Fiber-Base duality
and
Global Symmetry Enhancement

Futoshi Yagi (KIAS)

Based on arXiv: 1411.2450
V. Mitev, E.Pomoni, M.Taki, FY

§ 1 Introduction

Five dimensional gauge theory

Perturbatively non-renormalizable.

(Gauge coupling constant) = $[\text{Mass}]^{-1/2}$
Weakly coupled at Low energy
Strongly coupled at High energy

Is it well defined quantum theory?

Can we compute anything non-trivial??

Five dimensional *N=1* supersymmetric gauge theory

Supersymmetry + gauge symmetry

⇒ Possible low energy effective theory at Coulomb phase is highly constrained

Quantum correction to gauge coupling is 1-loop exact

5D *N=1* SUSY SU(2) gauge theory with N_f flavor

UV fixed point exists for $N_f \leqq 7$ ’96 Seiberg

(N_f =8 is subtle)

- **1-loop computation**

$$\frac{1}{g_{\mathrm{eff}}(a)^2} = \frac{1}{g_0{}^2} + (8 - N_f)|a| \qquad (a \geq 0)$$

a : Coulomb moduli parameter

- **Brane setup is proposed.**

(String theory gives “UV description”)

5D *N=1* SUSY SU(2) gauge theory with N_f flavor

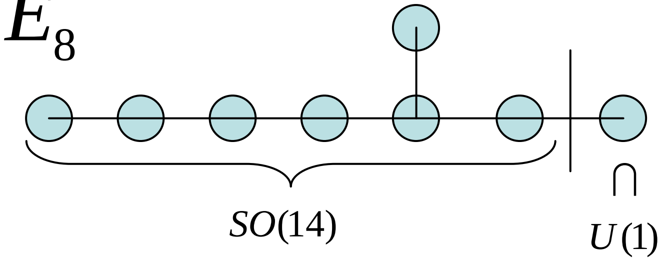
Global symmetry enhancement at UV fixed point

'96 Seiberg

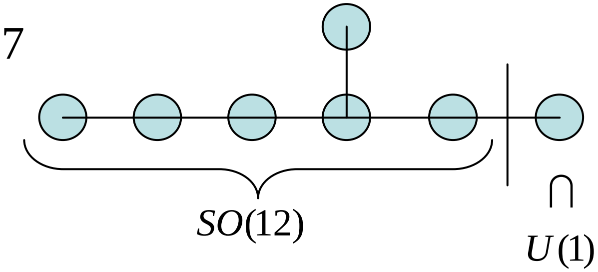
$$\underbrace{SO(2N_f)}_{\text{Flavor symmetry}}\times \underbrace{U(1)}_{\text{Instanton symmetry}}\subset E_{N_f+1}$$

cf. Heterotic E_8 x E_8

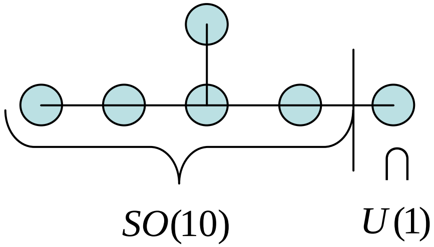
E_8



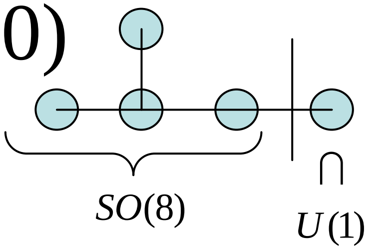
E_7



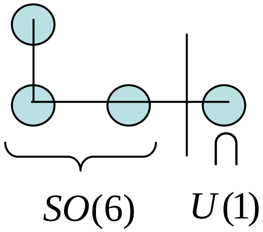
E_6



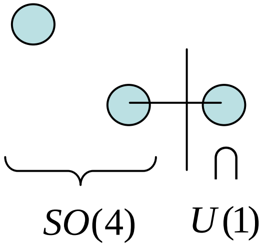
$E_5 = SO(10)$



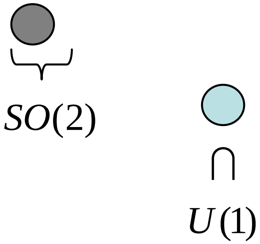
$E_4 = SU(5)$



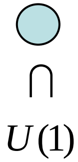
$E_3 = SU(2) \times SU(3)$



$E_2 = U(1) \times SU(2)$



$E_1 = SU(2)$



Goal of this talk

Deeper understanding of the
global symmetry enhancement of
5D $N=1$ SU(2) gauge theory with N_f flavor

Plan of this talk

√ **§ 1 Introduction**

§ 2 Problem to be solved

§ 3 Pure SU(2) case (N_f=0)

§ 4 Higher flavor generalization (N_f>0)

§ 5 Summary

§ 2 Problem to be solved

Superconformal index

$$I(x,y,q,m)=Tr\left[(-1)^F e^{-\beta\{Q,S\}} x^{2(j_1+R)} y^{2j_2} q^k \prod_{i=1}^{N_f} m_i^{H_i}\right]$$

F : Fermion number

Q,S : Superconformal $(Q=Q^{A=1}_{m=2}, S=Q^{\dagger})$

$(\{Q,S\}=\varepsilon_0-2j_1-3R \qquad \varepsilon_0$: Dilatation$)$

j_1, j_2 : Spin

R : R symmetry

k : Instanton number

H_i : Flavor symmetry

-Defined for superconformal field theory realized at the UV fixed point

-Only BPS particle (multiplet invariant under part of SUSY) contributes (Generalization of Witten Index)

- Superconformal index should respect global symmetry of the theory

Superconformal index
→ Partition function on S^4 x S^1

Superconformal index

$$I(x,y,q,m)=\int[da]\,Z^{5D}{}_{Nek}(q,m,a;\varepsilon_1,\varepsilon_2)Z^{5D}{}_{Nek}(q^{-1},m,a;\varepsilon_1,\varepsilon_2)$$

$q=(\beta\Lambda_{4D})^4$: instanton factor

m : mass parameters

a : Coulomb moduli parameter

$x=e^{-\beta(\varepsilon_1+\varepsilon_2)/2},\quad y=e^{-\beta(\varepsilon_1-\varepsilon_2)/2}\quad(\beta:S^1\text{ circumference})$

’12 H.Kim, S.Kim, K.Lee
’14 C.Hwang, J.Kim, S.Kim, J.Park

$Z^{5D}{}_{Nek}(q,m,a;\varepsilon_1,\varepsilon_2)$: **Five dimensional Nekrasov partition function**

Superconformal index

$$I(x,y,q,m)=\int[da]\,Z^{5D}{}_{Nek}(q,m,a;\varepsilon_1,\varepsilon_2)Z^{5D}{}_{Nek}(q^{-1},m,a,\varepsilon_1,\varepsilon_2)$$

is written in terms of the characters of E_{N_f+1}

’12 H.Kim, S.Kim, K.Lee
’14 C.Hwang, J.Kim, S.Kim, J.Park

Invariant under the Weyl transformation of E_{N_f+1}

e.g. For pure SYM $\quad E_1 = SU(2): \quad q \leftrightarrow q^{-1}$

$$\text{How about } Z^{5D}{}_{Nek}(q,m,a;\varepsilon_1,\varepsilon_2)?$$

Is Nekrasov partition function invariant ?

- **It should be invariant**

IR theory (prepotential) should respect enhanced symmetry at UV

Is Nekrasov partition function invariant ?

- **It should be invariant**

IR theory (prepotential) should respect enhanced symmetry at UV

$\Updownarrow$ **Paradox?**

- **Naïve computation implies No….?**

$e.g.$ For pure SYM

$$\mathrm{E}_1 = SU(2): \quad q \leftrightarrow q^{-1}$$

Instanton factor

$$Z^{5D}{}_{Nek}(q^{-1}, a, \varepsilon_1, \varepsilon_2) \neq Z^{5D}{}_{Nek}(q, a, \varepsilon_1, \varepsilon_2)?$$

Positive power in q

Is Nekrasov partition function invariant ?

✓ **· It should be invariant**

IR theory (prepotential) should respect enhanced symmetry at UV

⇕ **Paradox?**

~~**·Naïve computation implies No….?**~~

~~$e.g.$ For pure SYM~~

~~$E_1 = SU(2): \quad q \leftrightarrow q^{-1}$~~

~~Instanton factor~~

~~$Z^{5D}{}_{Nek}(q^{-1},a,\varepsilon_1,\varepsilon_2) \neq Z^{5D}{}_{Nek}(q,a,\varepsilon_1,\varepsilon_2)?$~~

~~Positive power in q~~

Resolution

5D Nekrasov partition function is invariant

$$Z^{5D}{}_{Nek}(q^{-1}, a'(a,q), \varepsilon_1, \varepsilon_2) = Z^{5D}{}_{Nek}(q, a, \varepsilon_1, \varepsilon_2)$$

Coulomb moduli parameter is also transformed under the E_{N_f+1} Weyl symmetry

Resolution

5D Nekrasov partition function is invariant

$$Z^{5D}{}_{Nek}(q^{-1},a'(a,q),\varepsilon_1,\varepsilon_2)=Z^{5D}{}_{Nek}(q,a,\varepsilon_1,\varepsilon_2)$$

Coulomb moduli parameter is also transformed under the E_{N_f+1} Weyl symmetry

How to understand / derive the transformation for Coulomb moduli parameter?

⇨ **Fiber-base duality**

§ 3 Pure SU(2) case

Two equivalent ways to realized 5D SU(2) gauge theory in string / M theory

M-theory on Calabi-Yau

Fiber-Base duality

Fiber CP^1 ⇔ Base CP^1

’97 Katz, Mayr, Vafa

IIB string with 5-branes

S-duality

D5 brane ⇔ NS5 brane

‘97 Aharony, Hanany, Kol

S-duality

~~Fiber-Base duality~~

and

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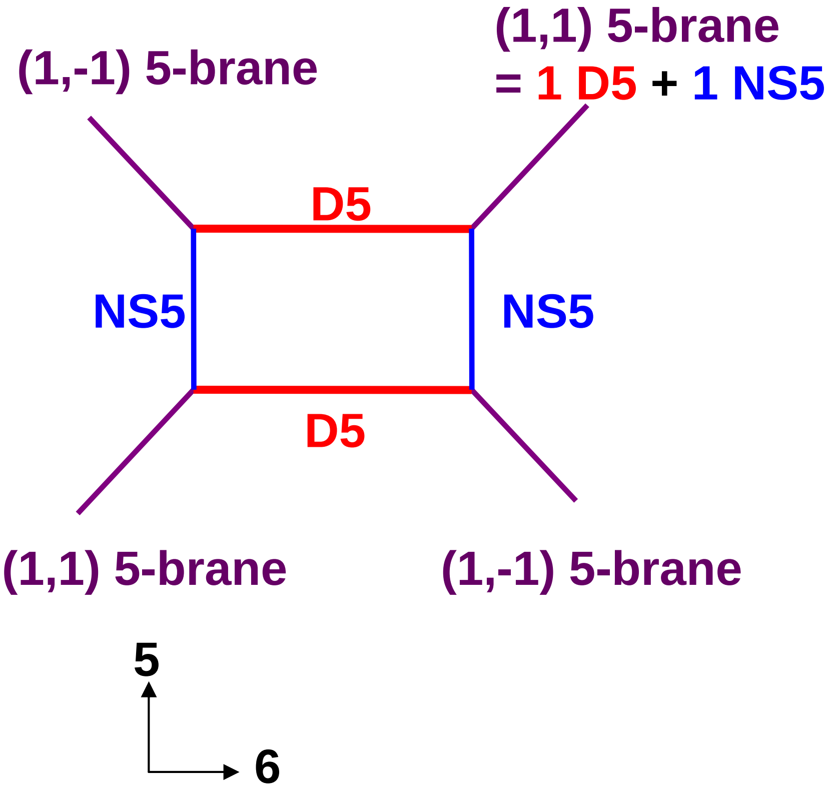
Pure SU(2) SYM



(1,-1) 5-brane

(1,1) 5-brane
= 1 D5 + 1 NS5

D5

NS5

NS5

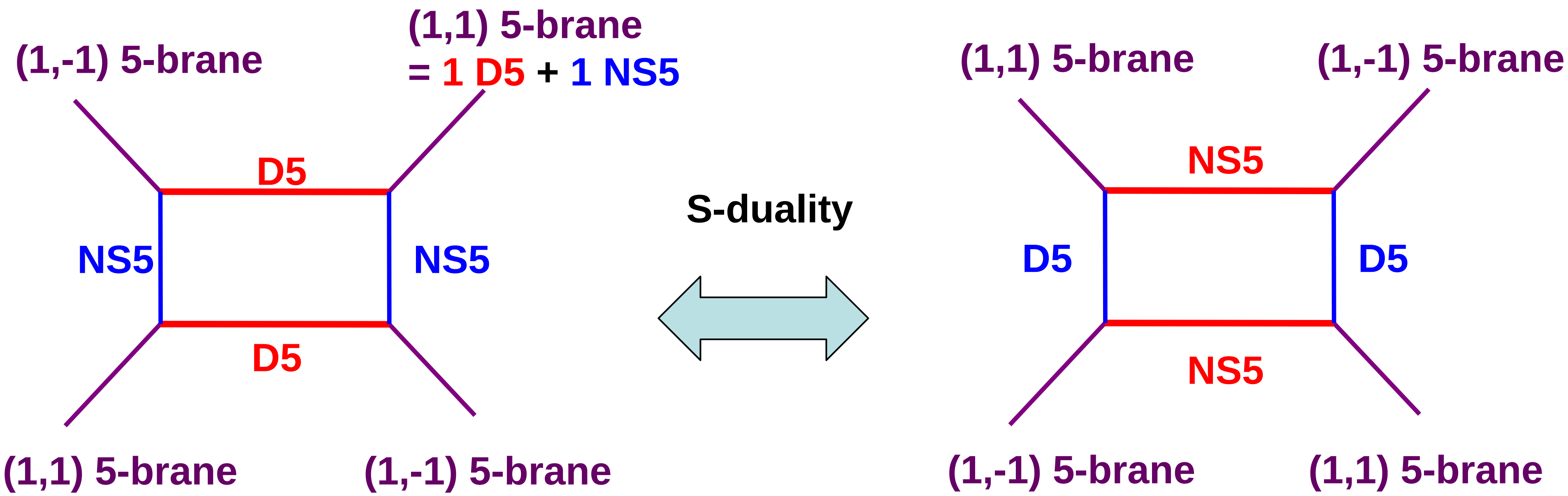
D5

(1,1) 5-brane

(1,-1) 5-brane

NS5	0 1 2 3 4 5
D5	0 1 2 3 4 6

S-duality for pure SU(2) SYM



Pure SU(2) SYM

(1,-1) 5-brane

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= 1 D5 + 1 NS5

D5

NS5

NS5

D5

(1,1) 5-brane

(1,-1) 5-brane

S-duality

Pure SU(2) SYM

(1,1) 5-brane

(1,-1) 5-brane

NS5

D5

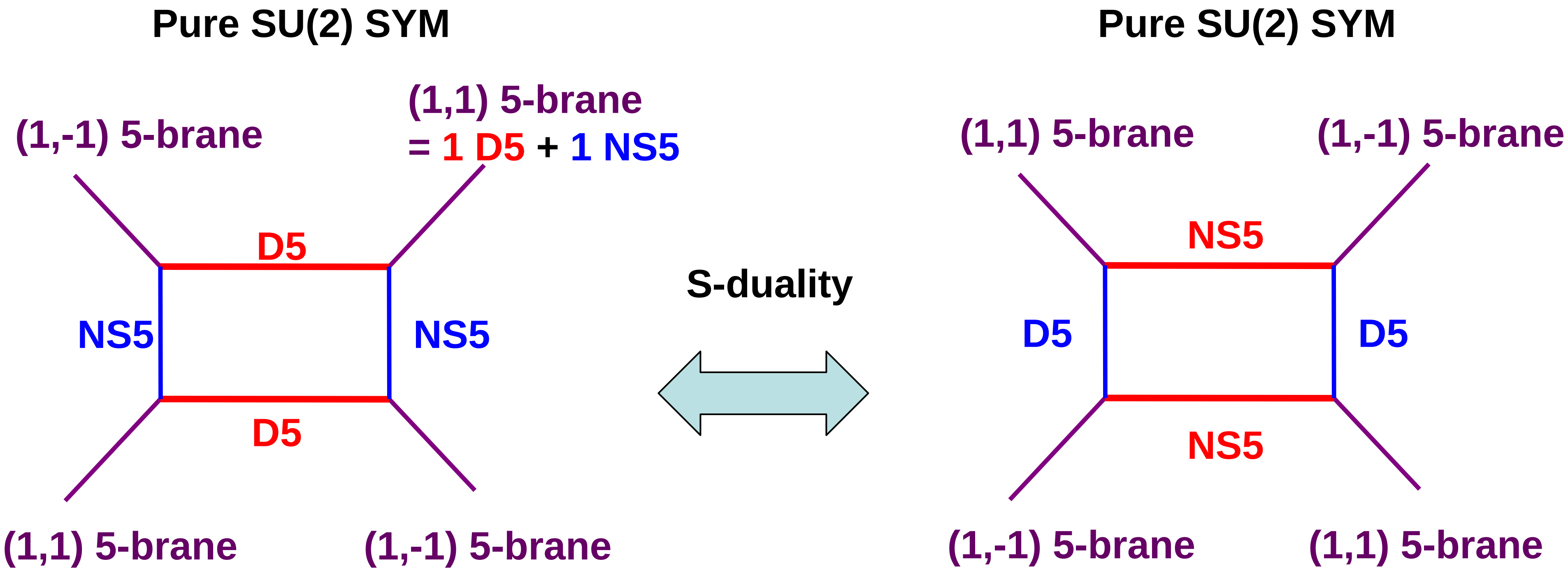
D5

NS5

(1,-1) 5-brane

(1,1) 5-brane

S-duality for pure SU(2) SYM

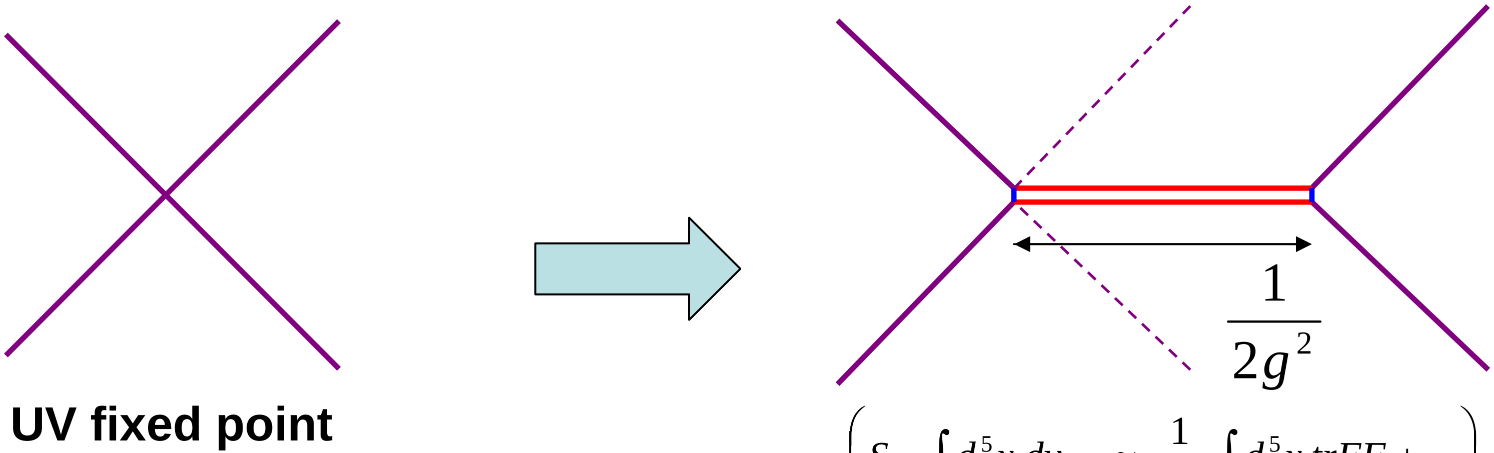


Gauge coupling
Coulomb moduli parameters

are transformed

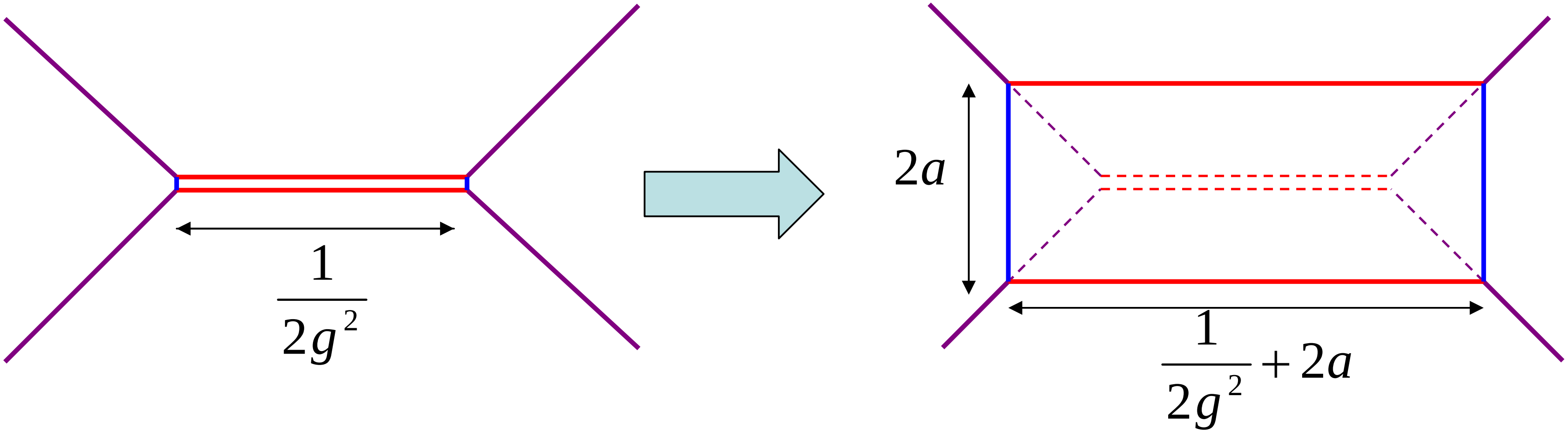
Can be read of
from the diagram

Global deformation (change the boundary condition) = Gauge coupling, (Mass parameter)



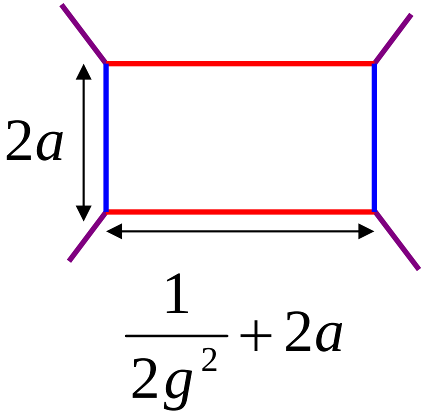
$$\left(S = \int d^5x\, dy \cdots \approx \frac{1}{2g^2} \int d^5x\, \mathrm{tr} FF + \cdots \right)$$

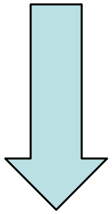
Local deformation (does not change the boundary condition) = Coulomb moduli parameter



S-duality transformation:

$$2a \Leftrightarrow \frac{1}{2g^2} + 2a$$





$$q \Rightarrow q^{-1} \quad \left(q = e^{-\frac{\beta}{2g_{5D}^2}} = (\beta\Lambda_{4D})^4 \right)$$

← **Weyl Symmetry for *E*$_1$ = *SU*(2) !!**

'97 Aharony,Hanany,Kol

$$a \Rightarrow a + \frac{1}{4g^2}$$

← **Coulomb moduli parameter is also transformed!**

The combination $\tilde{A} = q^{\frac{1}{4}} e^{-\beta a} = e^{-\beta\left(a+\frac{1}{8g^2}\right)}$ **is invariant under**

$$q \Rightarrow q^{-1}, \qquad a \Rightarrow a + \frac{1}{4g^2}$$

Nekrasov partition function should be rewritten as

$$Z^{5D}{}_{\mathrm{Nek}}(q, a, \varepsilon_1, \varepsilon_2) = Z_{\mathrm{pert}}(a, \varepsilon_1, \varepsilon_2) \sum_{k=0}^{\infty} Z_k(a, \varepsilon_1, \varepsilon_2) q^k$$

Original form

$$= \sum_{n=0}^{\infty} \tilde{Z}_n(q, \varepsilon_1, \varepsilon_2) \tilde{A}^n$$

New form

Nekrasov partition function for pure *SU*(2)

$$Z_{Nek}(q,\tilde{A};x,y)=1+\frac{(1+x^2)y}{(x-y)(1-xy)}\chi_{[1]}(q)\tilde{A}^2$$
$$+\left[\frac{(x^2+x+x^4y+y+x^2y+x^3y^2+xy^2)y}{(1-xy)(1+xy)(x-y)(x+y)}\right.$$
$$\left.\times\left(\frac{x^4+1}{x^2}+\frac{x}{(1-xy)(x-y)}\chi_{[2]}(q)\right)\right]\tilde{A}^4+\cdots$$

$$xy=e^{-\beta\varepsilon_1},\quad \frac{y}{x}=e^{+\beta\varepsilon_2},$$

Character of $E_1=SU(2)$: $\chi_{[\ell]}(q)\equiv\sum_{m=0}^{\ell}q^{\ell-2m}$

Manifestly E_1 invariant!!

Comment 1

The new variable $\tilde{A}$ is essentially the effective coupling constant in the decompactified five dimensional theory

$$\tilde{A} = q^{\frac{1}{4}} e^{-\beta a} = e^{-\frac{\beta}{8}\left(\frac{1}{g^2}+8a\right)} = e^{-\frac{\beta}{8 g_{\mathrm{eff}}(a)^2}} \qquad (a > 0)$$

We have reinterpreted this variable as “shifted Coulomb moduli parameter”

$$\tilde{A} = q^{\frac{1}{4}} e^{-\beta a} = e^{-\beta\left(a+\frac{1}{8g^2}\right)} \equiv e^{-\beta a'}$$

Comment 2

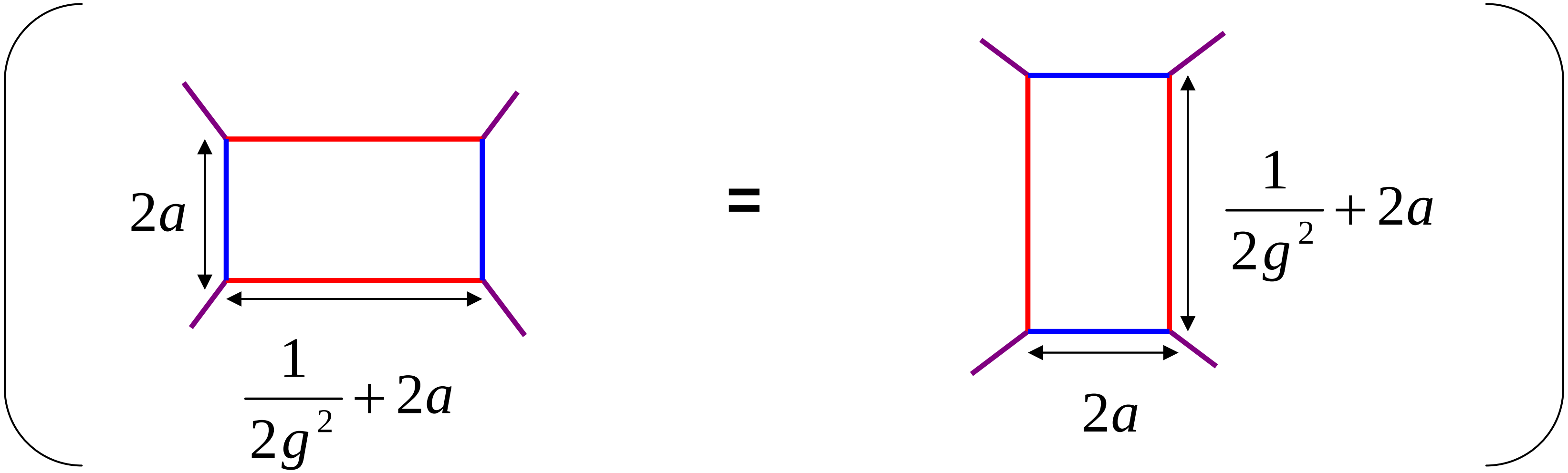
Instanton/perturbative part of the Nekrasov partition function is NOT invariant under the E_1 Weyl symmetry separately.

$$\because e^{-\beta a} \Rightarrow e^{-\beta a} q^{\frac{1}{2}}$$

(1-loop part → instanton contribution)

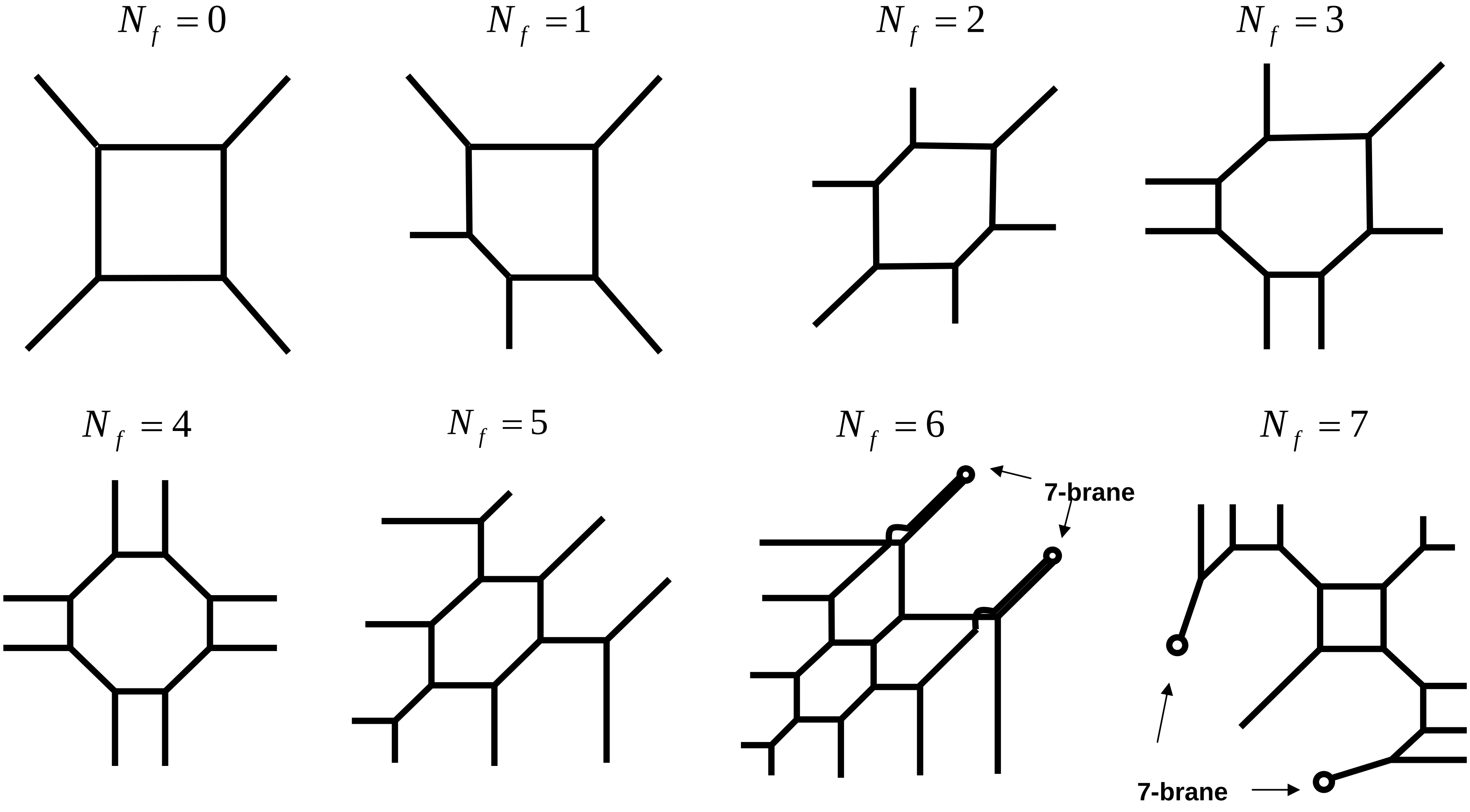
Comment 3

By using the topological vertex method, we can show the invariance under E_1 invariance **at all order** **at least in the case of** $\varepsilon_1 = -\varepsilon_2 \quad (x=1)$



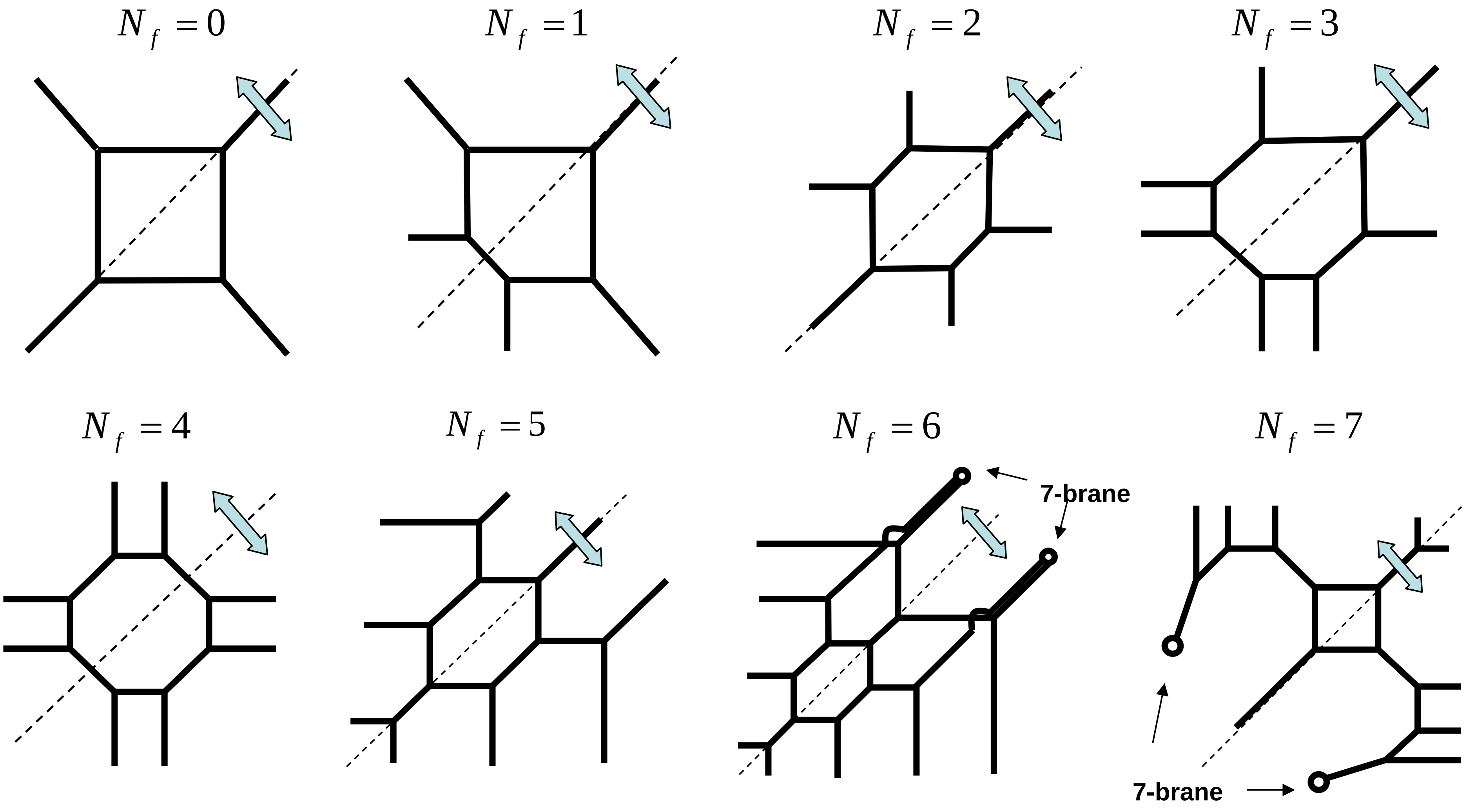
§ 4 Higher flavor generalization

Brane setup



’09 Benini-Benvenuti-Tachikawa

Brane setup



’09 Benini-Benvenuti-Tachikawa

Weyl symmetry of flavor SO(2N_f) (act on N_f masses)

+

Symmetry induced from S-duality (exchange masses and gauge coupling)

⇒

Weyl symmetry of enhanced E_{N_f+1} symmetry

We obtain manifestly E_{Nf+1} invariant Nekrasov partition function

$$\boxed{Z_{Nek}(X,\tilde{A};x,y)=1-\frac{xy}{(1-xy)(x-y)}\left(\sum_R \chi_R^{E_{N_f+1}}(X)+c(x,y)\right)\tilde{A}+\cdots}$$

$$\tilde{A}=e^{-\beta a}q^{\frac{2}{8-N_f}}=e^{-\beta\left(a+\frac{1}{(8-N_f)g^2}\right)}$$

$\chi_R^{E_{N_f+1}}(X)$ **:**character of E_{Nf+1} in the representation *R*

Summary

SO($2N_f$) + S-duality = E_{N_f+1}

Nekrasov partition function is invariant under E_{N_f+1}

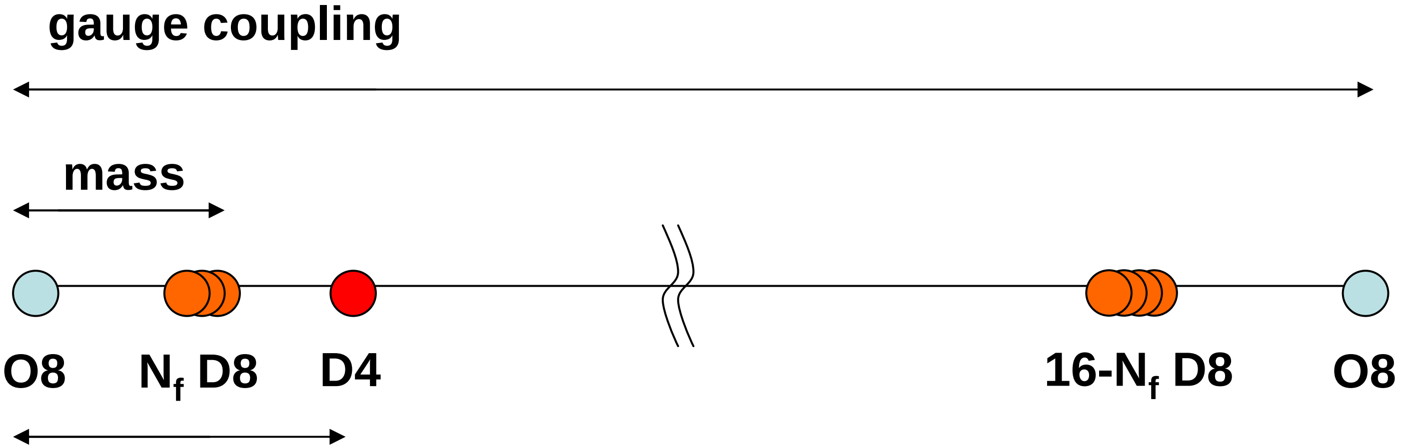
Future work

Related to E_n as U-duality group ? ?

Brane setup

'96 Seiberg

Type IIA String on S^1/Z_2 (line segment)



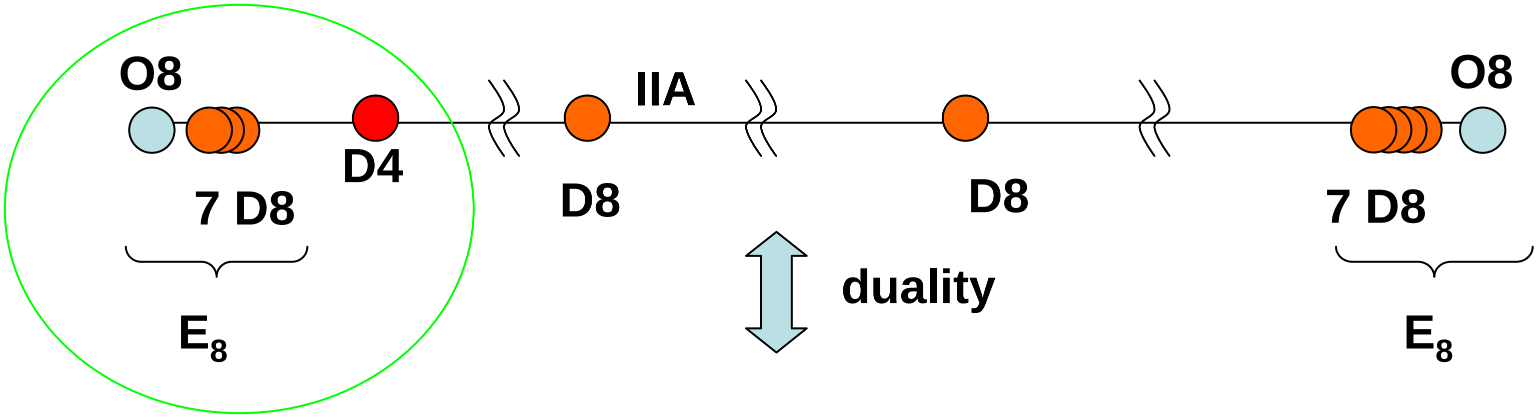
Coulomb moduli (vacuum expectation value of adjoint scalar)

World volume theory on D4 = Sp(1) gauge theory

D4 - O8: Vector multiplet (gauge field)
D4 - D8: Hypermultiplet (fundamental matter)

UV fixed point:

infinite gauge coupling, massless, vanishing Coulomb moduli



Global symmetry enhancement at UV fixed point

$$SO(2N_f) \times U(1) \subset E_{N_f+1}$$

Flavor symmetry ↗ ↖ Instanton symmetry

cf. Heterotic E_8 x E_8