Higher direct images of logarithmic canonical sheaves

Sándor Kovács

University of Washington

ICM Satellite Conference, Daejeon, Korea
August 8, 2014

Question (Problem B in Grothendieck's '58 ICM address)

Let $f : X \to Y$ be a proper birational morphism of non-singular varieties over an algebraically closed field k. Is it true that $R^q f_* \mathscr{O}_X = 0$ for $q > 0$?

Corollary

If $\mathscr{E}$ is a locally free sheaf on Y, then

$$H^i(Y, \mathscr{E}) \simeq H^i(X, f^* \mathscr{E})$$

for all i.

Corollary

Arithmetic genus is a birational invariant of smooh projective varieties.

Corollary (Hironaka '64)

"Yes", in characteristic zero.

This is a corollary of Hironaka's resolution of singularities theorem.

Grothendieck also comments that it is essential that both X and Y be non-singular.

- This is true in the sense that allowing for instance Y to be arbitrarily singular leads to easy counter-examples. (Say, a cone over any smooth projective non-rational curve).
- On the other hand, there is a well-behaving class of singularities (rational singularities) for which this condition still holds.

Rational singularities were defined for surfaces by Artin in '66 and extended to arbitrary dimensions by Bruns in '74.

Definition (*of a rational resolution*)

Let $f : X \to Y$ be a proper birational morphism with X non-singular, everything defined over an algebraically closed field k. Then f is called a *rational resolution* if

- $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism, i.e., Y is normal,
- $R^i f_*\mathscr{O}_X = 0$ for all $i > 0$, and
- $R^i f_*\omega_X = 0$ for all $i > 0$.

Remark

The first two conditions together are equivalent to asking that $\mathscr{O}_Y \to Rf_*\mathscr{O}_X$ is an isomorphism (in the appropriate derived category) and in characteristic zero the last condition follows from the Grauert-Riemenschneider vanishing theorem.

Definition

Y has *rational singularities* if $\exists$ a rational resolution $f : X \to Y$.

Theorem

If $f : X \to Y$ is a rational resolution, then $\forall g : Z \to Y$ resolution of singularities is also a rational resolution.

Proof (for char $k = 0$)

By [Hironaka64] there exists a resolution of Y that dominates both X and Z:

$$\begin{tikzcd} & W \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] \arrow[dd, "h"] & \\ X \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

$$\mathscr{O}_Y \simeq Rf_*\mathscr{O}_X \simeq Rf_* \underbrace{R\pi_*\mathscr{O}_W}_{\mathscr{O}_X \simeq} \simeq Rh_*\mathscr{O}_W \simeq Rg_* \underbrace{R\sigma_*\mathscr{O}_W}_{\mathscr{O}_Z \simeq} \simeq Rg_*\mathscr{O}_Z.$$

$\square$

In $\operatorname{char} k > 0$ this is a recent theorem of Chatzistamatiou-Rülling.

Definition

Two integral schemes, X and Y are said to be *properly birational* if there exists an integral scheme Z and proper birational morphims

$$X \longleftarrow Z \longrightarrow Y.$$

Theorem (Chatzistamatiou-Rülling '11)

Let k be a perfect field, S a k-scheme, $\phi : X \to S, \psi : Y \to S$ integral S-schemes that are smooth over k and properly birational over S. Then

- $R^i\phi_*\mathscr{O}_X \simeq R^i\psi_*\mathscr{O}_Y$ *for all i, and*
- $R^i\phi_*\omega_X \simeq R^i\psi_*\omega_Y$ *for all i.*

Corollary

If $f : X \to Y$ is a proper birational morphism of smooth k-schemes, then $R^if_*\mathscr{O}_X = 0$ and $R^if_*\omega_X = 0$ for all $i > 0$.

Goal

Extend these results to pairs.

Definition

Let Y be a normal variety over a field k and B a reduced effective divisor on Y. Then (Y, B) is called a *reduced pair*.

Definition

A reduced pair (X, D) is called an *snc pair* if X is smooth over k and D is an snc divisor. For a reduced pair (Y, B) the *snc locus of* (Y, B) is the largest open set $U \subseteq Y$ such that $(U, B|_U)$ is an snc pair. The snc locus of (Y, B) is denoted by $\mathrm{snc}(Y, B) := U$.

An snc pair should be considered the "smooth" object among pairs.

Definition

Let (Y, B) be a reduced pair. A *resolution of singularities* of (Y, B), denoted by $f:(X, D) \to (Y, B)$, is a morphism $f: X \to Y$ such that

- f is proper and birational,
- $D = f_*^{-1}B$ is the birational transform of B, and
- (X, D) is an snc pair.

The notions of *rational resolutions* and *rational singularities* have been extended to pairs. This has been done independently and somewhat differently by Schwede-Takagi and Kollár-K___.
Here I will discuss the latter.

Definition

A resolution of singularities $f:(X,D) \to (Y,B)$ is called a *rational resolution* if

- $\mathscr{O}_Y(-B) \simeq f_*\mathscr{O}(-D)$,
- $R^i f_*\mathscr{O}_X(-D) = 0$ for $i > 0$,
- $R^i f_*\omega_X(D) = 0$ for $i > 0$.

Remark

As in the absolute (=non-pair) case, the first two conditions together are equivalent to asking that $\mathscr{O}_Y(-B) \simeq Rf_*\mathscr{O}(-D)$ (in the appropriate derived category)
and the last condition follows from a logarithmic version of the Grauert-Riemenschneider vanishing theorem.

Definition (first approximation)

A *rational pair* should be a reduced pair (Y,B) that admits a rational resolution $f:(X,D) \to (Y,B)$.

Example

Let $(Y, B) = (\mathbb{A}^2, (xy = 0))$ and $(X, D) = (Bl_O Y, f_*^{-1} B)$ where $f : X \to Y$ is the blowing up of the origin. Apply Rf_* to

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$$

and observe that $f_* \mathscr{O}_D \simeq \mathscr{O}_{D,P} \oplus \mathscr{O}_{D,Q}$, where $\{P, Q\} = D \cap \mathrm{Ex}(f)$, so the natural map $\nu : f_* \mathscr{O}_X \to f_* \mathscr{O}_D$ is **not** surjective, and hence $0 \neq \operatorname{coker} \nu \subseteq R^1 f_* \mathscr{O}_X(-D) \neq 0$.

In other words, an snc pair admits rational (the identity) and non-rational resolutions as well, and hence every pair admits non-rational resolutions.
It is preferable to have a definition that holds for all “resolutions”.
For this reason we will limit the resolutions that are allowed for checking rationality.

Definition

Let (X, D) be an snc pair. A *stratum* of (X, D) is an irreducible component of an arbitrary intersection of some components of D. (This includes X as the empty intersection.)

Definition

A resolution of singularities $f:(X, D) \to (Y, B)$ is called a *thrifty resolution* if

- f is an isomorphism over the general point of every stratum of $\mathrm{snc}(Y, B)$, and
- f is an isomorphism at the general point of every stratum of (X, D), and

Essentially, a resolution is thrifty if we only resolve what we must.

Observation

If $f:(X,D) \to (Y,B)$ is a thrifty resolution, then it induces a bijection between the strata of (X,D) and the strata of $\mathrm{snc}(Y,B)$. Furthermore, the induced morphism on a stratum of (X,D) is birational onto the closure of the corresponding stratum of $\mathrm{snc}(Y,B)$.

Definition

A birational map (not necessarily a morphism) $f:(X,D) \dashrightarrow (Y,B)$ is called *thrifty* if there exist open sets $U \subseteq X$, $V \subseteq Y$ such that U contains the general point of every stratum of $\mathrm{snc}(X,D)$, similarly V contains the general point of every stratum of $\mathrm{snc}(Y,B)$, and $f|_U : U \xrightarrow{\simeq} V$ is an isomorphism.

Observation

A resolution of singularities $f:(X,D) \to (Y,B)$ is a thrifty resolution if and only if it is a thrifty rational map.

Definition

A reduced pair (Y, B) is called a *rational pair* if it admits a thrifty rational resolution.

This is actually not too far from simply asking for a rational resolution (without requiring that it be thrifty).

Theorem (Erickson)

If a log resolution $f:(X, D) \to (Y, B)$ *(i.e., such that* $D \cup \mathrm{Ex}(f)$ *is an snc divisor), then if* f *is rational, then it is thrifty.*

In some cases "thrifty" and "rational" are the same.

Theorem (Kollár-K___)

Let (Y, Σ) *be a dlt pair and let* $B = \lfloor \Sigma \rfloor$*. Assume that* $\operatorname{char} k = 0$*. Then a resolution of singularities* $f:(X, D) \to (Y, B)$ *is rational if and only if it is thrifty.*

Theorem (Kollár)

In characteristic zero every reduced pair admits a thrifty resolution and any two thrifty resolutions can be dominated by a third.

Corollary

In characteristic zero if (Y, B) admits a thrifty rational resolution, then every thrifty resolution of (Y, B) is rational.

Is this also true in positive characteristic?

Theorem (Main Theorem)

Let k be a perfect field, S a k-scheme, (X, D) and (Y, B) reduced pairs over S with structure morphisms $\phi : X \to S$ and $\psi : Y \to S$. Assume that (X, D) and (Y, B) are snc pairs as k-schemes and properly birational over S. Finally assume that the induced birational map $(X, D) \dashrightarrow (Y, B)$ is thrifty. Then

- $R\phi_*\mathscr{O}_X(-D) \simeq R\psi_*\mathscr{O}_Y(-B)$, *and*
- $R\phi_*\omega_X(D) \simeq R\psi_*\omega_Y(B)$.

This improves [Chatzistamatiou-Rülling11] in two ways:

- it is a log version, i.e., allows for boundary divisors D and B, and
- the isomorphism is between the total derived push-forwards (in the derived category), not just between the cohomology sheaves.

Corollary

If $f_i : (X_i, D_i) \to (Y, B)$ *for* $i = 1, 2$ *are two thrifty resolutions of* (Y, B)*, then* f_1 *is rational if and only if* f_2 *is rational.*

Corollary

In arbitrary characteristic if (Y, B) *admits a thrifty rational resolution, then every thrifty resolution of* (Y, B) *is rational.*