}_B^C = 0, \quad P_A^B + \bar{P}_A^B = \delta_A^B. \]

It follows

\[ P_A^B V_{Bp} = V_A^p, \quad \bar{P}_A^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}, \quad \bar{P}_A^B V_{Bp} = 0, \quad P_A^B \bar{V}_{B\bar{p}} = 0. \]

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However, our emphasis lies on the ‘projectors’ rather than the “generalized metric".
The DFT-vielbeins generate a pair of two-index projectors,

\[ P_{AB} := V^a_A V^p_B, \quad P^A_B P^C_B = P^C_A, \quad \bar{P}_{AB} := \bar{V}^{\bar{a}}_A \bar{V}^{\bar{p}}_B, \quad \bar{P}^A_B \bar{P}^C_B = \bar{P}^C_A, \]

which are symmetric, orthogonal and complementary to each other,

\[ P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P^A_B \bar{P}^C_B = 0, \quad P^A_B + \bar{P}^A_B = \delta^A_B. \]

It follows

\[ P^A_B V^p_B = V^a_A, \quad \bar{P}^A_B \bar{V}^{\bar{p}}_B = \bar{V}^{\bar{a}}_A, \quad \bar{P}^A_B V^p_B = 0, \quad P^A_B \bar{V}^{\bar{p}}_B = 0. \]

Note also

\[ \mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}. \]

However, our emphasis lies on the ‘projectors’ rather than the "generalized metric".
Surely we also get the six-index projectors

\[ \mathcal{P}_{CABDEF} := P_{C}^{D}P_{[A}^{[E}P_{B]^{F]} + \frac{2}{D-1} P_{C[A}P_{B]}^{[E}P^{F]}D, \quad \mathcal{P}_{C}^{D}P_{DEF}^{GHI} = \mathcal{P}_{CAB}^{GHI}, \]
\[ \bar{\mathcal{P}}_{CABDEF} := \bar{P}_{C}^{D}\bar{P}_{[A}^{[E}\bar{P}_{B]^{F]} + \frac{2}{D-1} \bar{P}_{C[A}\bar{P}_{B]}^{[E}\bar{P}^{F]}D, \quad \bar{\mathcal{P}}_{C}^{D}\bar{P}_{DEF}^{GHI} = \bar{\mathcal{P}}_{CAB}^{GHI}, \]

which are symmetric and traceless,

\[ \mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}, \quad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}, \]
\[ \mathcal{P}^{A}_{ABDEF} = 0, \quad \mathcal{P}^{AB}\mathcal{P}_{ABCDEF} = 0, \quad \bar{\mathcal{P}}^{A}_{ABDEF} = 0, \quad \bar{\mathcal{P}}^{AB}\bar{\mathcal{P}}_{ABCDEF} = 0, \]

and play crucial roles in the construction of the completely covariant derivatives and curvatures.
Having all the ‘right’ field-variables prepared, we now discuss their derivatives or ‘semi-covariant derivatives’.
Semi-covariant derivatives

- For each gauge symmetry we assign a corresponding connection,
  - $\Gamma_A$ for the DFT-diffeomorphism (generalized Lie derivative),
  - $\Phi_A$ for the ‘unbarred’ local Lorentz symmetry, $\text{Spin}(1, D-1)_L$,
  - $\bar{\Phi}_A$ for the ‘barred’ local Lorentz symmetry, $\text{Spin}(D-1, 1)_R$.

- Combining all of them, we introduce master ‘semi-covariant’ derivative

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A.$$
Semi-covariant derivatives

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Combining all of them, we introduce master ‘semi-covariant’ derivative

$$D_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A.$$
It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A, \quad D_A = \partial_A + \Phi_A + \bar{\Phi}_A.$$ 

The former is the ‘semi-covariant’ derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^n \Gamma_{CA_i}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n}.$$ 

And the latter is the covariant derivative for the $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ local Lorenz symmetries.
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$$\nabla_A = \partial_A + \Gamma_A, \quad D_A = \partial_A + \Phi_A + \bar{\Phi}_A.$$  

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The former is the ‘semi-covariant’ derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

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And the latter is the covariant derivative for the \( \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \) local Lorenz symmetries.
By definition, the master derivative annihilates all the ‘constants’,

\[ \mathcal{D}_A \mathcal{J}_{BC} = \nabla_A \mathcal{J}_{BC} = \Gamma_{AB}^D \mathcal{J}_{DC} + \Gamma_{AC}^D \mathcal{J}_{BD} = 0, \]
\[ \mathcal{D}_A \eta_{pq} = \mathcal{D}_A \eta_{pq} = \Phi_{Ap}^r \eta_{rq} + \Phi_{Aq}^r \eta_{pr} = 0, \]
\[ \mathcal{D}_A \bar{\eta}_{\bar{p}\bar{q}} = \mathcal{D}_A \bar{\eta}_{\bar{p}\bar{q}} = \bar{\Phi}_{A\bar{p}} \bar{r} \eta_{\bar{r}q} + \bar{\Phi}_{A\bar{q}} \bar{r} \eta_{\bar{p}\bar{r}} = 0, \]
\[ \mathcal{D}_A C_{+\alpha\beta} = \mathcal{D}_A C_{+\alpha\beta} = \Phi_{A\alpha} \delta C_{+\delta\beta} + \Phi_{A\beta} \delta C_{+\alpha\delta} = 0, \]
\[ \mathcal{D}_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = \mathcal{D}_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = \bar{\Phi}_{A\bar{\alpha}} \bar{\delta} \bar{C}_{+\bar{\delta}\bar{\beta}} + \bar{\Phi}_{A\bar{\beta}} \bar{\delta} \bar{C}_{+\bar{\alpha}\bar{\delta}} = 0, \]

including the gamma matrices,

\[ \mathcal{D}_A (\gamma^p)^\alpha_\beta = \mathcal{D}_A (\gamma^p)^\alpha_\beta = \Phi_{A\beta} q (\gamma^q)^\alpha_\beta + \Phi_{A\alpha} \delta (\gamma^p)^\delta_\beta - (\gamma^p)^\alpha_\delta \Phi_{A\delta} \beta = 0, \]
\[ \mathcal{D}_A (\bar{\gamma}^\bar{p})^{\bar{\alpha}}_{\bar{\beta}} = \mathcal{D}_A (\bar{\gamma}^\bar{p})^{\bar{\alpha}}_{\bar{\beta}} = \bar{\Phi}_{A\bar{\beta}} \bar{q} (\bar{\gamma}^\bar{q})^{\bar{\alpha}}_{\bar{\beta}} + \bar{\Phi}_{A\bar{\alpha}} \bar{\delta} (\bar{\gamma}^\bar{p})^{\bar{\delta}}_{\bar{\beta}} - (\bar{\gamma}^\bar{p})^{\bar{\alpha}}_{\bar{\delta}} \bar{\Phi}_{A\bar{\delta}} \bar{\beta} = 0. \]
It follows then that the connections are all anti-symmetric,

$$
\Gamma_{ABC} = -\Gamma_{ACB},
$$

$$
\Phi_{Apq} = -\Phi_{Aqp}, \quad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha},
$$

$$
\Phi_{A\bar{p}\bar{q}} = -\Phi_{A\bar{q}\bar{p}}, \quad \Phi_{A\bar{\alpha}\bar{\beta}} = -\Phi_{A\bar{\beta}\bar{\alpha}},
$$

and as usual,

$$
\Phi_A^{\alpha\beta} = \frac{1}{4} \Phi_{A pq} (\gamma^{pq})^{\alpha\beta}, \quad \Phi_{A\bar{\alpha}\bar{\beta}} = \frac{1}{4} \Phi_{A\bar{p}\bar{q}} (\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}\bar{\beta}}.
$$
Further, the master derivative is compatible with the whole NS-NS sector,

\[ \mathcal{D}_A d = \nabla_A d := -\frac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \frac{1}{2} \Gamma^B_{BA} = 0, \]

\[ \mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma^C_{AB} V_{Cp} + \Phi_{Apq} V_{Bq} = 0, \]

\[ \mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma^C_{AB} \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}\bar{q}} \bar{V}_{B\bar{q}} = 0. \]

It follows that

\[ \mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0, \quad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0, \]

and the connections are related to each other,

\[ \Gamma^A_{BC} = V^p_B D_A V_{Cp} + \bar{V}^\bar{p}_B D_A \bar{V}_{C\bar{p}}, \]

\[ \Phi_{Apq} = V^p_B \nabla_A V_{Bq}, \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^\bar{p}_B \nabla_A \bar{V}_{B\bar{q}}. \]
Further, the master derivative is compatible with the whole NS-NS sector,

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\[ \mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma^C_{AB} V_{Cp} + \Phi_{Apq} V_{Bq} = 0, \]

\[ \mathcal{D}_A \tilde{V}_{B\bar{p}} = \partial_A \tilde{V}_{B\bar{p}} + \Gamma^C_{AB} \tilde{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}\bar{q}} \tilde{V}_{B\bar{q}} = 0. \]

It follows that

\[ \mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0, \quad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0, \]

and the connections are related to each other,

\[ \Gamma_{ABC} = V_B^p D_A V_{Cp} + \tilde{V}_{B\bar{p}} D_A \tilde{V}_{C\bar{p}}, \]

\[ \Phi_{Apq} = V^B_{\ p} \nabla_A V_{Bq}, \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \tilde{V}^{B\bar{p}} \nabla_A \tilde{V}_{B\bar{q}}. \]
The connections assume the following most general forms:

\[
\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A^p V_B^q + \tilde{\Delta}_{C\bar{p}q} \bar{V}_A^{\bar{p}} \bar{V}_B^{\bar{q}},
\]

\[
\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq},
\]

\[
\tilde{\Phi}_{A\bar{p}q} = \tilde{\Phi}^0_{A\bar{p}q} + \tilde{\Delta}_{A\bar{p}q}.
\]

Here \( \Gamma^0_{CAB} \) is the torsionless DFT-Christoffel connection which we fixed earlier,

\[
\Gamma^0_{CAB} = 2 \left( P \partial_C P \bar{P} \right)_{[AB]} + 2 \left( \bar{P}_A^D P_B^E - P_A^D P_B^E \right) \partial_D P_{EC} - \frac{4}{D-1} \left( \bar{P}_C[A P_B]^D + P_C[A P_B]^D \right) \left( \partial_D d + \left( P \partial^E P \bar{P} \right)_{[ED]} \right),
\]

and, with the corresponding derivative, \( \nabla^0_A = \partial_A + \Gamma^0_A \),

\[
\Phi^0_{Apq} = V_B^p \nabla_A^0 V_B^q = V_B^p \partial_A V_B^q + \Gamma^0_{ABC} V_B^p V_C^q,
\]

\[
\tilde{\Phi}^0_{A\bar{p}q} = \bar{V}_B^{\bar{p}} \nabla_A^0 \bar{V}_B^{\bar{q}} = \bar{V}_B^{\bar{p}} \partial_A \bar{V}_B^{\bar{q}} + \Gamma^0_{ABC} \bar{V}_B^{\bar{p}} \bar{V}_C^{\bar{q}}.
\]
The connections assume the following most general forms:

\[ \Gamma^{CAB} = \Gamma^{0}_{CAB} + \Delta_{Cpq} V^{A}_{p} V^{q}_{B} + \tilde{\Delta}_{C\bar{p}q} \tilde{V}^{A}_{\bar{p}} \tilde{V}^{q}_{\bar{B}} , \]

\[ \Phi_{Apq} = \Phi^{0}_{Apq} + \Delta_{Apq} , \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^{0}_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} . \]

Further, the extra pieces, \( \Delta_{Apq} \) and \( \bar{\Delta}_{A\bar{p}\bar{q}} \), correspond to the torsion of SDFT, which must be covariant and, in order to maintain \( D_A d = 0 \), must satisfy

\[ \Delta_{Apq} V^{A}_{p} = 0 , \quad \bar{\Delta}_{A\bar{p}\bar{q}} \tilde{V}^{A}_{\bar{p}} = 0 . \]

Otherwise they are arbitrary.

As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

\[ \bar{\rho}^{\gamma_{pq}} \psi_{A} , \quad \bar{\psi}_{\bar{p}}^{\gamma_{A}} \psi_{\bar{q}} , \quad \bar{\rho}^{\gamma_{Apq}} \rho , \quad \bar{\psi}_{\bar{p}}^{\gamma_{Apq}} \psi^{\bar{p}} , \]

where we set \( \psi_{A} = \tilde{V}^{A}_{\bar{p}} \psi_{\bar{p}} , \quad \gamma_{A} = V^{A}_{p} \gamma_{p} . \)
The connections assume the following most general forms:

\[ \Gamma_{CAB} = \Gamma_{CAB}^0 + \Delta_{Cpq} V_A^p V_B^q + \bar{\Delta}_{C\bar{p}q} \bar{V}_A^\bar{p} \bar{V}_B^\bar{q}, \]

\[ \Phi_{Apq} = \Phi_{Apq}^0 + \Delta_{Apq}, \]

\[ \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}_{A\bar{p}\bar{q}}^0 + \bar{\Delta}_{A\bar{p}\bar{q}}. \]

Further, the extra pieces, \( \Delta_{Apq} \) and \( \bar{\Delta}_{A\bar{p}\bar{q}} \), correspond to the torsion of SDFT, which must be covariant and, in order to maintain \( \mathcal{D}_A d = 0 \), must satisfy

\[ \Delta_{Apq} V^A^p = 0, \quad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0. \]

Otherwise they are arbitrary.

As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

\[ \bar{\rho} \gamma_{pq} \psi_A, \quad \bar{\psi}_p \gamma_A \psi_q, \quad \bar{\rho} \gamma_{Apq} \rho, \quad \bar{\psi}_p \gamma_{Apq} \psi^{\bar{p}}, \]

where we set \( \psi_A = \bar{V}_A^\bar{p} \psi^{\bar{p}}, \quad \gamma_A = V_A^p \gamma_p. \)
The connections assume the following most general forms:

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\]

\[
\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq},
\]

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\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}}.
\]

Further, the extra pieces, \(\Delta_{Apq}\) and \(\bar{\Delta}_{A\bar{p}\bar{q}}\), correspond to the torsion of SDFT, which must be covariant and, in order to maintain \(D_A d = 0\), must satisfy

\[
\Delta_{Apq} V^A = 0, \quad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^A = 0.
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As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

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\]

where we set \(\psi_A = \bar{V}_A^{\bar{p}} \psi_{\bar{p}}, \quad \gamma_A = V_A^p \gamma_p\).
The usual curvatures for the three connections,

\[ R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}, \]

\[ F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr} \Phi_{Bqr} - \Phi_{Bpr} \Phi_{Aqr}, \]

\[ \bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}} \bar{\Phi}_{B\bar{r}q} - \bar{\Phi}_{B\bar{p}\bar{r}} \bar{\Phi}_{A\bar{r}q}, \]

are, from \([\mathcal{D}_A, \mathcal{D}_B] V_{Cp} = 0\) and \([\mathcal{D}_A, \mathcal{D}_B] \bar{V}_{C\bar{p}} = 0\), related to each other,

\[ R_{ABCD} = F_{CDpq} V_{A^p} V_{B^q} + \bar{F}_{CD\bar{p}\bar{q}} \bar{V}_{A^\bar{p}} \bar{V}_{B^\bar{q}}. \]

However, the crucial object in DFT is

\[ S_{ABCD} := \frac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma_{EABC}^E \Gamma_{ECD} \right), \]

which we named the semi-covariant “Riemann” curvature.
The usual curvatures for the three connections,

\[ R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}, \]

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\[ \bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}r} \bar{\Phi}^{r}\bar{q} - \bar{\Phi}_{B\bar{p}r} \bar{\Phi}^{r}\bar{q}, \]

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However, the crucial object in DFT is

\[ S_{ABCD} := \frac{1}{2} \left( R_{ABCD} + R_{CDAB} - \Gamma^E_{AB} \Gamma_{ECD} \right), \]

which we named the semi-covariant “Riemann” curvature.
Properties of the semi-covariant curvature

- Precisely the same symmetric property as the ordinary Riemann curvature,

\[ S_{ABCD} = \frac{1}{2} \left( S_{[AB][CD]} + S_{[CD][AB]} \right), \]

\[ S^0_{[ABC]D} = 0. \]

- Projection property,

\[ P^A_I \bar{P}^B_J P^K_C \bar{P}^L_D S_{ABCD} = 0. \]

- Under arbitrary variation of the connection, \( \delta \Gamma_{ABC} \), it transforms as

\[ \delta S_{ABCD} = \mathcal{D}_{[A} \delta \Gamma_{B]} CD + \mathcal{D}_{[C} \delta \Gamma_{D]} AB - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^{E} CD - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^{E} AB, \]

\[ \delta S^0_{ABCD} = \mathcal{D}_{[A} \delta \Gamma^0_{B]} CD + \mathcal{D}_{[C} \delta \Gamma^0_{D]} AB. \]
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\[ P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} \equiv 0. \]

Under arbitrary variation of the connection, \( \delta \Gamma_{ABC} \), it transforms as

\[ \delta S_{ABCD} = D_{[A} \delta \Gamma_{B]CD} + D_{[C} \delta \Gamma_{D]AB} - \frac{3}{2} \Gamma_{[ABE]} \delta \Gamma^{E} \cdot CD - \frac{3}{2} \Gamma_{[CDE]} \delta \Gamma^{E} \cdot AB, \]

\[ \delta S^0_{ABCD} = D_{[A} \delta \Gamma^0_{B]CD} + D_{[C} \delta \Gamma^0_{D]AB}. \]
Precisely the same symmetric property as the ordinary Riemann curvature,

\[ S_{ABCD} = \frac{1}{2} \left( S_{[AB][CD]} + S_{[CD][AB]} \right), \]

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\]

\[
\delta S^0_{ABCD} = \mathcal{D}_{[A} \delta \Gamma^0_{B]}CD + \mathcal{D}_{[C} \delta \Gamma^0_{D]}AB.
\]
In general, as discussed earlier in this talk, under DFT-diffeomorphisms the variation of the semi-covariant derivative contains an *anomalous part* dictated by the six-index projectors,

\[
\delta_X (\nabla_C T_{A_1 \cdots A_n}) \equiv \hat{\mathcal{L}}_X (\nabla_C T_{A_1 \cdots A_n}) + \sum_i 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}^{BFDE} \partial_F \partial_{[D} X_{E]} T_{B \cdots},
\]

and hence,

\[
\delta_X \neq \hat{\mathcal{L}}_X.
\]

However, the characteristic property of our master semi-covariant derivative is that, contracted with the projectors, vielbeins as well as gamma matrices, it can generate various fully covariant quantities, as listed below.
In general, as discussed earlier in this talk, under DFT-diffeomorphisms the variation of the semi-covariant derivative contains an *anomalous part* dictated by the six-index projectors,

\[
\delta_X \left( \nabla_C T_{A_1 \ldots A_n} \right) \equiv \mathcal{L}_X \left( \nabla_C T_{A_1 \ldots A_n} \right) + \sum_i 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}^{BFDE} \partial_F \partial_{[DX_E]} T_{B \ldots },
\]

and hence,

\[
\delta_X \neq \mathcal{L}_X.
\]

However, the characteristic property of our master semi-covariant derivative is that, contracted with the projectors, vielbeins as well as gamma matrices, it can generate various fully covariant quantities, as listed below.
For $O(D, D)$ tensors: we recall

\[ P_C^D \bar{P}_{A_1}^{B_1} \bar{P}_{A_2}^{B_2} \cdots \bar{P}_{A_n}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n}, \]

\[ \bar{P}_C^D P_{A_1}^{B_1} P_{A_2}^{B_2} \cdots P_{A_n}^{B_n} \nabla_D T_{B_1 B_2 \cdots B_n}, \]

\[ P^{AB} \bar{P}_{C_1}^{D_1} \bar{P}_{C_2}^{D_2} \cdots \bar{P}_{C_n}^{D_n} \nabla_A T_{B D_1 D_2 \cdots D_n}, \]

\[ \bar{P}^{AB} P_{C_1}^{D_1} P_{C_2}^{D_2} \cdots P_{C_n}^{D_n} \nabla_A T_{B D_1 D_2 \cdots D_n} \]

\[ \left\{ \begin{array}{l}
\text{Divergences,} \\
\text{Laplacians.}
\end{array} \right. \]
Projector-aided, fully covariant derivatives

- For Spin$(1, D-1)_L \times Spin(D-1, 1)_R$ tensors:

  \[ \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \ldots \bar{q}_n}, \quad \mathcal{D}_{\bar{p}} T_{q_1 q_2 \ldots q_n}, \]

  \[ \mathcal{D}^p T_{p\bar{q}_1 \bar{q}_2 \ldots \bar{q}_n}, \quad \mathcal{D}^{\bar{p}} T_{\bar{p}q_1 q_2 \ldots q_n}, \]

  \[ \mathcal{D}_p \mathcal{D}^p T_{\bar{q}_1 \bar{q}_2 \ldots \bar{q}_n}, \quad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_{q_1 q_2 \ldots q_n}, \]

  where we set

  \[ \mathcal{D}_p := V^A_p \mathcal{D}_A, \quad \mathcal{D}_{\bar{p}} := \bar{V}^A_{\bar{p}} \mathcal{D}_A. \]

  These are the pull-back of the previous results using the DFT-vielbeins.
Dirac operators for fermions, $\rho^\alpha$, $\psi_\rho^\alpha$, $\rho'^\alpha$, $\psi'_\rho^\alpha$:

$$\gamma^p D_p \rho = \gamma^A D_A \rho,$$
$$\gamma^p D_p \psi_\rho = \gamma^A D_A \psi_\rho,$$
$$D_p \bar{\rho},$$
$$D_p \bar{\psi}_\rho = D_A \psi^A,$$
$$\bar{\psi}^A \gamma_\rho (D_A \psi_q - \frac{1}{2} D_q \psi_A),$$

$$\bar{\gamma}^\bar{p} D_{\bar{p}} \rho' = \bar{\gamma}^A D_A \rho',$$
$$\bar{\gamma}^\bar{p} D_{\bar{p}} \psi'_\rho = \bar{\gamma}^A D_A \psi'_\rho,$$
$$D_{\bar{p}} \rho',$$
$$D_{\bar{p}} \psi'_\rho = D_A \psi'^A,$$
$$\bar{\psi}'^A \bar{\gamma}_{\bar{p}} (D_A \psi'_q - \frac{1}{2} D_q \psi'_A).$$

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Projector-aided, fully covariant derivatives

- For $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ bi-fundamental spinorial fields, $T^\alpha_{\bar{\beta}}$:

  \[
  D_+ T := \gamma^A D_A T + \gamma^{(D+1)} D_A \bar{T} \gamma^A,
  \]

  \[
  D_- T := \gamma^A D_A T - \gamma^{(D+1)} D_A \bar{T} \gamma^A.
  \]

- Especially for the torsionless case, the corresponding operators are nilpotent

  \[
  (D_0^+)^2 T \equiv 0, \quad (D_0^-)^2 T \equiv 0,
  \]

  and hence, they define $O(D, D)$ covariant cohomology.

- The field strength of the R-R potential, $C^\alpha_{\bar{\alpha}}$, is then defined by

  \[
  F := D_0^+ C.
  \]

- Thanks to the nilpotency, the R-R gauge symmetry is simply realized

  \[
  \delta C = D_0^+ \Delta \quad \implies \quad \delta F = D_0^+ (\delta C) = (D_0^+)^2 \Delta \equiv 0.
  \]
For Spin\((1, D-1)_L \times \text{Spin}(D-1, 1)_R\) bi-fundamental spinorial fields, \(\mathcal{T}^\alpha{}_{\bar{\beta}}:\)

\[
\begin{align*}
\mathcal{D}_+ \mathcal{T} &:= \gamma^A \mathcal{D}_A \mathcal{T} + \gamma^{(D+1)} \mathcal{D}_A \mathcal{T} \bar{\gamma}^A, \\
\mathcal{D}_- \mathcal{T} &:= \gamma^A \mathcal{D}_A \mathcal{T} - \gamma^{(D+1)} \mathcal{D}_A \mathcal{T} \bar{\gamma}^A.
\end{align*}
\]

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(\mathcal{D}_0^+)^2 \mathcal{T} \equiv 0, \quad (\mathcal{D}_0^-)^2 \mathcal{T} \equiv 0,
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\delta C = \mathcal{D}_0^+ \Delta \quad \implies \quad \delta \mathcal{F} = \mathcal{D}_0^+ (\delta C) = (\mathcal{D}_0^+)^2 \Delta \equiv 0.
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Projector-aided, fully covariant derivatives

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For $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ bi-fundamental spinorial fields, $\mathcal{T}^\alpha_{\bar{\beta}}$:

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\mathcal{D}_+ \mathcal{T} := \gamma^A D_A \mathcal{T} + \gamma^{(D+1)} D_A \mathcal{T} \gamma^A , \\
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Especially for the torsionless case, the corresponding operators are nilpotent

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Thanks to the nilpotency, the R-R gauge symmetry is simply realized

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\delta C = \mathcal{D}_+^0 \Delta \quad \implies \quad \delta \mathcal{F} = \mathcal{D}_+^0 (\delta C) = (\mathcal{D}_+^0)^2 \Delta \equiv 0 .
\]
Scalar curvature:

\[(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \cdot\]

“Ricci” curvature:

\[S_{pq} + \frac{1}{2}D_{r}\bar{\Delta}_{pq}^r + \frac{1}{2}D_{r}\Delta_{q}p^r , \]

where we set

\[S_{pq} := V^A_p \bar{V}^B_q S_{AB} , \quad S_{AB} = S_{ACB}^C . \]
Scalar curvature:

\[
(P^{AB} p^{CD} - \bar{p}^{AB} \bar{p}^{CD}) S_{ACBD}.
\]

“Ricci” curvature:

\[
S_{p\bar{q}} + \frac{1}{2} \mathcal{D}_{\bar{r}} \Delta p^{\bar{r}} + \frac{1}{2} \mathcal{D}_{r} \Delta \bar{q}^{r},
\]

where we set

\[
S_{p\bar{q}} := V^{A}_{p} \bar{V}^{B}_{\bar{q}} S_{AB}, \quad S_{AB} = S_{ACB}^{C}.
\]
Combining all the results above, we are now ready to spell

\[ \mathcal{N} = 2 \ D = 10 \ \text{Supersymmetric Double Field Theory} \]
Lagrangian:

\[ \mathcal{L}_{\text{Type II}} = e^{-2d} \left[ \frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \bar{\mathcal{F}}) - i \bar{\rho} \mathcal{F} \rho' + i \bar{\psi} \gamma_q \mathcal{F} \bar{\gamma} \bar{\psi}' q 
\right. 
\left. + i \frac{1}{2} \bar{\rho} \gamma^p \mathcal{D}_p^* \rho - i \bar{\psi} \bar{\gamma}^p \mathcal{D}_p^* \psi - i \frac{1}{2} \bar{\psi} \gamma^q \mathcal{D}_q^* \psi - i \frac{1}{2} \bar{\rho}' \bar{\gamma}^p \mathcal{D}_p^* \rho' + i \bar{\psi}' \mathcal{D}_p^* \rho' + i \bar{\psi}' \mathcal{D}_q^* \psi' q \right]. \]

where \( \bar{\mathcal{F}}^\alpha_\alpha \) denotes the charge conjugation, \( \bar{\mathcal{F}} := \bar{C}^{-1} \mathcal{F}^T C_+ \).

As they are contracted with the DFT-vielbeins properly, every term in the Lagrangian is fully covariant.
\( \mathcal{N} = 2 \quad D = 10 \) SDFT \[ 1210.5078 \]

- **Lagrangian:**

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+ i \frac{1}{2} \bar{\rho} \gamma^p D^*_p \rho - i \bar{\psi} \bar{\gamma}^q D^*_q \psi - i \frac{1}{2} \bar{\psi} \bar{\gamma}^q D^*_q \psi \right].
\]

where \( \bar{\mathcal{F}} \alpha \) denotes the charge conjugation, \( \bar{\mathcal{F}} := \bar{C}_+^{-1} \mathcal{F}^T C_+ \).

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\( \mathcal{N} = 2 \) \( D = 10 \) SDFT \([1210.5078]\)

**Lagrangian:**

\[
\mathcal{L}_{Type\,II} = e^{-2d} \left[ \frac{1}{8} \left( P^{AB} P^{CD} - \tilde{P}^{AB} \tilde{P}^{CD} \right) S_{ACBD} + \frac{1}{2} \text{Tr}(\mathcal{F} \tilde{\mathcal{F}}) - i \bar{\psi}_p \gamma_q \mathcal{F} \tilde{\gamma} \psi' q \\
+ i \frac{1}{2} \bar{\rho} \gamma^p D^*_p \rho - i \bar{\psi} \tilde{\gamma} D^*_q \psi - i \frac{1}{2} \bar{\psi} \tilde{\gamma} \bar{\rho} \gamma^q D^*_q \psi - i \frac{1}{2} \bar{\psi} \tilde{\gamma} \bar{\rho} \gamma^q D^*_q \psi' + i \bar{\psi} \tilde{\gamma} \bar{\rho} \gamma^q D^*_q \psi' \right].
\]

**Torsions:** The semi-covariant curvature, \( S_{ABCD} \), is given by the connection,

\[
\Gamma_{ABC} = \Gamma^0_{ABC} + i \frac{1}{3} \bar{\rho} \gamma_{ABC} \rho - 2i \bar{\rho} \gamma_{BC} \psi_A - i \frac{1}{3} \bar{\psi} \tilde{\gamma} \gamma_{ABC} \psi p + 4i \bar{\psi} B \gamma A \psi C \\
+ i \frac{1}{3} \bar{\rho}' \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \rho - 2i \bar{\rho}' \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \psi - i \frac{1}{3} \bar{\psi} \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \psi' + i \bar{\psi} \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \psi',
\]

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:

\( D^*_A \) for the unprimed fermions and \( D'^*_A \) for the primed fermions, set by

\[
\Gamma^*_{ABC} = \Gamma_{ABC} - i \frac{11}{96} \bar{\rho} \gamma_{ABC} \rho + i \frac{5}{4} \bar{\rho} \gamma_{BC} \psi_A + i \frac{5}{24} \bar{\psi} \tilde{\gamma} \gamma_{ABC} \psi p - 2i \bar{\psi} B \gamma A \psi C + i \frac{5}{2} \bar{\rho}' \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \psi' A,
\]

\[
\Gamma'^*_{ABC} = \Gamma_{ABC} - i \frac{11}{96} \bar{\rho}' \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \rho' + i \frac{5}{4} \bar{\psi} \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \psi' A + i \frac{5}{24} \bar{\psi}' \tilde{\gamma} \gamma_{ABC} \psi' p - 2i \bar{\psi}' B \tilde{\gamma} \bar{\rho} \gamma^p D^*_p \psi',
\]
\[ N = 2 \ D = 10 \ \text{SDFT} \ [1210.5078] \]

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+ i \frac{1}{2} \bar{\rho} \gamma^p D^*_p \rho - i \bar{\psi} \bar{p} D^*_p \bar{\rho} - i \frac{1}{2} \bar{\psi} \bar{p} \gamma^q D_q^* \psi \bar{p} - i \frac{1}{2} \bar{\rho}' \bar{\gamma} \bar{p} D^*_p \rho' + i \bar{\psi}' \bar{p} D^*_p \rho' + i \frac{1}{2} \bar{\psi}' \bar{p} \bar{\gamma} \bar{q} D^*_q \psi' p \right].
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+ i \frac{1}{3} \bar{\rho}' \bar{\gamma}_{ABC} \rho' - 2i \bar{\rho}' \bar{\gamma}_{BC} \psi' A - i \frac{1}{3} \bar{\psi}' \bar{p} \bar{\gamma}_{ABC} \psi' \bar{p} + 4i \bar{\psi}' B \bar{\gamma} A \psi' C,
\]

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\]
The $\mathcal{N} = 2$ supersymmetry transformation rules are

\[ \delta_\varepsilon d = -i \frac{1}{2} (\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho') , \]

\[ \delta_\varepsilon V_{Ap} = i \tilde{V}_A \tilde{q} (\bar{\varepsilon}'\bar{\gamma}q \psi_p - \bar{\varepsilon}\gamma p \psi_q) , \]

\[ \delta_\varepsilon \tilde{V}_{A\bar{p}} = i V_A q (\bar{\varepsilon}\gamma q \psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}} \psi_{\bar{q}}) , \]

\[ \delta_\varepsilon C = i \frac{1}{2} (\gamma^p \varepsilon \psi_p - \varepsilon \bar{p}' - \psi_p \varepsilon' \bar{\gamma} \bar{p} + \rho \varepsilon') + C \delta_\varepsilon d - \frac{1}{2} (\tilde{V}_A q \delta_\varepsilon V_{Ap}) \gamma^{(d+1)} \gamma^p C \bar{\gamma} q , \]

\[ \delta_\varepsilon \rho = -\gamma^p \hat{D}_p \varepsilon + i \frac{1}{2} \gamma^p \varepsilon \psi_p \rho' - i \gamma^p \psi_q \varepsilon' \bar{\gamma} q \psi_p , \]

\[ \delta_\varepsilon \rho' = -\bar{\gamma} \bar{p} \hat{D}'_{\bar{p}} \varepsilon' + i \frac{1}{2} \bar{\gamma} \bar{p} \varepsilon' \bar{\psi}_q \rho - i \bar{\gamma} q \psi_p \varepsilon \gamma p q , \]

\[ \delta_\varepsilon \psi_{\bar{p}} = \hat{D}_{\bar{p}} \varepsilon + (\bar{\mathcal{F}} - i \frac{1}{2} \gamma q \rho \bar{\psi}_q + i \frac{1}{2} \psi q \bar{\rho}' \bar{\gamma} q) \bar{\gamma}_{\bar{p}} \varepsilon' + i \frac{1}{4} \varepsilon \psi_{\bar{p}} \rho + i \frac{1}{2} \psi_{\bar{p}} \bar{\varepsilon} \rho , \]

\[ \delta_\varepsilon \psi_{p}' = \hat{D}'_p \varepsilon' + (\bar{\mathcal{F}} - i \frac{1}{2} \bar{\gamma} \bar{q} \rho' \bar{\psi}_q + i \frac{1}{2} \psi' p \bar{\rho} \gamma q) \gamma p \varepsilon + i \frac{1}{4} \psi' \varepsilon p \rho' + i \frac{1}{2} \psi' p \bar{\varepsilon}' \rho' , \]

where

\[ \hat{\Gamma}_{ABC} = \Gamma_{ABC} - i \frac{17}{48} \bar{\rho} \gamma ABC \rho + i \frac{5}{2} \bar{\rho} \gamma BC \psi_A + i \frac{1}{4} \bar{\psi} \rho \gamma ABC \psi_{\bar{B}} - 3 i \tilde{\psi}_{B} \bar{\gamma}_A \psi_{C} , \]

\[ \hat{\Gamma}'_{ABC} = \Gamma_{ABC} - i \frac{17}{48} \bar{\rho}' \bar{\gamma} ABC \rho' + i \frac{5}{2} \bar{\rho}' \bar{\gamma} BC \psi_A' + i \frac{1}{4} \bar{\psi}' p \bar{\gamma} ABC \psi_{p} - 3 i \bar{\psi}_{B} \gamma A \psi_{C} . \]
The Lagrangian is \textbf{pseudo} : It is necessary to impose a \textbf{self-duality} of the R-R field strength by hand,

\[ \tilde{F}_- := \left( 1 - \gamma^{(D+1)} \right) \left( F - i \frac{1}{2} \rho \bar{\rho}' + i \frac{1}{2} \gamma^\rho \psi_q \bar{\psi}' \gamma^q \right) = 0. \]
The Lagrangian is **pseudo**: It is necessary to impose a **self-duality** of the R-R field strength by hand,

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\]
Under the $\mathcal{N} = 2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

$$\delta_\epsilon \mathcal{L}_{\text{Type II}} \simeq -\frac{1}{8} e^{-2d} \bar{V}^A q \delta_\epsilon V_A p \text{Tr} \left( \gamma^p \tilde{\mathcal{F}}_\gamma \bar{\gamma}_\bar{q} \tilde{\mathcal{F}}_\gamma \right),$$

where

$$\tilde{\mathcal{F}}_\gamma := \left( 1 - \gamma^{(D+1)} \right) \left( \mathcal{F} - i\frac{1}{2} \rho \bar{\rho}' + i\frac{1}{2} \gamma^p \psi \bar{\psi} \psi' \bar{\psi} \right).$$

This verifies, to the full order in fermions, the supersymmetric invariance of the action, modulo the self-duality.

For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_\epsilon \tilde{\mathcal{F}}_\gamma = -i \left( \tilde{D}_p \rho + \gamma^p \tilde{D}_p \psi \bar{\rho} - \gamma^p \mathcal{F} \bar{\gamma}_\bar{p} \psi' \bar{p} \right) \bar{\epsilon}' \gamma \bar{p} - i \gamma^p \epsilon \left( \tilde{D}'_p \bar{\rho}' + \tilde{D}'_p \psi' \bar{\gamma}' \bar{p}' - \bar{\psi} \gamma_p \mathcal{F} \bar{\gamma} \bar{p}' \right).$$
Under the $N=2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

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where

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For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

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Equations of Motion for Bosons

- **DFT-vielbein:**

\[ S_{p\bar{q}} + \text{Tr}(\gamma_\rho \mathcal{F} \bar{\gamma}_\bar{q} \mathcal{F}) + i \bar{\rho} \gamma_\rho \tilde{D}_q \rho + 2i \bar{\psi} q \tilde{D}_p \rho - i \bar{\psi} \gamma_\rho \tilde{D}_q \psi \rho + i \bar{\rho}' \bar{\gamma}_\bar{q} \tilde{D}_p \rho' + 2i \bar{\psi}' p \tilde{D}_q \rho' - i \bar{\psi}' q \bar{\gamma}_\bar{q} \tilde{D}_p \psi' q = 0. \]

This is DFT-generalization of Einstein equation.

- **DFT-dilaton:**

\[ \mathcal{L}_{\text{Type II}} = 0. \]

Namely, the on-shell Lagrangian vanishes!

- **R-R potential:**

\[ \mathcal{D}_-^0 \left( \mathcal{F} - i \rho \bar{\rho}' + i \gamma' \bar{\psi} \bar{\gamma}' \bar{\gamma} \bar{s} \right) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( \mathcal{D}_+^0 \),

\[ \mathcal{D}_-^0 \left( \mathcal{F} - i \rho \bar{\rho}' + i \gamma' \bar{\psi} \bar{\gamma}' \bar{\gamma} \bar{s} \right) = \mathcal{D}_-^0 \left( \mathcal{F}^{(D+1)} \right) = -\gamma^{(D+1)} \mathcal{D}_+^0 \mathcal{F} = -\gamma^{(D+1)} (\mathcal{D}_+^0)^2 \mathcal{C} = 0. \]

- **The 1.5 formalism** works: The variation of the Lagrangian induced by that of the connection is trivial,

\[ \delta \mathcal{L}_{\text{Type II}} = \delta \mathcal{F}_{ABC} \times 0. \]
Equations of Motion for Bosons

- **DFT-vielbein:**

\[ S_{\rho\bar{q}} + \text{Tr}(\gamma_\rho \bar{F} \gamma_q \bar{F}) + i \bar{\rho} \gamma_\rho \bar{D}_q \rho + 2i \bar{\psi}_q \bar{D}_p \rho - i \bar{\psi} \gamma_\rho \bar{D}_q \psi \bar{p} + i \bar{\rho}' \gamma_q \bar{D}_p \rho' + 2i \bar{\psi}'_p \bar{D}_q \rho' - i \bar{\psi}'q \gamma_q \bar{D}_p \psi'q = 0. \]

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- **R-R potential:**

\[ \mathcal{D}_- \left( \mathcal{F} - i \rho \rho' + i \gamma' \psi \bar{\gamma}' \bar{\gamma} \right) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( \mathcal{D}^0_+ \),

\[ \mathcal{D}_- \left( \mathcal{F} - i \rho \rho' + i \gamma' \psi \bar{\gamma}' \bar{\gamma} \right) = \mathcal{D}_0^0 \left( \gamma^{(D+1)} \mathcal{F} \right) = -\gamma^{(D+1)} \mathcal{D}_+^0 \mathcal{F} = -\gamma^{(D+1)} (\mathcal{D}_+^0)^2 \mathcal{C} = 0. \]

- The **1.5 formalism** works: The variation of the Lagrangian induced by that of the connection is trivial,

\[ \delta \mathcal{L}_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \]
Equations of Motion for Bosons

- **DFT-vielbein:**

\[ S_{p\bar{q}} + \text{Tr}(\gamma_p F \bar{\gamma}_q \bar{F}) + i \bar{\rho} \gamma_p \bar{D}_q \rho + 2i \bar{\psi}_q \bar{D}_p \rho - i \bar{\psi} \gamma_p \bar{D}_q \psi + i \bar{\rho}' \bar{\gamma}_q \bar{D}_p \rho' + 2i \bar{\psi}'_p \bar{D}_q \rho' - i \bar{\psi}'_q \bar{\gamma}_q \bar{D}_p \psi' = 0. \]

This is **DFT-generalization of Einstein equation.**

- **DFT-dilaton:**

\[ \mathcal{L}_{\text{Type II}} = 0. \]

Namely, the on-shell Lagrangian vanishes!

- **R-R potential:**

\[ D_0^+ \left( F - i \rho \bar{\rho}' + i \gamma' \psi \bar{\gamma}' \bar{\gamma} \bar{\gamma} \right) = 0, \]

which is automatically met by the self-duality, together with the nilpotency of \( D^0_+ \),

\[ D_-^0 \left( F - i \rho \bar{\rho}' + i \gamma' \psi \bar{\gamma}' \bar{\gamma} \bar{\gamma} \right) = D_-^0 \left( \gamma^{(D+1)} F \right) = -\gamma^{(D+1)} D^0_+ F = -\gamma^{(D+1)} (D^0_+)^2 C = 0. \]

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- **DFT-dilaton:**
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  Namely, the on-shell Lagrangian vanishes!

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  \[ \mathcal{D}_0^- \left( F - i \rho \rho' + i \gamma^r \psi_\bar{s} \bar{\psi}'_r \bar{\gamma}^{\bar{s}} \right) = 0, \]
  which is automatically met by the self-duality, together with the nilpotency of \( \mathcal{D}_+^0 \),
  \[ \mathcal{D}_-^0 \left( F - i \rho \rho' + i \gamma^r \psi_\bar{s} \bar{\psi}'_r \bar{\gamma}^{\bar{s}} \right) = \mathcal{D}_-^0 \left( \gamma^{(D+1)} F \right) = -\gamma^{(D+1)} \mathcal{D}_+^0 F = -\gamma^{(D+1)} (\mathcal{D}_+^0)^2 \mathcal{C} = 0. \]

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Equations of Motion for Bosons

- **DFT-vielbein:**

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\[ \mathcal{D}^0_-(\bar{F} - \bar{i} \rho \rho' + \bar{i} \gamma' \bar{\psi}_s \bar{\psi}'_r \bar{\gamma}'_s) = \mathcal{D}^0_-(\gamma^{(D+1)} \bar{F}) = -\gamma^{(D+1)} \mathcal{D}^0_+ \bar{F} = -\gamma^{(D+1)}(\mathcal{D}^0_+)^2 \mathcal{L} = 0. \]

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\[ \delta \mathcal{L}_{\text{Type II}} = \delta \Gamma_{ABC} \times 0. \]
Equations of Motion for Fermions

- DFT-dilatinos,
  \[ \gamma^p \tilde{\nabla}_p \rho - \tilde{\nabla}_p \psi^p - \nabla \rho' = 0, \quad \tilde{\gamma}^\rho \tilde{\nabla}_\rho \rho' - \tilde{\nabla}_\rho \psi^\rho - \nabla \rho = 0. \]

- Gravitinos,
  \[ \tilde{\nabla}_\rho + \gamma^p \tilde{\nabla}_p \psi_\rho - \gamma^p \nabla \tilde{\gamma}_\rho \psi^\rho = 0, \quad \tilde{\nabla}_\rho' + \tilde{\gamma}^\rho \tilde{\nabla}_\rho \psi^\rho' - \tilde{\gamma}^\rho \nabla \tilde{\gamma}_\rho \psi_\rho = 0. \]
DFT-dilatinos,
\[ \gamma^p \tilde{\nabla}_p \rho - \tilde{\nabla}_{\bar{p}} \bar{\psi}^\bar{p} - \mathcal{F} \rho' = 0, \quad \bar{\gamma}^\bar{p} \tilde{\nabla}_{\bar{p}} \rho - \tilde{\nabla}_p \psi^p - \bar{\mathcal{F}} \rho = 0. \]

Gravitinos,
\[ \tilde{\nabla}_{\bar{p}} \rho + \gamma^p \tilde{\nabla}_p \psi_{\bar{p}} - \gamma^p \mathcal{F} \bar{\gamma}_{\bar{p}} \psi'_{\rho} = 0, \quad \tilde{\nabla}_p \rho' + \bar{\gamma}^\bar{p} \tilde{\nabla}_{\bar{p}} \psi'_{p} - \bar{\gamma}^\bar{p} \bar{\mathcal{F}} \gamma_p \psi_{\bar{p}} = 0. \]
Truncation to $\mathcal{N} = 1 \ D = 10$ SDFT [1112.0069]

- Turning off the primed fermions and the R-R sector truncates the $\mathcal{N} = 2 \ D = 10$ SDFT to $\mathcal{N} = 1 \ D = 10$ SDFT,

$$\mathcal{L}_{\mathcal{N}=1} = e^{-2d} \left[ \frac{1}{8} \left( p^{AB} p^{CD} - \bar{p}^{AB} \bar{p}^{CD} \right) S_{ACBD} + i \frac{1}{2} \bar{\rho} \gamma^{A} D^{*}_{A} \rho - i \bar{\psi}^{A} D^{*}_{A} \rho - i \frac{1}{2} \bar{\psi}^{B} \gamma^{A} D^{*}_{A} \psi_{B} \right].$$

- $\mathcal{N} = 1$ Local SUSY:

$$
\begin{align*}
\delta_{\varepsilon} d & = -i \frac{1}{2} \bar{\varepsilon} \rho, \\
\delta_{\varepsilon} V_{Ap} & = -i \bar{\varepsilon} \gamma_{p} \psi_{A}, \\
\delta_{\varepsilon} \bar{V}_{A\bar{p}} & = i \bar{\varepsilon} \gamma_{A} \psi_{\bar{p}}, \\
\delta_{\varepsilon} \rho & = -\gamma^{A} \hat{D}_{A} \varepsilon, \\
\delta_{\varepsilon} \psi_{\bar{p}} & = \bar{V}^{A}_{A} \hat{D}_{A} \varepsilon - i \frac{1}{4} (\bar{\rho} \psi_{\bar{p}}) \varepsilon + i \frac{1}{2} (\bar{\varepsilon} \rho) \psi_{\bar{p}}.
\end{align*}
$$
Commutator of supersymmetry reads

\[ [\delta \varepsilon_1, \delta \varepsilon_2] \equiv \hat{L} \chi_3 + \delta \varepsilon_3 + \delta_{\mathfrak{so}(1,9)} L + \delta_{\mathfrak{so}(9,1)} R + \delta_{\text{trivial}}. \]

where

\[ X_3^A = i \bar{\varepsilon}_1 \gamma^A \varepsilon_2, \quad \varepsilon_3 = i \frac{1}{2} \left[ (\bar{\varepsilon}_1 \gamma^p \varepsilon_2) \gamma^p \rho + (\bar{\rho} \varepsilon_2) \varepsilon_1 - (\bar{\rho} \varepsilon_1) \varepsilon_2 \right], \quad \text{etc.} \]

and \( \delta_{\text{trivial}} \) corresponds to the fermionic equations of motion.
Now I am going to sketch the parametrization of the DFT-field-variables in terms of Riemannian ones, the diagonal gauge of $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$, the reduction of SDFT to SUGRA, and the ‘unification’ of IIA and IIB.

Nevertheless, we emphasize that SDFT can describe not only Riemannian (SUGRA) backgrounds but also new type of non-Riemannian (“metric-less”) string theory backgrounds. 

Note also ‘global’ aspects of interest in DFT:

- **T-fold** Hull
- “non-geometry” Berman-Cederwall-Perry, Papadopoulos
- Scherk-Schwarz Geissbuhler, Grana-Marques-Aldazabal-Rosabal, Dibitetto-Fernandez-Melgarejo-Marques-Roest, Berman-Lee

*cf. Gomis-Ooguri*
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Recall the defining algebraic properties of the DFT-vielbeins,

\[ V_{Ap} V^{Aq} = \eta_{pq}, \quad \bar{V}_{\bar{A}\bar{p}} \bar{V}^{\bar{A}\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}, \quad V_{Ap} \bar{V}^{A\bar{q}} = 0, \quad V_{Ap} V^{Bp} + \bar{V}_{\bar{A}\bar{p}} \bar{V}^{B\bar{p}} = \mathcal{J}_{AB}. \]

We may parametrize the solution in terms of Riemannian variables.

Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

\[ V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_{\mu}^p \\ (B + e)_{\nu p} \end{pmatrix}, \quad \bar{V}_{\bar{A}\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}^\mu \\ (B + \bar{e})_{\nu \bar{p}} \end{pmatrix}. \]

Here \( e_{\mu}^p \) and \( \bar{e}_{\nu \bar{p}} \) are two copies of the \( D \)-dimensional vielbeins, or zehnbeins, corresponding to the same spacetime metric,

\[ e_{\mu}^p e_{\nu}^q \eta_{pq} = -\bar{e}_{\mu}^\bar{p} \bar{e}_{\nu}^\bar{q} \bar{\eta}_{\bar{p}\bar{q}} = g_{\mu \nu}, \]

and further, \( B_{\mu p} = B_{\mu \nu} (e^{-1})_{\nu}^p, \quad B_{\mu \bar{p}} = B_{\mu \nu} (\bar{e}^{-1})_{\bar{p}}^\nu. \)
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and further, \( B_{\mu p} = B_{\mu \nu} (e^{-1})_p^\nu, \quad B_{\mu \bar{p}} = B_{\mu \nu} (\bar{e}^{-1})_{\bar{p}}^\nu. \)
Instead, we may choose an alternative parametrization,

\[ V_A^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^\mu \nu \\ (\tilde{e}^{-1})^\mu \nu \end{pmatrix}, \quad \bar{V}_A^\bar{p} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^\mu \nu \\ (\tilde{e}^{-1})^\mu \nu \end{pmatrix}, \]

where \( \beta^\mu \nu = \beta^\mu \nu (\tilde{e}^{-1})^\nu \), \( \beta^\mu \bar{\nu} = \beta^\mu \nu (\tilde{e}^{-1})^\nu \), and \( \tilde{e}^\mu \), \( \tilde{e}^\mu \bar{p} \) correspond to a pair of T-dual vielbeins for winding modes,

\[ \tilde{e}^\mu \ p \tilde{e}^\nu \ q \eta^{pq} = -\tilde{e}^\mu \ p \tilde{e}^\nu \ q \eta^{\bar{p} \bar{q}} = (g - B g^{-1} B)^{-1} \mu \nu. \]

Note that in the T-dual winding mode sector, the \( D \)-dimensional curved spacetime indices are all upside-down: \( \tilde{x}_\mu, \tilde{e}^\mu \ p, \tilde{e}^\mu \bar{p}, \beta^\mu \nu \) (cf. \( x^\mu, e_\mu \ p, e_\mu \bar{p}, B_{\mu \nu} \)).
Two parametrizations:

\[
V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix}
(e^{-1})_{\rho}^{\mu} \\
(B + e)_{\nu p}
\end{pmatrix}, \quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
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(B + \bar{e})_{\nu \bar{p}}
\end{pmatrix}
\]

versus

\[
V_{A^p} = \frac{1}{\sqrt{2}} \begin{pmatrix}
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\end{pmatrix}, \quad \bar{V}_{A^\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
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\end{pmatrix}.
\]

In connection to the section condition, \( \partial^A \partial_A \equiv 0 \), the former matches well with the choice, \( \frac{\partial}{\partial \bar{x}_\mu} \equiv 0 \), while the latter is natural when \( \frac{\partial}{\partial x_\mu} \equiv 0 \).

Yet if we consider dimensional reductions from \( D \) to lower dimensions, there is no longer preferred parametrization.

c.f. “\( \beta \)-gravity” Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun et al.
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versus

\[ V_{A\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})_{\mu}^\nu \\ (\bar{e}^{-1})_{\nu}^\mu \end{pmatrix}, \quad \bar{V}_{A\bar{\mu}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})_{\mu}^{\bar{\nu}} \\ (\bar{e}^{-1})_{\bar{\nu}}^\mu \end{pmatrix}. \]

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versus

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\]

However, let me stress that to maintain the clear $O(D, D)$ covariant structure, it is necessary to work with the parametrization-independent, and $O(D, D)$ covariant, DFT-vielbeins, $V_{\alpha p}, V_{\alpha \bar{p}}$, rather than the Riemannian variables, $e_{\mu p}, B_{\mu \nu}$.
Parametrization: Reduction to Generalized Geometry

Two parametrizations:

\[ V_{A\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_{\mu}^{\rho} \\ (B + e)_{\nu}^{\rho} \end{pmatrix}, \quad \bar{V}_{A\bar{\nu}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{\nu}}^{\rho} \\ (B + \bar{e})_{\nu}^{\rho} \end{pmatrix} \]

versus

\[ V_{A\rho} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})_{\mu}^{\rho} \\ (\bar{e}^{-1})_{\nu}^{\rho} \end{pmatrix}, \quad \bar{V}_{A\bar{\nu}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \bar{e})_{\mu}^{\rho} \\ (\bar{e}^{-1})_{\nu}^{\rho} \end{pmatrix}. \]

However, let me stress that to maintain the clear \( O(D, D) \) covariant structure, it is necessary to work with the parametrization-independent, and \( O(D, D) \) covariant, DFT-vielbeins, \( V_{A\mu}, \bar{V}_{A\bar{\nu}} \), rather than the Riemannian variables, \( e_{\mu}^{\rho}, B_{\mu\nu} \).
From now on, let us restrict ourselves to the former parametrization and impose
\[ \frac{\partial}{\partial \tilde{x}^\mu} \equiv 0. \]

This reduces (S)DFT to Generalized Geometry

For example, the \(\mathcal{O}(D, D)\) covariant Dirac operators become

\[ \sqrt{2} \gamma^A D_A \rho \equiv \gamma^m \left( \partial_m \rho + \frac{1}{4} \omega_{mnp} \gamma^{np} \rho + \frac{1}{24} H_{mnp} \gamma^{np} \rho - \partial_m \phi \rho \right), \]

\[ \sqrt{2} \gamma^A D_A \psi \equiv \gamma^m \left( \partial_m \psi + \frac{1}{4} \omega_{mnp} \gamma^{np} \psi + \bar{\omega}_{mpq} \psi \bar{q} + \frac{1}{24} H_{mnp} \gamma^{np} \psi + \frac{1}{2} H_{mpq} \psi \bar{q} - \partial_m \phi \psi \right), \]

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From now on, let us restrict ourselves to the former parametrization and impose
\[ \frac{\partial}{\partial x_\mu} \equiv 0. \]

This reduces (S)DFT to Generalized Geometry

Hitchin; Grana, Minasian, Petrini, Waldram

For example, the \( \mathcal{O}(D, D) \) covariant Dirac operators become

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Since the two zehnbeins correspond to the same spacetime metric, they are related by a Lorentz rotation,

\[(e^{-1} \bar{e})^p {}^{\bar{p}}(e^{-1} \bar{e})^q {}^{\bar{q}} \eta_{p \bar{q}} = -\eta_{pq}\].

Further, there is a spinorial representation of this Lorentz rotation,

\[S_{e} \gamma^{\bar{p}} S_{e}^{-1} = \gamma^{(D+1)} \gamma^p (e^{-1} \bar{e})_p {}^{\bar{p}},\]

such that

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\]
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\]

This identification with the ordinary IIA/IIB SUGRAs can be established, if we ‘fix’ the two zehnbeins equal to each other,

\[
e_{\mu}^\rho \equiv \bar{e}_{\mu}^\bar{\rho},
\]

using a $\text{Pin}(D - 1, 1)_R$ local Lorentz rotation which may or may not flip the $\text{Pin}(D - 1, 1)_R$ chirality,

\[
\mathbf{c}' \rightarrow \det(e^{-1} \bar{e}) \mathbf{c}'.
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Namely, the $\text{Pin}(D - 1, 1)_R$ chirality changes iff $\det(e^{-1} \bar{e}) = -1$. 
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However, the theory contains two ‘types’ of Riemannian solutions, as classified above.

Conversely, any solution in type IIA and type IIB supergravities can be mapped to a solution of $\mathcal{N} = 2 \, D = 10$ SDFT of fixed chirality e.g. $\mathbf{c} \equiv \mathbf{c}' \equiv +1$.

In conclusion, the single unique $\mathcal{N} = 2 \, D = 10$ SDFT unifies type IIA and IIB SUGRAs. Further it allows non-Riemannian solutions.
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Unification of type IIA and IIB SUGRAs

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- In conclusion, the single unique $\mathcal{N} = 2 \ D = 10$ SDFT unifies type IIA and IIB SUGRAs. Further it allows non-Riemannian solutions.
Setting the diagonal gauge,

\[ e_\mu^p \equiv \bar{e}_\mu \bar{p} \]

with \( \eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}} \), \( \bar{\gamma}^p = \gamma^{(D+1)} \gamma^p \), \( \bar{\gamma}^{(D+1)} = -\gamma^{(D+1)} \), breaks the local Lorentz symmetry,

\[ \text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \implies \text{Spin}(1, D-1)_D. \]

And it reduces SDFT to SUGRA:

\[ \mathcal{N} = 2 \quad D = 10 \quad \text{SDFT} \implies 10D \quad \text{Type II democratic SUGRA} \]

Bergshoeff, et al.; Coimbra, Strickland-Constable, Waldram

\[ \mathcal{N} = 1 \quad D = 10 \quad \text{SDFT} \implies 10D \quad \text{minimal SUGRA} \]

Chamseddine; Bergshoeff et al.
Diagonal gauge fixing and Reduction to SUGRA

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After the diagonal gauge fixing, we may parameterize the R-R potential as

\[ C \equiv \left( \frac{1}{2} \right)^{\frac{D+2}{4}} \sum'_{p} \frac{1}{p!} \, C_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p} \]

and obtain the field strength,

\[ \mathcal{F} := \mathcal{D}_+^0 C \equiv \left( \frac{1}{2} \right)^{\frac{D+2}{4}} \sum'_{p} \frac{1}{(p+1)!} \, \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}} \]

where \( \sum'_{p} \) denotes the odd \( p \) sum for Type IIA and even \( p \) sum for Type IIB, and

\[ \mathcal{F}_{a_1 a_2 \cdots a_p} = p \left( D_{[a_1} C_{a_2 \cdots a_p]} - \partial_{[a_1} \phi C_{a_2 \cdots a_p]} \right) + \frac{p!}{3!(p-3)!} \, H_{[a_1 a_2 a_3 C_{a_4 \cdots a_p}]} \]

The pair of nilpotent differential operators, \( \mathcal{D}_+^0 \) and \( \mathcal{D}_-^0 \), reduce to a ‘twisted K-theory’ exterior derivative and its dual, after the diagonal gauge fixing,

\[ \mathcal{D}_+^0 \quad \implies \quad d + (H - d\phi) \wedge \]

\[ \mathcal{D}_-^0 \quad \implies \quad * \left[ d + (H - d\phi) \wedge \right] * \]
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\[ F := \mathcal{D}_+^0 C \equiv \left( \frac{1}{2} \right)^{\frac{D}{4}} \sum' p \frac{1}{(p+1)!} F_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}} \]

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In this way, ordinary SUGRA $\equiv$ gauge-fixed SDFT,

$$\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R \implies \text{Spin}(1, D-1)_D.$$
The diagonal gauge, $e_\mu^p \equiv \bar{e}_\mu \bar{p}$, is incompatible with the vectorial $O(D, D)$ transformation rule of the DFT-vielbein.

In order to preserve the diagonal gauge, it is necessary to modify the $O(D, D)$ transformation rule.
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In order to preserve the diagonal gauge, it is necessary to modify the $O(D, D)$ transformation rule.
The $O(D, D)$ rotation must accompany a compensating $\text{Pin}(D-1, 1)_R$ local Lorentz rotation, $\bar{L}_{\bar{q} \bar{p}}$, $S_{\bar{L} \bar{\alpha} \bar{\beta}}$ which we can construct explicitly as below.

$$\bar{L} = \bar{e}^{-1} \left[ a^t - (g + B) b^t \right] \left[ a^t + (g - B) b^t \right]^{-1} \bar{e}, \quad \bar{\gamma}^{\bar{q}} \bar{L}_{\bar{q} \bar{p}} = S_{\bar{L}}^{-1} \bar{\gamma}^{\bar{p}} S_{\bar{L}},$$

where $a$ and $b$ are parameters of a given $O(D, D)$ group element,

$$M_{A}^{B} = \begin{pmatrix} a^{\mu \nu} & b^{\mu \sigma} \\ c_{\rho \nu} & d_{\rho \sigma} \end{pmatrix}. $$
Modified $O(D, D)$ Transformation Rule After The Diagonal Gauge Fixing

<table>
<thead>
<tr>
<th>Original</th>
<th>Transformation</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>$\rightarrow$</td>
<td>$d$</td>
</tr>
<tr>
<td>$V_A^p$</td>
<td>$\rightarrow$</td>
<td>$M_A^B V_B^p$</td>
</tr>
<tr>
<td>$\bar{V}_A^\bar{p}$</td>
<td>$\rightarrow$</td>
<td>$M_A^B \bar{V}<em>B^\bar{q} \bar{L}</em>{\bar{q}\bar{p}}$</td>
</tr>
<tr>
<td>$C^\alpha \bar{\alpha}$, $F^\alpha \bar{\alpha}$</td>
<td>$\rightarrow$</td>
<td>$C^\alpha \bar{\beta} (S_{\bar{L}}^{-1})^\bar{\beta} \bar{\alpha}$, $F^\alpha \bar{\beta} (S_{\bar{L}}^{-1})^\bar{\beta} \bar{\alpha}$</td>
</tr>
<tr>
<td>$\rho^\alpha$</td>
<td>$\rightarrow$</td>
<td>$\rho^\alpha$</td>
</tr>
<tr>
<td>$\rho' \bar{\alpha}$</td>
<td>$\rightarrow$</td>
<td>$(S_{\bar{L}})^{\bar{\alpha}} \bar{\beta} \rho'^{\bar{\beta}}$</td>
</tr>
<tr>
<td>$\psi^{\alpha}_{\bar{p}}$</td>
<td>$\rightarrow$</td>
<td>$(\bar{L}^{-1})<em>{\bar{p}}^q \psi^{\alpha}</em>{q}$</td>
</tr>
<tr>
<td>$\psi'_{\bar{p}} \bar{\alpha}$</td>
<td>$\rightarrow$</td>
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</tr>
</tbody>
</table>

- All the barred indices are now to be rotated. Consistent with Hassan
- The R-R sector can be also mapped to $O(D, D)$ spinors.

Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach
If and only if \( \det(\bar{L}) = -1 \), the modified \( O(D, D) \) rotation flips the chirality of the theory, since

\[
\bar{\gamma}^{(D+1)} S L = \det(\bar{L}) S L \bar{\gamma}^{(D+1)}.
\]

Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under \( O(D, D) \) T-duality.

However, since \( \bar{L} \) explicitly depends on the parametrization of \( V_{\alpha \rho} \) and \( \bar{V}_{\alpha \rho} \) in terms of \( g_{\mu \nu} \) and \( B_{\mu \nu} \), it is impossible to impose the modified \( O(D, D) \) transformation rule from the beginning on the parametrization-independent covariant formalism.

The chirality flipping is an artifact of the diagonal gauge fixing.
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Worldsheet Perspective
The section condition is equivalent to the ‘coordinate gauge symmetry’, 1304.5946

\[ x^M \sim x^M + \varphi \partial^M \varphi' . \]

A ‘physical point’ is one-to-one identified with a ‘gauge orbit’ in coordinate space.

The coordinate gauge symmetry can be concretely realized on worldsheet, 1307.8377

\[ S = \frac{1}{4\pi \alpha'} \int d^2 \sigma \ L , \quad L = -\frac{1}{2} \sqrt{-h} h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M A_{jM} , \]

where

\[ D_i X^M = \partial_i X^M - A^M_i , \quad A^M_i \partial_M \equiv 0 . \]

The Lagrangian is quite symmetric thanks to the auxiliary gauge field, \( A^M_i \):

- String worldsheet diffeomorphisms plus Weyl symmetry (as usual)
- \( O(D, D) \) T-duality
- Target spacetime diffeomorphisms
- The coordinate gauge symmetry

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh
String propagates in doubled-yet-gauged spacetime

- The section condition is equivalent to the ‘coordinate gauge symmetry’, 1304.5946

\[ x^M \sim x^M + \varphi \partial^M \varphi'. \]

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- The coordinate gauge symmetry can be concretely realized on worldsheet, 1307.8377

\[ S = \frac{1}{4\pi \alpha'} \int d^2 \sigma \, \mathcal{L}, \quad \mathcal{L} = -\frac{1}{2} \sqrt{-h} h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M A_{jM}, \]

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Jeong-Hyuck Park Stringy Differential Geometry and Supersymmetric Double Field Theory
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\textit{c.f.} Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh
For example, under target spacetime ‘finite’ diffeomorphism \textit{à la} Zwiebach-Hohm

\[ L_{M}^{N} := \partial_{A}X'^{B}, \quad \bar{L} := \mathcal{J}L^{t}\mathcal{J}^{-1}, \]
\[ F := \frac{1}{2}(L\bar{L}^{-1} + \bar{L}^{-1}L), \quad \bar{F} := \mathcal{J}F^{t}\mathcal{J}^{-1} = \frac{1}{2}(L^{-1}\bar{L} + \bar{L}L^{-1}) = F^{-1}, \]

each field transforms as

\[ X^{M} \quad \rightarrow \quad X'^{M}(X), \]
\[ \mathcal{H}_{MN}(X) \quad \rightarrow \quad \mathcal{H}'_{MN}(X') = \bar{F}^{K}_{M}\bar{F}^{L}_{N}\mathcal{H}_{KL}(X), \]
\[ A^{M} \quad \rightarrow \quad A'^{M} = A^{N}F_{N}^{M} + dX^{N}(L - F)_{N}^{M} \quad : \quad A'^{M}\partial'^{N}_{M} \equiv 0, \]
\[ DX^{M} \quad \rightarrow \quad D'X'^{M} = DX^{N}F_{N}^{M}, \]

such that the worldsheet action remains invariant, up to total derivatives.
The Equation Of Motion for $X^L$ can be conveniently organized in terms of our DFT-Christoffel connection:

$$\frac{1}{\sqrt{-h}} \partial_i \left( \sqrt{-h} D^i X^M \mathcal{H}_{ML} + \epsilon^{ij} \partial_j A_{jL} \right) - 2 \Gamma_{LMN} \left( PD_i X^M \right)^L \left( \bar{P} D^i X \right)^N = 0,$$

which is comparable to the geodesic motion of a point particle, $\ddot{Y}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{Y}^\mu \dot{Y}^\nu = 0$.

The EOM of $A_i^M$ implies a priori,

$$\delta A_{iM} \left( \mathcal{H}^M_{\ N} D^i X^N + \frac{1}{\sqrt{-h}} \epsilon^{ij} D_j X^M \right) = 0.$$

Especially, for the case of the ‘non-degenerate’ Riemannian background, a complete self-duality follows

$$\mathcal{H}^M_{\ N} D^i X^N + \frac{1}{\sqrt{-h}} \epsilon^{ij} D_j X^M = 0.$$

Finally, the EOM of $h_{ij}$ gives the Virasoro constraints,

$$\left( D_i X^M D_j X^N - \frac{1}{2} h_{ij} D_k X^M D^k X^N \right) \mathcal{H}_{MN} = 0.$$
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String propagates in doubled-yet-gauged spacetime

After parametrization, $X^M = (\tilde{Y}_\mu, Y^\nu)$, $\mathcal{H}_{MN}(G, B)$, and integrating out $A^M_i$, it can produce either the standard string action for the ‘non-degenerate’ Riemannian case,

$$\frac{1}{4\pi\alpha'} \mathcal{L} \equiv \frac{1}{2\pi\alpha'} \left[ -\frac{1}{2} \sqrt{-h} h^{ij} \partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \frac{1}{2} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu \right],$$

with the bonus of the topological term introduced by Giveon-Rocek, Hull.
Stringy Differential Geometry and Supersymmetric Double Field Theory

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\[
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\]

with the bonus of the topological term introduced by Giveon-Roceck, Hull

or chiral actions for ‘degenerate’ non-Riemannian cases, e.g. for \( \mathcal{H}_{AB} = J_{AB} \),

\[
\frac{1}{4\pi\alpha'} \mathcal{L} \equiv \frac{1}{4\pi\alpha'} \epsilon^{ij} \partial_i \tilde{Y}_\mu \partial_j Y^\mu, \quad \partial_i Y^\mu + \frac{1}{\sqrt{-h}} \epsilon_i^j \partial_j Y^\mu = 0.
\]

c.f. Gomis-Ooguri
U-duality
Parallel to the stringy differential geometry for $O(D, D)$ T-duality,
it is possible to construct M-theoretic differential geometry for each U-duality group.
\[ A_{N-1} \equiv \mathfrak{sl}(N) \]

\[ D_{N-1} \equiv \mathfrak{so}(N-1, N-1) \]

\[ E_{N-1} \]

\[ E_N \]

**Table:** Dynkin diagrams for \( A_{N-1} \), \( D_{N-1} \), \( E_{N-1} \) and \( E_N \)

- **\( E_{11} \):** conjectured to be the ultimate duality group. West
- **\( E_{10} \):** Damour, Nicolai, Henneaux and further \( E_n \) (\( n \leq 8 \)) “Exceptional Field Theory”
- **\( D_{10} \):** Double Field Theory
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- \( A_{10} \): U-gravity
Geometric Constitution of U-gravity
Notation.

Small Latin alphabet letters denote the $\text{SL}(N)$ vector indices, i.e.
$a, b, c, \ldots = 1, 2, \ldots, N$. 
Extended-yet-gauged spacetime.

The spacetime is formally extended, being $\frac{1}{2}N(N - 1)$-dimensional. The coordinates carry a pair of anti-symmetric $\text{SL}(N)$ vector indices,

$$x^{ab} = -x^{ba} = x^{[ab]},$$

and hence so does the derivative,

$$\partial_{ab} = -\partial_{ba} = \partial_{[ab]} = \frac{\partial}{\partial x^{ab}}, \quad \partial_{ab} x^{cd} = \delta_a^c \delta_b^d - \delta_a^d \delta_b^c.$$

However, the spacetime is gauged: the coordinate space is equipped with an equivalence relation (‘Coordinate Gauge Symmetry’),

$$x^{ab} \sim x^{ab} + \frac{1}{(N-4)!} \epsilon^{abc_1 \ldots c_{N-4} de} \phi_{c_1 \ldots c_{N-4}} \partial_{de} \varphi,$$

where $\phi_{c_1 \ldots c_{N-4}}$ and $\varphi$ are arbitrary functions in U-gravity.

Each equivalence class, or gauge orbit defined by the equivalence relation represents a single physical point, and diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.
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**Extended-yet-gauged spacetime.**

The spacetime is formally extended, being $\frac{1}{2}N(N - 1)$-dimensional. The coordinates carry a pair of anti-symmetric $\text{SL}(N)$ vector indices,

$$\chi^{ab} = -\chi^{ba} = \chi^{[ab]} ,$$

and hence so does the derivative,

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However, the spacetime is gauged: the coordinate space is equipped with an equivalence relation (‘Coordinate Gauge Symmetry’),

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Each equivalence class, or gauge orbit defined by the equivalence relation represents a single physical point, and diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.
Realization of the coordinate gauge symmetry.

The equivalence relation is realized in U-gravity by enforcing that, arbitrary functions and their arbitrary derivatives are invariant under the coordinate gauge symmetry shift,

\[ \Phi(x + \Delta) = \Phi(x), \quad \Delta^{ab} = \frac{1}{(N-4)!} \epsilon^{abc_1 \cdots c_{N-4}de} \phi_{c_1 \cdots c_{N-4}} \partial_{de} \varphi. \]
The invariance under the coordinate gauge symmetry is, in fact, equivalent to a section condition, \( c.f. \) Berman-Perry for \( N = 5 \):

\[
\partial_{[ab}\partial_{cd]} \equiv 0.
\]

Acting on arbitrary functions, \( \Phi, \Phi' \), and their products, the section condition leads to

\[
\partial_{[ab}\partial_{cd]} \Phi = \partial_{[ab}\partial_{c]d} \Phi = 0 \quad \text{(weak constraint)},
\]

\[
\partial_{[ab}\Phi\partial_{cd]} \Phi' = \frac{1}{2} \partial_{[ab}\Phi\partial_{c]d} \Phi' - \frac{1}{2} \partial_{d[a\Phi\partial_{bc}] \Phi'} = 0 \quad \text{(strong constraint)}.
\]
**Section condition.**

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Geometric Constitution of U-gravity

**Diffeomorphism.**

U-gravity diffeomorphism is generated by a generalized Lie derivative,

c.f. Berman-Perry for $N = 5$

$$\hat{\mathcal{L}}_X T^{a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q} := \frac{1}{2} X^{cd} \partial_{cd} T^{a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q}$$

$$+ \frac{1}{2} \left( \frac{1}{2} p - \frac{1}{2} q + \omega \right) \partial_{cd} X^{cd} T^{a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q}$$

$$- \sum_{i=1}^{p} T^{a_1 \cdots c \cdots a_p b_1 b_2 \cdots b_q} \partial_{cd} X^{a_i d}$$

$$+ \sum_{j=1}^{q} \partial_{b_j d} X^{cd} T^{a_1 a_2 \cdots a_p b_1 \cdots c \cdots b_q}.$$

Here we let the tensor density, $T^{a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q}$, carry the ‘total’ weight, $\frac{1}{2} p - \frac{1}{2} q + \omega$, such that each upper or lower index contributes to the total weight by $+\frac{1}{2}$ or $-\frac{1}{2}$ respectively, while $\omega$ corresponds to a possible ‘extra’ weight.
Geometric Constitution of U-gravity

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\]

Note

\[\hat{\mathcal{L}}_X \delta^a{}_b = 0,\]

and the commutator,

\[
[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] = \hat{\mathcal{L}}_{[X, Y]_G}, \quad [X, Y]_G^{ab} = \frac{1}{2} X^{cd} \partial_{cd} Y^{ab} - \frac{3}{2} X^{[ab} \partial_{cd} Y^{cd]} - (X \leftrightarrow Y).
\]
The only geometric object in $\text{SL}(N)$ U-gravity is a metric, or U-metric, which is a generic non-degenerate $N \times N$ symmetric matrix, obeying surely the section condition,

$$M_{ab} = M_{ba} = M_{(ab)}.$$

Like in Riemannian geometry, the U-metric with its inverse, $M^{ab}$, may freely lower or raise the positions of the $N$-dimensional $\text{SL}(N)$ vector indices.

**Integral measure.**

While the U-metric has no extra weight, its determinant, $M \equiv \det(M_{ab})$, acquires an extra weight, $\omega = 4 - N$. The duality invariant integral measure is then

$$|M|^{\frac{1}{4-N}}.$$
**U-metric.**

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Semi-covariant derivative and semi-covariant Riemann curvature.

We define a semi-covariant derivative,

\[ \nabla_{cd} T^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} := \partial_{cd} T^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} + \frac{1}{2} \left( \frac{1}{2} p - \frac{1}{2} q + \omega \right) \Gamma^{e}_{cde} T^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} - \sum_{i=1}^{p} T^{a_1 \ldots e \ldots a_p b_1 b_2 \ldots b_q} \Gamma^{a_i}_{cde} + \sum_{j=1}^{q} \Gamma^{e}_{cdb} T^{a_1 a_2 \ldots a_p b_1 \ldots e \ldots b_q} , \]

and a semi-covariant Riemann curvature,

\[ S_{abcd} := 3 \partial_{[ab} \Gamma_{e][cd]}^{e} + 3 \partial_{[cd} \Gamma_{e][ab]}^{e} + \frac{1}{4} \Gamma^{abe}_{e} \Gamma^{cdf}_{e} + \frac{1}{2} \Gamma^{abe}_{e} \Gamma^{cdf}_{e} + \Gamma_{ab[c} \Gamma_{d]ef}^{f} + \Gamma_{cd[a} \Gamma_{b]ef}^{f} + \Gamma_{ea[c} \Gamma_{d]fb}^{e} - \Gamma_{eb[c} \Gamma_{d]fa}^{e} . \]
We define a semi-covariant derivative,
\[ \nabla_{cd} T^{a_1a_2\cdots a_p b_1b_2\cdots b_q} := \partial_{cd} T^{a_1a_2\cdots a_p b_1b_2\cdots b_q} + \frac{1}{2} \left( \frac{1}{2} p - \frac{1}{2} q + \omega \right) \Gamma^{e}_{cde} T^{a_1a_2\cdots a_p b_1b_2\cdots b_q} 
\]
\[ - \sum_{i=1}^{p} T^{a_1\cdots e\cdots a_p b_1b_2\cdots b_q} \Gamma^{ae}_{cde} + \sum_{j=1}^{q} \Gamma^{e}_{cdbj} T^{a_1a_2\cdots a_p b_1\cdots e\cdots b_q}, \]
and a semi-covariant Riemann curvature,
\[ S_{abcd} := 3 \partial_{[ab} \Gamma_{e][cd]} e + 3 \partial_{[cd} \Gamma_{e][ab]} e + \frac{1}{4} \Gamma_{abe} e \Gamma_{cdf} f + \frac{1}{2} \Gamma_{abe} f \Gamma_{cdf} e 
\]
\[ + \Gamma_{[ab} \Gamma_{cde]} ef f + \Gamma_{[cd} \Gamma_{ae]} ef f + \Gamma_{ea} e \Gamma_{d]fb} f - \Gamma_{eb} e \Gamma_{d][fa} f. \]

The semi-covariant derivative obeys the Leibniz rule and annihilates the Kronecker delta symbol,
\[ \nabla_{cd} \delta_a^b = 0. \]
Semi-covariant derivative and semi-covariant Riemann curvature.

We define a semi-covariant derivative,
\[
\nabla_{cd} T_{a_1 a_2 \cdots a_p}^{b_1 b_2 \cdots b_q} := \partial_{cd} T_{a_1 a_2 \cdots a_p}^{b_1 b_2 \cdots b_q} + \frac{1}{2} \left( \frac{1}{2} p - \frac{1}{2} q + \omega \right) \Gamma_{cde}^{e} T_{a_1 a_2 \cdots a_p}^{b_1 b_2 \cdots b_q} \\
- \sum_{i=1}^{p} T_{a_1 \cdots a_p}^{b_1 b_2 \cdots b_q} \Gamma_{cde}^{a_i} + \sum_{j=1}^{q} \Gamma_{cde}^{b_j} T_{a_1 a_2 \cdots a_p}^{b_1 \cdots e \cdots b_q},
\]
and a semi-covariant Riemann curvature,
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+ \Gamma_{ab[c}^{e} \Gamma_{d]}^{ef} + \Gamma_{cd[a}^{e} \Gamma_{b]}^{ef} + \Gamma_{ea[c}^{f} \Gamma_{d]}^{fb} - \Gamma_{eb[c}^{f} \Gamma_{d]}^{fa}.
\]
A crucial defining property of the semi-covariant Riemann curvature is that, under arbitrary transformation of the connection it transforms as total derivative,
\[
\delta S_{abcd} = 3 \nabla_{[ab} \delta \Gamma_{e][cd]}^{e} + 3 \nabla_{[cd} \delta \Gamma_{e][ab]}^{e}.
\]
**Semi-covariant derivative and semi-covariant Riemann curvature.**

We define a semi-covariant derivative,

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\nabla_{cd} T^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} := \partial_{cd} T^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} + \frac{1}{2} ( \frac{1}{2} p - \frac{1}{2} q + \omega ) \Gamma_{cde}^e T^{a_1 a_2 \ldots a_p b_1 b_2 \ldots b_q} \\
- \sum_{i=1}^{p} T^{a_1 \ldots \cdot a_p b_1 b_2 \ldots b_q} \Gamma_{cde}^i + \sum_{j=1}^{q} \Gamma_{cdb}^e T^{a_1 a_2 \ldots a_p b_1 \ldots \cdot e \ldots \cdot b_q},
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\[
S_{abcd} := 3 \partial_{[ab} \Gamma_{e][cd]}^e + 3 \partial_{[cd} \Gamma_{e][ab]}^e + \frac{1}{4} \Gamma_{abe}^e \Gamma_{cdf}^f + \frac{1}{2} \Gamma_{abe}^f \Gamma_{cdf}^e \\
+ \Gamma_{ab[e} \Gamma_{d]ef}^f + \Gamma_{cd[a} \Gamma_{b]ef}^f + \Gamma_{ea[c} \Gamma_{d]fb}^e - \Gamma_{eb[c} \Gamma_{d]fa}^e.
\]

Further, the semi-covariant Riemann curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature, including the Bianchi identity,

\[
S_{abcd} = S_{[ab][cd]} = S_{cdab}, \quad S_{[abc]d} = 0.
\]
Connection.

\[
\Gamma_{abcd} = A_{abcd} + \frac{1}{2}(A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad})
\]

\[
+ \frac{1}{N-2} (M_{ac}A^e_{(bd)e} - M_{ad}A^e_{(bc)e} + M_{bd}A^e_{(ac)e} - M_{bc}A^e_{(ad)e}) ,
\]

where

\[
A_{abcd} := -\frac{1}{2} \partial_{ab} M_{cd} + \frac{1}{2(N-4)} M_{cd} \partial_{ab} \ln |M| .
\]

This connection is the unique solution to the following five constraints:

1. \[
\Gamma_{abcd} + \Gamma_{abdc} = 2A_{abcd} ,
\]
2. \[
\Gamma^{abcd} + \Gamma^{bacd} = 0 ,
\]
3. \[
\Gamma^{abcd} + \Gamma^{bca} + \Gamma^{cab} = 0 ,
\]
4. \[
\Gamma^{cab} + \Gamma^{cba} = 0 ,
\]
5. \[
P^{abcd}_{efgh} \Gamma_{efgh} = 0 .
\]
\[ \Gamma_{abcd} = A_{abcd} + \frac{1}{2} (A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \]

\[ + \frac{1}{N-2} \left( M_{ac} A_{e(bd)e} - M_{ad} A_{e(bc)e} + M_{bd} A_{e(ac)e} - M_{bc} A_{e(ad)e} \right), \]

where

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2. \[ \Gamma_{abc}^d + \Gamma_{bac}^d = 0, \]
3. \[ \Gamma_{abc}^d + \Gamma_{bca}^d + \Gamma_{cab}^d = 0, \]
4. \[ \Gamma_{cab}^c + \Gamma_{cba}^c = 0, \]
5. \[ P_{abcd}^{efgh} \Gamma_{efgh} = 0. \]

Eq. (1) is equivalent to the U-metric compatibility condition,

\[ \nabla_{ab} M_{cd} = 0. \]
\[
\Gamma_{abcd} = A_{abcd} + \frac{1}{2} (A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \\
\quad + \frac{1}{N-2} \left( M_{ac} A^{e}_{(bd)e} - M_{ad} A^{e}_{(bc)e} + M_{bd} A^{e}_{(ac)e} - M_{bc} A^{e}_{(ad)e} \right),
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A_{abcd} := -\frac{1}{2} \partial_{ab} M_{cd} + \frac{1}{2(N-4)} M_{cd} \partial_{ab} \ln |M|.
\]

This connection is the unique solution to the following five constraints:

\[
\Gamma_{abcd} + \Gamma_{abdc} = 2A_{abcd}, \quad (1)
\]

\[
\Gamma_{abc}^{d} + \Gamma_{bac}^{d} = 0, \quad (2)
\]

\[
\Gamma_{abc}^{d} + \Gamma_{bca}^{d} + \Gamma_{cab}^{d} = 0, \quad (3)
\]

\[
\Gamma_{cab}^{c} + \Gamma_{cba}^{c} = 0, \quad (4)
\]

\[
P_{abcd}^{efgh} \Gamma_{efgh} = 0. \quad (5)
\]

Eq. (2) is natural from the skew-symmetric nature of the coordinates, \( x^{(ab)} = 0 \) and hence \( \partial_{(ab)} = \nabla_{(ab)} = 0. \)
Geometric Constitution of U-gravity

Connection.

\[ \Gamma_{abcd} = A_{abcd} + \frac{1}{2} (A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \]

\[ + \frac{1}{N-2} \left( M_{ac} A^{(bd)e} - M_{ad} A^{(bc)e} + M_{bd} A^{(ac)e} - M_{bc} A^{(ad)e} \right), \]

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4. \( \Gamma_{cab}^{\ c} + \Gamma_{cba}^{\ c} = 0 \) \( \quad (4) \)
5. \( \mathcal{P}_{abcd}^{\ efg} \Gamma_{efgh} = 0 \) \( \quad (5) \)

Eqs.\( (3, 4) \) make the semi-covariant derivative compatible with the generalized Lie derivative and the generalized bracket: \( \hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla), [X, Y]_G(\partial) = [X, Y]_G(\nabla). \)
Connection.

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\Gamma_{abcd} = A_{abcd} + \frac{1}{2} (A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \\
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\]
\[
\mathcal{P}_{abcd}^{efgh} \Gamma_{efgh} = 0. \quad (5)
\]

Eq.(5) is a projection condition which ensures the uniqueness.
Projection operator.

The above eight-index projection operator is explicitly,

\[
\mathcal{P}_{abcd}^{klmn} = \frac{1}{2} \delta^{[k}_{[a} \delta^{\ell}_{\ b]} \delta^{[m}_{[c} \delta^{n]}_{d]} + \frac{1}{2} \delta^{[k}_{[c} \delta^{\ell}_{\ d]} \delta^{[m}_{[a} \delta^{n]}_{b]} + \frac{1}{2} M_c[a \delta^m_{\ b]} M^{n[k} \delta^{\ell]}_d - \frac{1}{2} M_c[a \delta^m_{\ b]} M^{n[k} M^{\ell]}_d M^{m]}_d
\]

\[
+ \frac{1}{N-2} \left( \delta^{n}_{[a} M_b[c M^m[k} \delta^{\ell]}_d + \delta^{n}_{[c} M_d[a M^m[k} \delta^{\ell]}_b] - M_c[a M_b]_d M^m[k} M^{\ell]} n \right),
\]

which satisfies

\[
\mathcal{P}_{abcd}^{pqrs} \mathcal{P}_{pqrs}^{klmn} = \mathcal{P}_{abcd}^{klmn}, \quad \mathcal{P}_{abc}^{sklmn} = 0, \quad \mathcal{P}_{ab[cd]}^{klmn} = \mathcal{P}_{cd[ab]}^{klmn}.
\]
The above eight-index projection operator is explicitly,

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+ \frac{1}{N-2} \left( \delta_{[a}^{n} \mathcal{M}_{b]}^{[c} \mathcal{M}^{m[k} \delta_{d]}^{\ell]} + \delta_{[c}^{n} \mathcal{M}_{d]}^{[a} \mathcal{M}^{m[k} \delta_{b]}^{\ell]} - \mathcal{M}_{c[a} \mathcal{M}_{b]} \mathcal{M}^{m[k} \mathcal{M}^{n]} \right). \]

Crucially, the projection operator dictates the anomalous terms under diffeomorphism:

\[ (\delta X - \hat{\mathcal{L}}X)(\nabla_{ab} T^{c_1 \cdots c_p} d_1 \cdots d_q) = - \sum_{i=1}^{p} T^{c_1 \cdots c_i} d_1 \cdots d_q \Omega_{ab} e^{c_i} + \sum_{j=1}^{q} \Omega_{ab} e^{c_j} T^{c_1 \cdots c_p} d_1 \cdots d_q, \]

\[ (\delta X - \hat{\mathcal{L}}X) S_{abcd} = 2 \nabla_{e[a} \Omega_{b] cd} e^{e} + 2 \nabla_{e[c} \Omega_{d] ab} e^{e}, \]

where

\[ \Omega_{abcd} = \mathcal{P}_{abcd}^{klmn} \partial_{kl} \partial_{me} X^{ne}. \]
Complete covariantizations.

The semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized by (anti-)symmetrizing or contracting the $\text{SL}(N)$ vector indices properly,

\[
\nabla_{[ab} T_{c_1 c_2 \cdots c_q]} , \quad \nabla_{ab} T^a , \quad \nabla^a b T_{[ca]} + \nabla^a c T_{[ba]} , \quad \nabla^a b T_{(ca)} - \nabla^a c T_{(ba)} ,
\]

\[
\nabla_{ab} T^{abc_1 c_2 \cdots c_q} \quad \text{(divergence)} , \quad \nabla_{ab} \nabla^{[ab} T^{c_1 c_2 \cdots c_q]} \quad \text{(Laplacian)} ,
\]

and

\[
S_{ab} := S_{acb}^c = S_{ba} \quad \text{("Ricci" curvature)} ,
\]

\[
S := M^{ab} S_{ab} = S_{ab}^{ab} \quad \text{(scalar curvature)}.
\]
**Action.**

The action of $\text{SL}(N)$ U-gravity is given by the fully covariant scalar curvature,

$$\int_\Sigma M^{\frac{1}{4-N}} S,$$

where the integral is taken over a section, $\Sigma$.

**The Einstein equation of motion.**

The equation of motion corresponds to the vanishing of the ‘Einstein’ tensor,

$$S_{ab} + \frac{1}{2(N-4)} M_{ab} S = 0.$$

Diffeomorphism symmetry of the action implies a conservation relation,

$$\nabla^c [a S_{b}c + \frac{3}{8} \nabla_{ab} S = 0.$$
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$$\nabla^c [a S_b]_c + \frac{3}{8} \nabla_{ab} S = 0.$$
Two inequivalent sections.

Up to $\text{SL}(N)$ rotations, there exist two inequivalent solutions to the section condition:

1. $\Sigma_{N-1}$ is an $(N-1)$-dimensional section given by
   \[ \partial_{\alpha\beta} = 0, \quad \partial_{\alpha N} \neq 0, \]
   where $\alpha, \beta = 1, 2, \cdots, N-1$.

2. $\Sigma_3$ is a three-dimensional section characterized by
   \[ \partial_{\mu i} = 0, \quad \partial_{ij} = 0, \quad \partial_{\mu\nu} \neq 0, \]
   where $\mu, \nu = 1, 2, 3$ and $i, j = 4, 5, \cdots, N$. We may further dualize
   \[ \tilde{x}_\mu \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho} x^{\nu\rho}, \quad \tilde{\partial}^\mu \tilde{x}_\nu = \delta^\mu_\nu. \]
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Geometric Constitution of U-gravity

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For a triplet of arbitrary functions, we note
\[ \partial_{[ab} \Phi \partial_{c][d} \Phi' \partial_{ef]} \Phi'' = 0 \quad \text{on} \quad \Sigma_{N-1}, \quad \partial_{[ab} \Phi \partial_{c][d} \Phi' \partial_{ef]} \Phi'' \neq 0 \quad \text{on} \quad \Sigma_3. \]

Since this is an $\text{SL}(N)$ covariant statement, the two sections are inequivalent.
Two inequivalent sections.

Up to $\text{SL}(N)$ rotations, there exist two inequivalent solutions to the section condition:

1. $\Sigma_{N-1}$ is an $(N - 1)$-dimensional section given by

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where $\alpha, \beta = 1, 2, \cdots, N - 1$.

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where $\mu, \nu = 1, 2, 3$ and $i, j = 4, 5, \cdots, N$. We may further dualize

$$\tilde{x}_\mu \equiv \frac{1}{2} \varepsilon_{\mu \nu \rho} x^{\nu \rho}, \quad \tilde{\partial}^\mu \tilde{x}_\nu = \delta^\mu_\nu.$$

Note: in the case of $\text{SL}(5)$, they correspond to $\mathcal{M}$-theory and type IIB theory respectively (with the compactification on seven-manifold).
Riemannian reductions.

1. Reduction to $\Sigma_{N-1}$ through $(N-1)$-dimensional Riemannian metric, $g_{\alpha\beta}$, a vector, $v^\alpha$, and a scalar, $\phi$,

$$M_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|} (-e^\phi + v^2) \end{pmatrix}, \quad |M|^{1_{4-N}} = e^{\frac{1}{4-N}\phi} \sqrt{|g|}.$$ 

The U-gravity scalar curvature reduces upon the section, $\Sigma_{N-1}$, to

$$S_{\Sigma_{N-1}} = 2 e^{-\phi} \left[ R_g - \frac{(N-3)(3N-8)}{4(N-4)^2} \partial_\alpha \phi \partial^\alpha \phi + \frac{N-2}{N-4} \Delta \phi + \frac{1}{2} e^{-\phi} (\nabla_\alpha v^\alpha)^2 \right].$$

The vector field can be dualized to an $(N-2)$-form potential.

2. Reduction to $\Sigma_3$, employing ‘dual’ upside-down notations,

$$M^{ab} = \begin{pmatrix} \tilde{g}^{\mu\nu} & -\tilde{v}^{i\mu} \\ -\tilde{v}^{i\nu} & \sqrt{\tilde{g}} (e^{-\tilde{\phi}} \tilde{M}^{ij} + \tilde{v}^i \lambda \tilde{v}^j \lambda) \end{pmatrix}, \quad |M|^{1_{4-N}} = e^{\frac{N-3}{4-N} \tilde{\phi}} \sqrt{\tilde{g}}.$$ 

The U-gravity scalar curvature reduces upon the section, $\Sigma_3$, to

$$S_{\Sigma_3} = -2 R_{\tilde{g}} + \frac{(N-3)(3N-8)}{2(N-4)^2} \tilde{\partial}_\mu \tilde{\phi} \tilde{\partial}_\mu \tilde{\phi} - \frac{4(N-3)}{N-4} \tilde{\Delta} \tilde{\phi} - \frac{1}{2} \tilde{\partial}_\mu \tilde{M}_{ij} \tilde{\partial}_\mu \tilde{M}^{ij} + e^{\tilde{\phi}} \tilde{M}^{ij} \nabla_\mu \tilde{v}^i \nabla_\nu \tilde{v}^j \nu,$$

which manifests $\text{SL}(N-3)$ S-duality.
**Geometric Constitution of U-gravity**

1. **Riemannian reductions.**

   Reduction to $\Sigma_{N-1}$ through $(N-1)$-dimensional Riemannian metric, $g_{\alpha\beta}$, a vector, $v^{\alpha}$, and a scalar, $\phi$,

   $$M_{ab} = \begin{pmatrix}
   \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_{\alpha} \\
   v_{\beta} & \sqrt{|g|} (-e^{\phi} + v^{2})
   \end{pmatrix}, \quad |M|^{\frac{1}{4-N}} = e^{\frac{1}{4-N}\phi} \sqrt{|g|}.$$

   The U-gravity scalar curvature reduces upon the section, $\Sigma_{N-1}$, to

   $$S|_{\Sigma_{N-1}} = 2e^{-\phi} \left[ R_{\tilde{g}} - \frac{(N-3)(3N-8)}{4(N-4)^2} \partial_{\alpha} \phi \partial^{\alpha} \phi + \frac{N-2}{N-4} \Delta \phi + \frac{1}{2} e^{-\phi} (\nabla_{\alpha} v^{\alpha})^{2} \right].$$

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   \end{pmatrix}, \quad |M|^{\frac{1}{4-N}} = e^{\frac{N-3}{4-N}\tilde{\phi}} \sqrt{|\tilde{g}|}.$$

   The U-gravity scalar curvature reduces upon the section, $\Sigma_{3}$, to

   $$S|_{\Sigma_{3}} = -2R_{\tilde{g}} + \frac{(N-3)(3N-8)}{2(N-4)^2} \tilde{\phi}^{\mu} \tilde{\phi}_{\mu} + \frac{4(N-3)}{N-4} \tilde{\Delta} \tilde{\phi} - \frac{1}{2} \tilde{\partial}^{\mu} \tilde{M}_{ij} \tilde{\partial}_{\mu} \tilde{M}^{ij} + e^{\phi} \tilde{M}_{ij} \nabla^{\mu} \tilde{v}_{i} \nabla_{\nu} \tilde{v}_{j},$$

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which manifests $\text{SL}(N-3)$ S-duality.
Non-Riemannian backgrounds.

When the upper left \((N - 1) \times (N - 1)\) block of the U-metric is degenerate – where \(g_{\alpha \beta} \sqrt{|g|}\) might have been positioned – the Riemannian metric ceases to exist upon \(\Sigma_{N-1}\). Nevertheless, \(SL(N)\) U-gravity has no problem with describing such a non-Riemannian background, as long as the whole \(N \times N\) U-metric is non-degenerate.

Similarly upon \(\Sigma_3\), U-gravity may allow the upper left \(3 \times 3\) block of the inverse of the U-metric to be degenerate.
Conclusion
Conclusion

Summary

- Riemannian geometry is for particle theory. String theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- Novel differential geometric ingredients:
  - Spacetime being extended-yet-gauged (section condition)
  - Semi-covariant derivative and semi-covariant curvature
  - Complete covariantizations of them through ‘projection’.

- $\mathcal{N} = 2 \, D = 10$ SDFT has been constructed to the full order in fermions. The theory unifies IIA and IIB SUGRAs, and allows non-Riemannian ‘metric-less’ backgrounds.

- Precisely parallel formulation for $\text{SL}(N)$ U-duality under the name, $\text{U-gravity}$.
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**Conclusion**

**Outlook**

- Further study and classification of the non-Riemannian, ‘metric-less’ backgrounds.
- Quantization of the string action on doubled-yet-gauged spacetime.
- $\mathcal{O}(10,10)$ covariant Killing spinor equation $\rightarrow$ SUSY and T-duality are compatible. Further generalization of ‘Generalized Complex structure’ or ‘G-structure’. Hitchin, Gualtieri, Gauntlett, Tomasiello, Rosa
- **DFT cosmology**? Cosmological constant reads $\Lambda e^{-2d} = \Lambda \sqrt{-\bar{g}} e^{-2\phi}$.
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