New algorithm to construct and triangulate convex polyhedral cone and it's application in higher order perturbative QFT calculatio

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Outline

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Muti-loop Calculation in QFT

In last ten years, there are very active developments in methods for higher-order perturbative calculation in QFT.

- Unitary Cut, On shell Helicity Amplitude, IBP, Symbol, Sector Decomposition, Mellin-Barnes(MB), Differential Equation(DE)
- Key step: to calculate loop integral (master integral, in more than one-loop case, master integral basis have not yet config out and therefore, sclar integral or even tensor integral)
- Sector Decomposition is a better way to separate all UV and IR divergences in loop integral in analytic form and passes the finite part to numerical calculation.

Geometric method for sector decomposition

- Sector Decompsotion is an old method and has been very actively developed. The most recently development is the Geometric method by T. kaneko and T. Ueda on 2010, Which beautifully and effectively translate the problem into a Geometric problem:
  "to construct and triangulate convex polyhedral cone"
- They (T. kaneko and T. Ueda) construct a computer program for their method by utilizing an algorithm from mathematican and they can finished the triple box(3-loop) by 53 hours CPU time.
- We develope a new algorithm and construct a computer program. We can finish the same work within 3-minuts CPU time on the same CPU.

Geometric method for sector decomposition

Sector Decomposition is a method used to separate divergences in loop integral. With α presentation of a propagator

$$\frac{1}{D_l^{a_l}} = -i \int_0^\infty \mathrm{d}\alpha_l \exp(iD_l \alpha_l a_l),$$

An h-loop integral with N propagators can be expressed as

$$G = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \cdots \mathrm{d}^d k_h}{D_1^{a_1} D_2^{a_2} \cdots D_N^{a_N}} = \int \mathrm{d}^d k \int \mathrm{d}^N \alpha \exp\left(i \sum_{l=1}^{N} D_l \alpha_l a_l \right)$$

After integration on loop momenta, it becomes

$$G = C \int_0^\infty \mathrm{d}^N \alpha \prod_l \alpha_l^{a_l - 1} U^{-d/2} \mathrm{e}^{-iF/U} \tag{1}$$

where U and F are homogeneous polynomials of α_i with the homogeneity degrees h and $h+1$, and C is a constant.

Let $\eta = \sum \alpha_l$, insert $\delta(\eta - \sum \alpha_l)$ into the integral, and make the transformation $\alpha_l = \eta \alpha_l'$. After the integration over η, the integral becomes

$$G = C' \int_0^1 \mathrm{d}^N \alpha \delta \left(1 - \sum \alpha_l\right) \prod_l \alpha_l^{a_l - 1} \frac{U^{a-(h+1)d/2}}{F^{a-hd/2}} \tag{2}$$

with $a = \sum a_l$.

One can always reach Eq.(2) with usual loop integral techniques. And this is where sector decomposition starts. In this integral, only the integration over α_i is remained, and the interval is now limited to $[0, 1]$ due to the delta function. And this is how sector decomposition works on it:

- separate the integration domain into N sectors $\Delta_{k,k=1,2,\dots,N}$, where Δ_k is defined by $\alpha_i \le \alpha_k, i \ne k$.
- do the transformation $\alpha_i' = \alpha_i / \alpha_k, i \ne k$ in Δ_k, and integrate over α_k with the delta function
- now, the integral in the integration domain Δ_k (labelled with G_k) becomes

$$G_k = C' \int_0^1 \mathrm{d}^{N-1} \alpha \prod_l \alpha_l^{a_l - 1} \frac{U_k^{a-(h+1)d/2}}{F_k^{a-hd/2}} \tag{3}$$

where U_k and F_k are obtained by setting α_k to 1 in U and F.

- Usually these Δ_k are called primary sectors. But they are not sufficient since the divergences are still hidden inside.
- Further decomposition is needed.
- Here we introduce the geometric method [Kaneko and Ueda (2010)], which can separate the divergence after one more decomposition (free from infinite recursion).
- For convenience, we rewrite G_k into

$$G_k = C' \int_0^1 \mathrm{d}^{N-1}\alpha \alpha^v U_k^\beta F_k^\gamma \qquad (4)$$

with $v = \{a_1 - 1, a_2 - 1, \ldots, a_{N-1} - 1\}$, $\alpha^v = \prod_l \alpha_l^{a_l - 1}$,
$\beta = a - (h+1)d/2$ and $\gamma = -(a - hd/2)$

This is how the geometric method works:

- A monomial in $U_k(F_k)$ has the form $c_b\alpha^b = c_b\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_{N-1}^{b_{N-1}}$, and can be characterized by a vector $b = \{b_1, b_2, \ldots, b_{N-1}\}$, thus we have two vector sets Z^U and Z^F
- do transformation $\alpha_i = \mathrm{e}^{-y_i}$, the Jacobian is $\mathrm{e}^{-\sum_i y_i}$, the integral becomes
$$G_k = C' \int_0^\infty \mathrm{d}^{N-1} y \mathrm{e}^{-v\cdot y} U_k^\beta F_k^\gamma \tag{5}$$
- now $U_k = \sum_{b\in Z^U} c_b \mathrm{e}^{-b\cdot y}$, $F_k = \sum_{b'\in Z^F} c_{b'} \mathrm{e}^{-b'\cdot y}$
- suppose $\mathrm{e}^{-b(b')\cdot y}$ is maximal in $U_k(F_K)$, label this domain as $\Delta_{bb'}$.
- extract these two term from U_k and F_k, and the integral can be further decomposed into

$$G_k = \sum_{b\in Z^U}\sum_{b'\in Z^F}\int_{\Delta_{bb'}} \mathrm{d}^{N-1} y \mathrm{e}^{-(v+b\beta+b'\gamma)\cdot y} \times \left[c_b + \sum_{d\in Z^U/\{b\}} c_d \mathrm{e}^{-(d-b)\cdot y}\right]^\beta \times \left[c_{b'} + \sum_{d'\in Z^F/\{b'\}} c_{d'} \mathrm{e}^{-(d'-b')\cdot y}\right]^\gamma \tag{6}$$

where $Z^U/\{b\}$ denotes subset of Z^U obtained by removing b from Z^U

- We have supposed that in the domain $\Delta_{bb'}$, $\mathrm{e}^{-b(b')\cdot y}$ is maximal in $U_k(F_K)$ for certain $b(b')$.
- $\mathrm{e}^{-(d-b)\cdot y} \leqslant 1$ and $\mathrm{e}^{-(d'-b')\cdot y} \leqslant 1$ for all $d(d')$
- $(d-b)\cdot y \geq 0$, $(d'-b')\cdot y \geq 0 \rightarrow$ restrictions on $\Delta_{bb'}$
- Last two terms in Eq.(6) is always finite when $\alpha_i = 0$
- All physical poles (if exist) have been factorized into first term
- Now we can expand last two term with respect to $\epsilon = (D-4)/2$ to required order and obtained a Laurent series of the whole integral in terms of ϵ, whose coefficients are expressed by finite integrals
- The question is how to obtain the domain $\Delta_{bb'}$

- Define $Z_b^{U(F)} \equiv \{v - b | v \in Z^{U(F)}/\{b\}\}$, $S \equiv Z_b^U \cup Z_{b'}^F \cup R^{N-1}$, where R^{N-1} is the standard basis of $(N-1)$-dimensional Euclidean space.
- Let $C(S)$ be the vector space generated by S, $C(S) \equiv \{\sum_i r_i v_i | r_i \geq 0, v_i \in S\}$, then $C(S)$ is just a convex polyhedral cone in $(N-1)$-dimensional Euclidean space.
- The dual cone of C(S) is defined by $C(S)^V \equiv \{y | v \cdot y \geq 0, \forall v \in C(S)\}$, $C(S)^V$ is also a convex polyhedral cone in $(N-1)$-dimensional Euclidean space.
- $\Delta_{bb'} = \{y | v \cdot y \geq 0, \forall v \in S\} = C(S)^V$. R^{N-1} is also in S because of the fact $y_i \geq 0$.
- Let V be the set of all edges of $C(S)^V$, then $C(S)^V$ can also be written as $\{\sum_i u_i v_i | u_i \geq 0, v_i \in V\}$
- If $C(S)^V$ is simplicial (has only $N-1$ edges), change the integration variables from y_i into u_i, $\Delta_{bb'}$ is obtained.
- If not, triangulate $C(S)^V$ (split it into several simplicial sub-cones), and change integration variables in each sub-cones (more sub-sectors).
- This is why this method is called 'geometric'

		Double box		Triple box	
Package	Strategy	No. of Sectors	Time(s)	No. of Sectors	Time(s)
FIESTA4	S	362	1.4	23783	831
	B	586	0.5	121195	127
	X	282	0.3	10259	44
	KU	266	13.7	6822	7472
	KU0	326	4.5	10556	6487
	KU2	266	61.9	-	-
SecDec3.0	X	320	1.4	11384	700
	G1	270	4.2	7871	574
T. Kaneko and T. Ueda		266	29.2	6568	123280
FDC-SD (Our result)		266	0.6	6568	116

Comparision with FIESTA4 [Smirnov (2015)], SecDec3.0 [Borowka et al. (2015)] and method proposers [Kaneko and Ueda (2010)]. Strategies KU, KU0, KU2, G1 and G2 are all based on the geometric method, while G2 is different in the strategy of primary sectors.

decomposition only, without the integration of finite coefficients

Summary

Thanks for your attention!

B. W. Harris and J. F. Owens, Phys. Rev. **D65**, 094032 (2002).

T. Kaneko and T. Ueda, Comput. Phys. Commun. **181**, 1352 (2010), 0908.2897.

A. V. Smirnov (2015), 1511.03614.

S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, and T. Zirke, Comput. Phys. Commun. **196**, 470 (2015), 1502.06595.