Introduction to Quantum Information Theory

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Outline

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Classical Information

States of **classical objects describe an objective reality.**

Examples:

- A classical bit **is** in the state 0 or 1
- A classical cat **is** either dead or alive.
- A classical particle **is** at position $\vec{x}$ and **has** the velocity $\vec{v}$.
- A classical electric field **has** the value $\vec{E}(\vec{r})$ at the position $\vec{r}$.
- Two given points in spacetime **have** a well-defined distance d.

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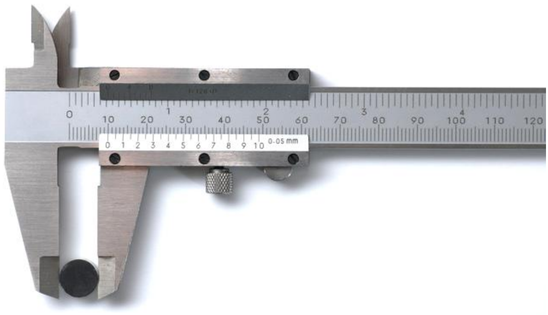
Classical information

- is a description of an objective reality.
- is independent of the observing subject.
- can be shared (copied).

Classical Information

Classical information can be obtained by measurement.



Ideal classical measurements are reproducible and do not change the state of the system.

[Fig.: Eugster, Wikimedia]

Quantum Information

States of quantum objects do not have an objective reality.

- A quantum bit can be in a 'superposition' of 0 and 1
- A quantum cat can be in a 'superposition' dead or alive.

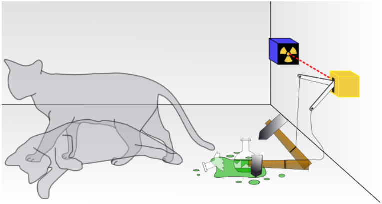
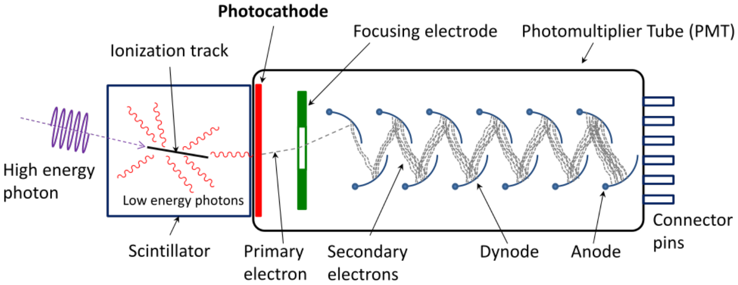


Fig.: Dhatfield, Wikimedia

- A quantum particle can be uncertain in space and momentum.
- A quantum field can be uncertain in its field value.
- In a (not yet existing) theory of quantum gravity, two points in spacetime may have an uncertain distance.

Quantum Measurement



Quantum measurements exchange information in both directions.

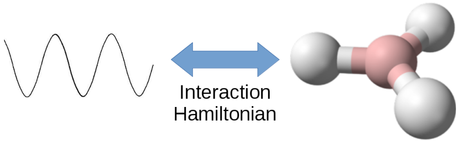


Fig.: Wikimedia / public domain

Interpretation of quantum mechanics

Wikipedia gives a list of the most important interpretations:

Interpretation	Author(s)	Deterministic?	Wavefunction real?	Unique history?	Hidden variables?	Collapsing wavefunctions?	Observer role?	Local?	Counterfactual definiteness?	Universal wavefunction exists?
Ensemble interpretation	Max Born, 1926	Agnostic	No	Yes	Agnostic	No	No	No	No	No
Copenhagen interpretation	Niels Bohr, Werner Heisenberg, 1927	No	No[1]	Yes	No	Yes[2]	Causal	No	No	No
de Broglie–Bohm theory	Louis de Broglie, 1927, David Bohm, 1952	Yes	Yes[3]	Yes[4]	Yes	No	No	No[15]	Yes	Yes
Quantum logic	Garrett Birkhoff, 1936	Agnostic	Agnostic	Yes[5]	No	No	Interpretational[6]	Agnostic	No	No
Time-symmetric theories	Satosi Watanabe, 1955	Yes	Yes	Yes	Yes	No	No	Yes	No	Yes
Many-worlds interpretation	Hugh Everett, 1957	Yes	Yes	No	No	No	No	Yes	Ill-posed	Yes
Consciousness causes collapse	Eugene Wigner, 1961	No	Yes	Yes	No	Yes	Causal	No	No	Yes
Stochastic interpretation	Edward Nelson, 1966	No	No	Yes	Yes[14]	No	No	No	Yes[14]	No
Many-minds interpretation	H. Dieter Zeh, 1970	Yes	Yes	No	No	No	Interpretational[7]	Yes	Ill-posed	Yes
Consistent histories	Robert B. Griffiths, 1984	No	No	No	No	No	No	Yes	No	Yes
Transactional interpretation	John G. Cramer, 1986	No	Yes	Yes	No	Yes[8]	No	No[12]	Yes	No
Objective collapse theories	Ghirardi–Rimini–Weber, 1986, Penrose interpretation, 1989	No	Yes	Yes	No	Yes	No	No	No	No
Relational interpretation	Carlo Rovelli, 1994	Agnostic	No	Agnostic[9]	No	Yes[10]	Intrinsic[11]	Yes[51]	No	No
QBism	Christopher Fuchs, Ruediger Schack, 2010	No	No[16]	Agnostic[17]	No	Yes[18]	Intrinsic[19]	Yes	No	No

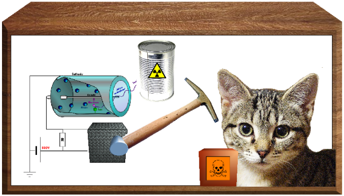
Ask N physics professors $\to$ N different opinions

Interpretation of quantum mechanics

Even 100 years after its discovery, the **interpretation of quantum mechanics** is still controversial, but it seems that:

- Classical objective reality does not exist in nature.
- What we perceive as reality is a relation between subject and object, created by physical interaction.
- Quantum states could be viewed as a 'catalogue' of possibilities of what could happen in an interaction.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\text{living cat}\rangle + |\text{dead cat}\rangle\Big)$$



Quantum amplitudes and state vectors

Standard Quantum Mechanics Formalism

Each classical configuration $c = 1, \ldots, N$ is associated with an **amplitude**

$$\psi_c \in \mathbb{C}$$

normalized by $\sum_c |\psi_c|^2 = 1$.

The list of all amplitudes $\{\psi_1, \ldots \psi_N\}$ can be regarded as a complex vector

$$|\psi\rangle = \{\psi_1, \ldots \psi_N\}$$

on the unit sphere $\langle\psi|\psi\rangle = 1$ of an N-dimensional vector space $\mathbb{C}^N$.

The qubit

Example

The two classical states 0,1 of a switch are associated with two complex amplitudes $\psi_0, \psi_1 \in \mathbb{C}$ normalized by $|\psi_0|^2 + |\psi_1|^2 = 1$.

These amplitudes are regarded as a complex vector:

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$$

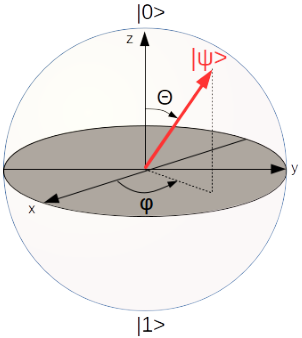
Because of the normalization they reside on the unit sphere of $\mathbb{C}^2$:

$$\langle\psi|\psi\rangle = \psi_0^*\psi_0 + \psi_1^*\psi_1 = 1.$$

This describes the (pure) state of a quantum bit **(qubit)**.

The qubit – Bloch sphere

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \in \mathbb{C}^2$$



- Normalization eliminates 1 degree of freedom
- Total phase factor $e^{i\eta}$ is not observable

$\Rightarrow$ 2 degrees of freedom left.

Bloch sphere representation of a qubit:

$$|\psi\rangle \; = \; \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{i\phi}\,|1\rangle$$

Unitary time evolution

As long as no measurement is carried out, the amplitudes evolve in time by means of the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi_t\rangle = \mathbf{H}|\psi_t\rangle \,,$$

where $\mathbf{H} = \mathbf{H}^\dagger$ is the Hamiltonian of the system.

Formal solution:

$$|\psi_t\rangle = \mathbf{U}_t|\psi_0\rangle,$$

where the time evolution operator

$$\mathbf{U}_t = \exp(-i\mathbf{H}t)$$

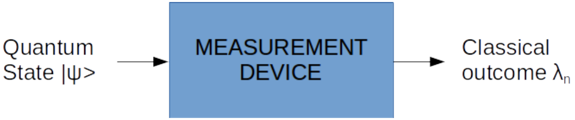
is unitary, i.e,

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{1}.$$

Projective measurement

Von-Neumann measurement postulate:

A measurement apparatus is described by a set of classical measurement results λ_n associated with orthogonal states $|\phi_n\rangle$.



Measuring a system in the state $|\psi\rangle$, the measurement apparatus will return λ_n with probability $p_n = |\langle\psi|\phi_n\rangle|^2$.

amplitudes $\neq$ probabilties

Artificial division into quantum and classical world

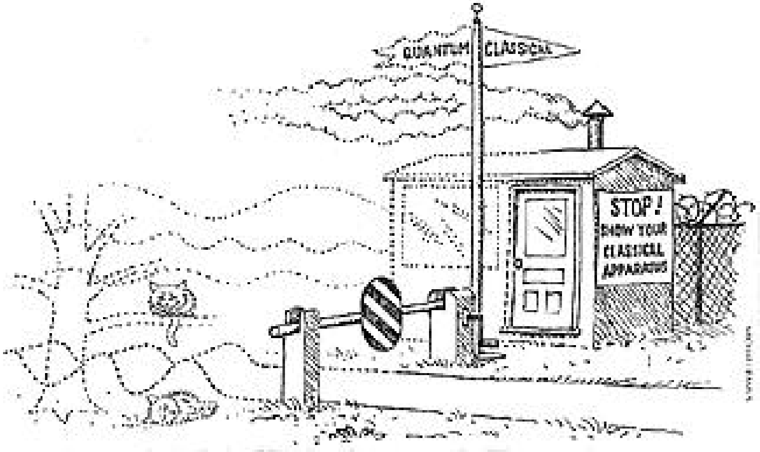
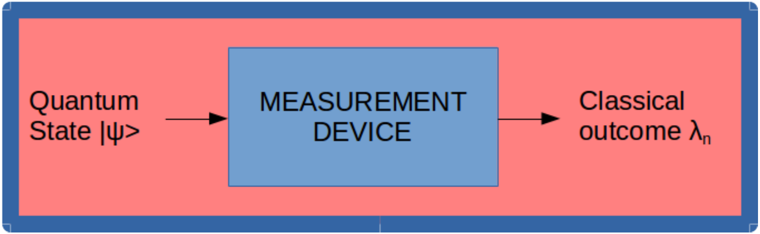


Figure by Zurek

The measurement postulate leads to an artificial division between the microscopic quantum and the macroscopic classical world.

Measurements create randomness



The measurement apparatus and the (human) observer are also quantum systems.

What is the state of the whole system?

We need a formalism that can handle **quantum amplitudes** and **statistical probabilities** on equal footing.

Outline

Quantum amplitudes and classical probabilities

Textbook quantum mechanics	**Classical statistics**
Amplitudes	**Probabilities**
quantum amplitudes ψ_i	probabilities p_i
state vectors $\|\psi\rangle$	probability distributions $\{p_i\}$
Uncertainty of what will happen in a measurement	*Ignorance of what did happen in a measurement*

$\Rightarrow$ **A common description is needed.**

$\Rightarrow$ **Use the notion of statistical ensembles.**

Concept of Ensembles

The probability p_1 that a bit is in the classical state 1 can be encoded by an **ensemble** of infinitely many bits, where 1 occurs with the frequency p_1.

```
001110010111010101110010001000110111001010011101011011011101011111100101111010110·
0001110000001010100101001001000100101001111010011100101110000001000001010111010100·
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111111110001111011110100100110111100000000101000100010111010100011110101001000010·
```

Idea: Consider **ensembles of qubits** instead of bits.

Quantum Ensembles

- **Classical bit:**

  Only two possibilities: 0 or 1

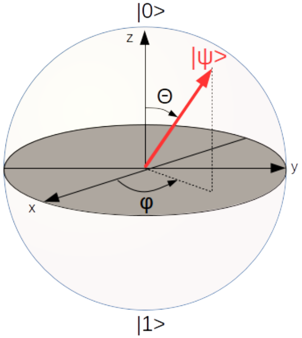
  $\Rightarrow$ Two probabilities p_0 and p_1.

- **Qubit:**

  Infinitely many possibilities:

  $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

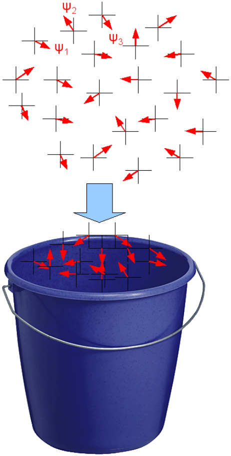
  $\Rightarrow$ Probability density $p(\psi) \simeq p(\theta, \phi)$.



Quantum Ensembles

Probabilistic ensemble of quantum states:

$|\psi_{(\theta,\phi)}\rangle$ occurs with probability density $p(\theta, \phi)$.



Quantum Ensembles

Probabilistic ensemble of quantum states:

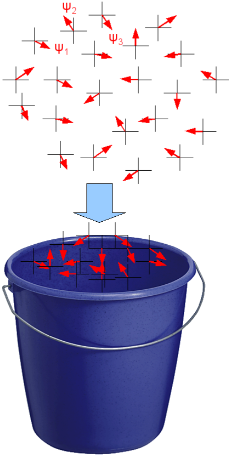
$|\psi_{(\theta,\phi)}\rangle$ occurs with probability density $p(\theta, \phi)$.

As we will see:

We cannot determine $p(\theta, \phi)$
by repeated measurements.

Different ensembles may represent the same measurement statistics, i.e. they are **equivalent** with respect to measurements.

Equivalence classes of ensembles!



Equivalence of Quantum Ensembles

Example: Projective measurement of an ensemble of qubits:

Let the measurement apparatus be characterized by two orthogonal vectors $|\phi_1\rangle, |\phi_2\rangle$ with tho corresponding possible outcomes λ_1 and λ_2.

Then λ_i is measured with the probability

$$p_i = \underbrace{\int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, p(\theta,\phi)}_{\text{sum over ensemble}} \underbrace{\Big|\langle\phi_i|\psi_{(\theta,\phi)}\rangle\Big|^2}_{\text{observation prob.}}$$

Equivalence of Quantum Ensembles

Reorganize this expression:

$$p_i = \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, p(\theta,\phi) \left| \langle \phi_i | \psi_{(\theta,\phi)} \rangle \right|^2$$

$$= \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, p(\theta,\phi) \langle \phi_i | \psi_{(\theta,\phi)} \rangle \langle \psi_{(\theta,\phi)} | \phi_i \rangle$$

$$= \langle \phi_i | \underbrace{\left(\int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, p(\theta,\phi) | \psi_{(\theta,\phi)} \rangle \langle \psi_{(\theta,\phi)} | \right)}_{2\times 2 \text{ matrix } \rho} | \phi_i \rangle$$

$$= \langle \phi_i | \rho | \phi_i \rangle = \mathrm{Tr}\Big[\underbrace{|\phi_i\rangle\langle\phi_i|}_{=:E_i} \rho \Big] = \mathrm{Tr}\Big[E_i \rho \Big]$$

Equivalence of Quantum Ensembles

Summary so far:

If we consider a measurement projecting on
$E_1 = |\phi_1\rangle\langle\phi_1|$, $E_2 = |\phi_2\rangle\langle\phi_2|$
with the outcomes λ_1 and λ_2,

then λ_i is measured with the probability

$$\boxed{p_i = \mathrm{Tr}\Big[E_i\,\rho\Big]}$$

$\Rightarrow$ All information that plays a role
is contained in the 2×2 matrix ρ,
called **statistical operator** or **density matrix**.

Statistical Operator / Density Matrix

Statistical operator:

$$\rho = \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, p(\theta,\phi)\, |\psi_{(\theta,\phi)}\rangle\langle\psi_{(\theta,\phi)}|$$

General compact notation:

$$\boxed{\rho = \int \mathrm{D}\psi\, \mathrm{p}(\psi)\, |\psi\rangle\langle\psi| \qquad\qquad \rho = \sum_\mathrm{i} \mathrm{p_i} |\psi_\mathrm{i}\rangle\langle\psi_\mathrm{i}|}$$

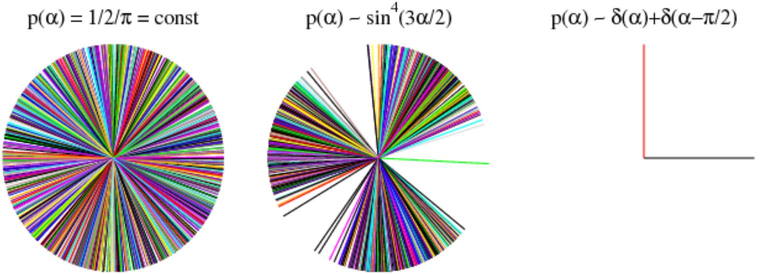
$\uparrow$ $\uparrow$

Vectors are normalized but not necessarily mutually orthogonal.

Statistical Operator / Density Matrix

Different ensembles may correspond to the same density matrix.

Example: $|\psi_\alpha\rangle = \cos\alpha|\uparrow\rangle + \sin\alpha|\downarrow\rangle, \qquad \rho = \int_0^{2\pi} \mathrm{d}\alpha\, p(\alpha)\, |\psi_\alpha\rangle\langle\psi_\alpha|$



$$\Rightarrow \ldots \Rightarrow \qquad \rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ in all cases.}$$

Statistical Operator – Properties

The statistical operator represents an equivalence class of different quantum ensembles which cannot distinguished by measurements.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \langle\psi_i|\psi_i\rangle = 1, \;\; p_i \in [0,1], \;\; \sum_i p_i = 1$$

Properties:

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Properties:

- ρ is a Hermitean operator:
$$\rho = \rho^\dagger$$
- The trace of ρ equals 1:
$$\mathrm{Tr}[\rho] = \sum_i p_i \mathrm{Tr}\Big[|\psi_i\rangle\langle\psi_i|\Big] = \sum_i p_i = 1$$
- Expectation values of ρ behave like probabilities:
$$\langle\Phi|\rho|\Phi\rangle = \sum_i p_i \underbrace{\langle\Phi|\psi_i\rangle\langle\psi_i|\Phi\rangle}_{\in[0,1]} \in [0,1]$$

Statistical Operator – Spectral decomposition

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad\qquad \rho = \int \mathrm{d}\alpha\, p(\alpha) |\psi_\alpha\rangle\langle\psi_\alpha|$$

Since ρ is Hermitean ($\rho = \rho^\dagger$) it has a set of orthonormal eigenvectors with real-valued eigenvalues, i.e., it has a **spectral decomposition**

$$\rho = \sum_{j=1}^{d} P_j \, |\Phi_j\rangle\langle\Phi_j|$$

with $\langle\Phi_i|\Phi_j\rangle = \delta_{ij}$ and $P_j \in \mathbb{R}$.

d=dimension of Hilbert space

Statistical Operator – Spectral decomposition

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \rho = \int \mathrm{d}\alpha\, p(\alpha) |\psi_\alpha\rangle\langle\psi_\alpha|$$

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d=dimension of Hilbert space

As all expectation values of ρ behave like probabilities, we can conclude that the **eigenvalues of ρ are like probabilities**, too:

$$P_j \in [0,1]\,, \qquad \sum_{j=1}^{d} P_j = 1$$

positive operator

Statistical Operator – Spectral decomposition

An operator ρ is called **positive** if one of the following equivalent statements holds:

- $\rho = \rho^\dagger$ and $\langle \rho\psi, \psi\rangle = \langle\psi|\rho|\psi\rangle \geq 0$
- $\rho = \rho^\dagger$ and all eigenvalues of ρ are non-negative.
- ρ can be written as $\rho = \mathbf{A}^\dagger \mathbf{A}$.

Density matrices are positive normalized operators.

Positive maps are functions mapping positive operators to other positive operators (density matrices onto density matrices).

*Note: **Completely positive** is more than **postive**... (later)*

Quantum states

- **Pure states**:
  Physics textbooks introduce quantum states as vectors $|\psi\rangle$.
  In our new formalism they are replaced by pure states of the form

$$\rho = |\psi\rangle\langle\psi|.$$

  Pure states represent the *maximal* knowledge that an observer can have about a quantum system.

- **Mixed states**:
  In quantum information theory, a **quantum state** generally refers to a mixed state represented by a statistical operator ρ.

  A mixed state represents the *partial* knowledge of an observer about a quantum system.

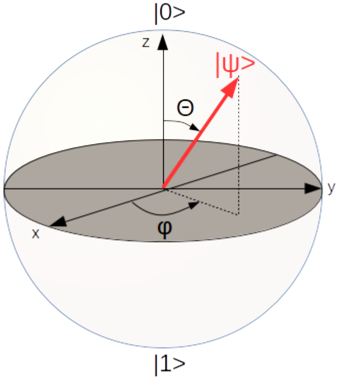
Quantum States

Quantum state of a qubit

Bloch ball representation:

$\rho = \frac{1}{2}\Big(\mathbf{1} + x\sigma^x + y\sigma^y + z\sigma^z\Big)$

The vector (x, y, z) on the Bloch ball can be interpreted as expectation value of $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.



Points on the sphere represent pure states.
Points inside the ball represent mixed states.
In the center we find the totally mixed state.

Quantum states

vector formalism:	*operator formalism:*
$\|\psi\rangle$	$\rho := \|\psi\rangle\langle\psi\|$
$\langle\psi\|\psi\rangle = 1$	$\mathrm{Tr}[\rho] = 1$
$i\partial_t\|\psi\rangle = H\|\psi\rangle$	$i\partial_t\rho = [H,\rho]$
$\langle \mathbf{A}\rangle_\psi = \langle\psi\|\mathbf{A}\|\psi\rangle$	$\langle \mathbf{A}\rangle_\rho = \mathrm{Tr}[\mathbf{A}\rho]$

coherent superposition

$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$

$|\alpha|^2 + |\beta|^2 = 1$

probabilistic mixture

$\psi = p_1\psi_1 + p_2\psi_2$

$p_1 + p_2 = 1$

Example: Pure and mixed qubit states

Coherent superposition (quantum uncertainty):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)$$

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\Big)$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Tr[\rho\sigma^z] = 0,\ Tr[\rho\sigma^x] = 1$$

Probabilistic mixture (subjective ignorance):

$$\rho = \frac{1}{2}(\rho_\uparrow + \rho_\downarrow) = \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\Big)$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Tr[\rho\sigma^z] = 0,\ Tr[\rho\sigma^x] = 0$$

Projective measurement in the density-matrix formulation

Consider again a von-Neumann measurement projecting on $E_1 = |\phi_1\rangle\langle\phi_1|, E_2 = |\phi_2\rangle\langle\phi_2|$ with the outcomes λ_1 and λ_2.

- **Textbook vector formalism:**

  The result λ_i occurs with probability $|\langle\psi|\phi_i\rangle|^2$. After the measurement the system is in the state $|\phi_i\rangle$.

- **Density matrix formalism:**

$$\rho \to \rho' = \sum_i E_i \, \rho \, E_i$$

Note that measurements generically generate mixed states!

Entropy and Information

Shannon information entropy

Recall classical probability theory:

The **information content** of an event with probability p is given by

$$H_p = -\log_2 p$$

The **average information content** (Shannon entropy) of a probability distribution $\{p_i\}$ is given by the expectation value of H_p:

$$H = \langle H_p \rangle_p = -\sum_i p_i \log_2 p_i$$

Shannon information entropy

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Different scientific cultures use different prefactors:

$H = -\sum_i p_i \log_2 p_i$	$H = -\sum_i p_i \ln p_i$	$H = -k_B \sum_i p_i \ln p_i$
Computer science	Mathematics	Physics

Von-Neumann entropy

Spectral decomposition:

$$\rho = \sum_i P_i |\Phi_i\rangle\langle\Phi_i|$$

The probability-like eigenvalues of ρ
allow us to define the entropy of a density operator as:

$$S_\rho = -\sum_{j=1}^{d} P_j \ln P_j = -\mathrm{Tr}\Big[\rho \ln \rho\Big]$$

- The von-Neumann entropy is basis-independent.
- The von-Neumann entropy varies in the range $0 \le S_\rho \le \ln d$.

Von-Neumann entropy of a quantum state

- **Pure state**:

$$\boxed{\rho = |\psi\rangle\langle\psi|}$$

ρ has the form of a $1d$ projector
Probability eigenvalues $\{1, 0, 0, \ldots\}$
$\Rightarrow$ **Zero entropy** $S_\rho = 0$.

- **Fully mixed state**:

$$\boxed{\rho = \frac{\mathbf{1}}{d}}$$

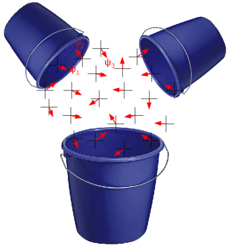
ρ is proportional to the identity matrix
Probability eigenvalues $\{1/d, 1/d, \ldots\}$
$\Rightarrow$ **Maximal entropy** $S_\rho = \ln d$.

Mixing of ensembles

$$S(\rho) = -\mathrm{Tr}\Big[\rho \ln \rho\Big]$$

If different ensembles $\rho_1, \rho_2, \ldots$ are put together, the resulting ensemble is simply

$$\rho = \sum_i \rho_i .$$



If different ensembles $\rho_1, \rho_2, \ldots$ are mixed with different probabilities (weights) q_i, we get

$$\rho = \sum_i q_i \, \rho_i .$$

Concavity under probabilistic mixing (the entropy cannot decrease):

$$\boxed{S\Big(\sum_i q_i \rho_i\Big) \;\geq\; \sum_i q_i S(\rho_i)}$$

Alternative information measures

Recall:

- The information of an event i with probability p_i is $H_i = -\ln p$.
- Shannon entropy $H = \sum_i p_i H_i = -\sum_i p_i \ln p_i$
- von-Neumann entropy $S = \langle -\ln \rho \rangle = -\mathrm{Tr}[\rho \ln \rho]$

Alternative information measures

Recall:

- The information of an event i with probability p_i is $H_i = -\ln p$.
- Shannon entropy $H = \sum_i p_i H_i = -\sum_i p_i \ln p_i$
- von-Neumann entropy $S = \langle -\ln \rho \rangle = -\mathrm{Tr}[\rho \ln \rho]$

The standard entropy is the arithmetic average (mean value) of the information distribution. But one may also consider higher moments

$$H^{(n)} = \sum_i p_i H_i^n \qquad S^{(n)} = \mathrm{Tr}[\rho(-\ln \rho)^n]$$

For example, $S^{(2)} - (S^{(1)})^2$ would be the **information variance**.

Renyi entropy

Knowing all moments or all cumulants, one could in principle recover the complete spectrum of the density matrix.

- Moment-generating function:

$$M(t) = \langle e^{-t \ln \rho} \rangle = \langle \rho^{-t} \rangle = \mathrm{Tr}[\rho^{1-t}]$$

- Cumulant-generating function:

$$K(t) = \ln M(t) = \ln \mathrm{Tr}[\rho^{1-t}]$$

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- Cumulant-generating function:

$$K(t) = \ln M(t) = \ln \mathrm{Tr}[\rho^{1-t}]$$

This is used to define the **Renyi entropy**

$$\boxed{S_q \;=\; \frac{1}{1-q} K(1-q) \;=\; \frac{1}{1-q} \ln \mathrm{Tr}[\rho^q]}$$

For $q \to 1$ this reduces to the von-Neumann entropy. The Renyi entropy incorporates all details of the information distribution, not only the mean.

Renyi entropy

For taking home:

In quantum information theory, the Renyi entropy
is so important because it is related to the

cumulatant-generating function
of the whole information distribution.

The ordinary entropy gives only the average information.

Information-preserving maps

An **information-preserving map** $\rho \mapsto \rho'$ leaves all information measures (von-Neumann entropy, Renyi entropy, ...) invariant.

$\Rightarrow$ This means that information-preserving maps preserve the spectrum (eigenvalues) of the density matrix, i.e.

$$\rho = \sum_{j=1}^{d} P_j \, |\Phi_j\rangle\langle\Phi_j| \qquad \mapsto \qquad \rho' = \sum_{j=1}^{d} P_j \, |\Phi'_j\rangle\langle\Phi'_j|$$

with $\langle\Phi_i|\Phi_j\rangle = \langle\Phi'_i|\Phi'_j\rangle = \delta_{ij}$ and the same P_j in both expressions.

Such a map has to be **unitary**:

$$\rho' = \mathbf{U}\rho\mathbf{U}^\dagger, \qquad \mathbf{U} = \sum_{k=1}^{d} |\Phi'_k\rangle\langle\Phi_k|, \qquad \mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{1}.$$

Information-preserving maps

Unitary operations...

- preserve all information-related aspects.
- neither create nor destroy entropy.
- are represented by unitary matrices $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$.
- can be viewed as 'rotations' in Hilbert space $\mathcal{H}$.
- have eigenvalues on the complex unit circle (phases)
- form the group $U(d)$, where d is the dimension of $\mathcal{H}$.
- can be generated by taking the imaginary exponential of Hermetian operators (generators, modular Hamiltonians).

Example:

Time evolution: $\rho(t) = \mathbf{U}(t)\,\rho(0)\mathbf{U}^\dagger(t) \quad \text{with } \mathbf{U}(t) = e^{-i\mathbf{H}t}$.

Quantum State Distance Measures

Quantum state distance measures

Problem: How similar are two given quantum states ρ and σ?

Various **distance measures** are known:

1. Trace distance
2. Quantum fidelity
3. Relative entropy

1. Trace distance

Define the ***p*-norm** of a matrix:

$$||\mathbf{A}||_p = \mathrm{Tr}\Big[|\mathbf{A}|^p\Big]^{1/p} \qquad \text{where} \qquad |\mathbf{A}| := \sqrt{\mathbf{A}\mathbf{A}^\dagger}$$

Special cases:

$$||\mathbf{A}||_1 = \mathrm{Tr}|\mathbf{A}|$$
$$||\mathbf{A}||_\infty = \text{maximal eigenvalue of } |\mathbf{A}|$$

1. Trace distance

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Special cases:

$$||\mathbf{A}||_1 = \mathrm{Tr}|\mathbf{A}|$$
$$||\mathbf{A}||_\infty = \text{maximal eigenvalue of } |\mathbf{A}|$$

Definition of the **trace distance**:

$$T(\rho,\sigma) := \frac{1}{2}||\rho-\sigma||_1 = \frac{1}{2}\mathrm{Tr}\left[\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}\right].$$

One can show [Nielsen] that the trace distance is the probability at which the states can be distinguished with an optimal measurement.

2. Quantum fidelity

Definition of the **quantum fidelity**

$$F(\rho,\sigma) = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right].$$

For pure states $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ one has $\sqrt{\rho} = \rho, \sqrt{\sigma} = \sigma$, hence

$$F(\rho,\sigma) = \sqrt{\langle\phi|\psi\rangle\langle\psi|\phi\rangle} = |\langle\phi|\psi\rangle|.$$

- The fidelity is unitary-invariant and behaves almost like a metric in the space of density matrices.
- Relation to the trace distance (Fuchs - van den Graaf):

$$1 - F(\rho,\sigma) \le D(\rho,\sigma) \le \sqrt{1 - F(\rho,\sigma)^2}\,.$$

3. Relative entropy

Classical relative entropy:

Suppose that we assume a 'wrong' probability distribution $\{q_i\}$ instead of the correct distribution $\{p_i\}$. Then the information mismatch of configuration i would be $(-\ln q_i) - (-\ln p_i) = \ln \frac{p_i}{q_i}$.

The average mismatch is known as the **relative entropy** or **Kullback-Leibler divergence**:

$$D(p||q) = \sum_i p_i \ln \frac{p_i}{q_i}$$

Quantum relative entropy:

$$S(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma).$$

$S(\rho\|\sigma)$ behaves almost like a metric, but it is non-symmetric.

Quantum Thermostatics

Quantum Thermostatics: Microcanonical ensemble

Gibbs postulate:

In an isolated system at thermal equilibrium S is maximal.

Classical physics: $S = \ln|\Omega|$

Quantum physics: $S(\rho) = \ln \dim\mathcal{H}$

$$\Rightarrow \quad \rho = \frac{\mathbf{1}}{\dim\mathcal{H}}$$

Microcanonical ensemble $\Leftrightarrow$ Fully mixed state

Example: Microcanonical qubit: $\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$

Quantum Thermostatics: Canonical ensemble

If energy is exchanged with a heat bath, the entropy is maximised under the constraint that the energy average $\overline{E} = \langle \mathbf{H} \rangle$ is constant.

$$\delta S[\rho] = \delta \mathrm{Tr}[\rho \ln \rho] = 0\,, \qquad \delta \mathrm{Tr}[\rho] = 0\,, \qquad \delta \mathrm{Tr}[\mathbf{H}\rho] = 0$$

Quantum Thermostatics: Canonical ensemble

If energy is exchanged with a heat bath, the entropy is maximised under the constraint that the energy average $\overline{E} = \langle \mathbf{H} \rangle$ is constant.

$$\delta S[\rho] = \delta \mathrm{Tr}[\rho \ln \rho] = 0\,, \qquad \delta \mathrm{Tr}[\rho] = 0\,, \qquad \delta \mathrm{Tr}[\mathbf{H}\rho] = 0$$

Introduce Lagrange multiplyer α, β:

$$\Rightarrow \delta\Big(\mathrm{Tr}[\rho \ln \rho] + \alpha \mathrm{Tr}[\rho] + \beta \mathrm{Tr}[\mathbf{H}\rho]\Big) = \mathrm{Tr}\Big[\delta\rho(\ln \rho + \alpha + \beta \mathbf{H})\Big] = 0\,.$$

Solution: $\rho \;=\; e^{-\alpha-\beta\mathbf{H}} \;\Rightarrow\; \rho = \frac{1}{Z} e^{-\mathbf{H}/k_B T}$ with $Z = \mathrm{Tr}\left[e^{-\beta\mathbf{H}}\right]$

$$\overline{E} \;=\; \langle \mathbf{H} \rangle \;=\; \mathrm{Tr}[\rho \mathbf{H}] \;=\; \frac{\mathrm{Tr}[\mathbf{H}e^{-\beta\mathbf{H}}]}{\mathrm{Tr}[e^{-\beta\mathbf{H}}]} \;=\; -\partial_\beta \ln Z$$

Generalized Measurements (POVMs)

Projective measurements

Recall projective von-Neumann measurement:

Textbook version:

A measurement is represented by a Hermitean operator

$$\mathbf{M} = \mathbf{M}^\dagger = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|$$

with an orthonormal set of eigenstates $|\phi_n\rangle$.

- The apparatus yields the result λ_n with probability $|\langle\psi|\phi_n\rangle|^2$ and projects $|\psi\rangle$ instantaneously on $|\phi_n\rangle$.
- Thus, all following measurements would give the same result.
- The expectation value of the measurement is $\langle\mathbf{M}\rangle = \langle\psi|\mathbf{M}|\psi\rangle$.

Projective measurements)

Recall projective von-Neumann measurement:

Advanced version:

A measurement is represented by a set of orthogonal projectors

$$E_n = |\phi_n\rangle\langle\phi_n|, \qquad E_n E_m = \delta_{nm} E_n$$

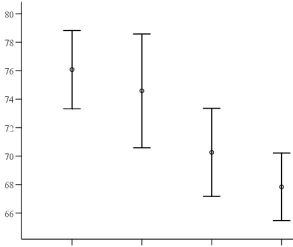
Upon measurement, the quantum state ρ is mapped to

$$\rho \;\mapsto\; \rho' = \sum_n E_n \, \rho \, E_n$$

Repeating the same measurement again does not change ρ'.

Realistic measurements

Realistic measurement

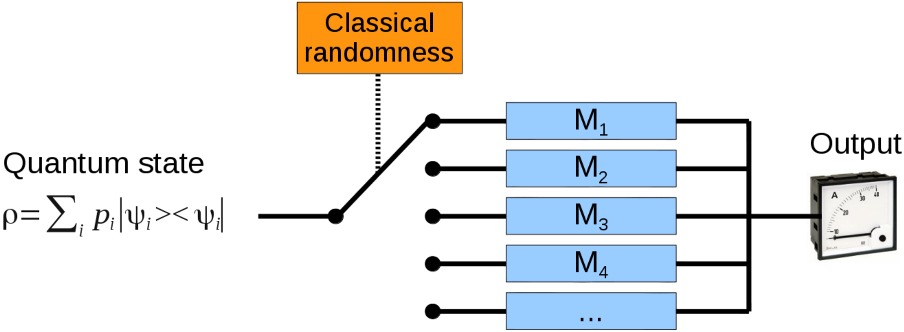


- exactly orthonormal eigenstates?
- result exactly reproduced on repitition?

Figure by Nijaki, Wikimedia

Realistic measurements

Simple model for a non-perfect measurement:



$\Uparrow$

Many slightly different perfect (projective) measurements.

Generalized measurements (POVMs)

Usual projective measurement:

$$\rho \to \rho' = \sum_n E_n \, \rho \, E_n$$

Generalized measurement (take $\mathbf{M}_j$ with probability q_j):

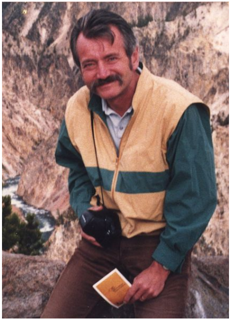
$$\rho \to \rho' = \sum_k q_k \sum_n E_n^{(k)} \, \rho \, E_n^{(k)}$$

- cannot be written as a projective measurement
- is a legal quantum operation (ρ' is again a density matrix)

Kraus Theorem

University of Würzburg:

- Nobel prize for X-rays
- Nobel prize for Quantum Hall effect
- Karl Kraus (1971): **Important theorem in quantum information theory.**



Kraus Theorem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces.

Let $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ be a quantum operation
(meaning that Φ applied to a density matrix is again a density matrix).

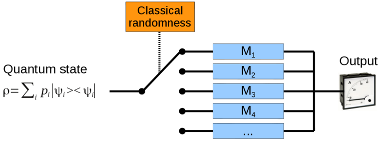
Then Φ can be written in the form

$$\boxed{\rho' = \Phi(\rho) = \sum_k \mathbf{B}_k \rho \mathbf{B}_k^\dagger}$$

with $\sum_k \mathbf{B}_k \mathbf{B}_k^\dagger = \mathbf{1}$.

The operators $\{\mathbf{B}_k\}$ are called **Kraus Operators**.
They are unique up to unitary transformations.

Generalized measurements (POVMs)



$$\rho \to \rho' = \color{red}{\sum_k q_k} \sum_n E_n^{(k)} \rho E_n^{(k)}$$

Applying the Kraus theorem there exist matrices $\{\mathbf{B}_k\}$ so that the above formula can be written as

$$\boxed{\rho' = \sum_k \mathbf{B}_k \rho \mathbf{B}_k^\dagger}$$

A measurement described by a set of operators $\{\mathbf{B}_k\}$ with $\sum_k \mathbf{B}_k \mathbf{B}_k{}^\dagger = \mathbf{1}$ is called a

Positive Operator-Valued Measurement **(POVM)**.

Generalized measurements (POVMs)

Check special case of projective von-Neumann measurements:

$$\rho \;\mapsto\; \rho' = \sum_k E_k \, \rho \, E_k$$

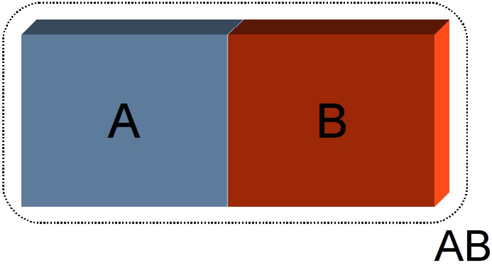
$$\rho \;\mapsto\; \rho' = \sum_k \mathsf{B}_k \rho {\mathsf{B}_k}^\dagger$$

$\Rightarrow \quad \mathsf{B}_k = E_k$ are just the orthogonal projection operators.

✓

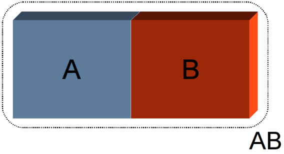
Outline

Bipartite systems



Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

Bipartite systems - Tensor product



Classical physics:

System A in config. c_A
System B in config. c_B $\Rightarrow$ System AB in config. $c_{AB} := (c_A, c_B)$.

Quantum physics:

System A has amplitudes ψ_{c_A}
System B has amplitudes ψ_{c_B} $\Rightarrow$ System AB has amplitudes $\psi_{c_{AB}} := \psi_{c_A}\psi_{c_B}$.

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

Do not confuse direct sum and tensor product

- **Direct sum** $\mathcal{H}_A \oplus \mathcal{H}_B$: Dimensions add up, vectors are simply concatenated:

$$|u\rangle \oplus |v\rangle = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \oplus \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \\ v^1 \\ v^2 \end{pmatrix}$$

classical

- **Tensor product** $\mathcal{H}_A \otimes \mathcal{H}_B$: Dimensions multiply, vector components are multiplied in all combinations:

$$|u\rangle \otimes |v\rangle = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \otimes \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \\ u_3 v_1 \\ u_3 v_2 \end{pmatrix}$$

quantum

Tensor product of operators

Likewise we can compute tensor product of operators (matrices) by forming products of all combinations of the components:

$$
\begin{aligned}
\mathbf{C} = \mathbf{A} \otimes \mathbf{B} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\
&= \begin{pmatrix}
a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} & a_{13}b_{11} & a_{13}b_{12} \\
a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} & a_{13}b_{21} & a_{13}b_{22} \\
a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} & a_{23}b_{11} & a_{23}b_{12} \\
a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} & a_{23}b_{21} & a_{23}b_{22} \\
a_{31}b_{11} & a_{31}b_{12} & a_{32}b_{11} & a_{32}b_{12} & a_{33}b_{11} & a_{33}b_{12} \\
a_{31}b_{21} & a_{31}b_{22} & a_{32}b_{21} & a_{32}b_{22} & a_{33}b_{21} & a_{33}b_{22}
\end{pmatrix}
\end{aligned}
$$

Tensor products in practice

Vectors of the form $|a\rangle \otimes |b\rangle$ are called **factorizable** or **product states**.
$\mathcal{H}_A \otimes \mathcal{H}_B$ is the set of all product states *plus* all their linear combinations.

Some useful rules:

Tensor products in practice

Vectors of the form $|a\rangle \otimes |b\rangle$ are called **factorizable** or **product states**.
$\mathcal{H}_A \otimes \mathcal{H}_B$ is the set of all product states *plus* all their linear combinations.

Some useful rules:

$$\Big(|a\rangle + |b\rangle\Big) \otimes \Big(|c\rangle + |d\rangle\Big) = |a\rangle \otimes |c\rangle + |a\rangle \otimes |d\rangle + |b\rangle \otimes |c\rangle + |b\rangle \otimes |d\rangle$$

$$(\lambda|a\rangle) \otimes (\mu|b\rangle) = \lambda\mu(|a\rangle \otimes |b\rangle)$$

$$\lambda \otimes \mathbf{A} \equiv \lambda\mathbf{A}\,, \qquad \lambda \otimes \mu \equiv \lambda\mu$$

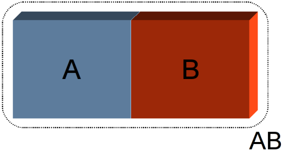
$$(\mathbf{A} \otimes \mathbf{B})(|u\rangle \otimes |v\rangle) = (\mathbf{A}|u\rangle) \otimes (\mathbf{B}|v\rangle)$$

$$(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$$

$$\mathrm{Tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{Tr}(\mathbf{A})\,\mathrm{Tr}(\mathbf{B})$$

$$\det(\mathbf{A} \otimes \mathbf{B}) = (\det \mathbf{A})^{dim(\mathcal{H}_B)}(\det \mathbf{B})^{dim(\mathcal{H}_A)}$$

Bipartite systems



There are two different aspects of quantum states with respect to a partition:

- **Entanglement** (quantum)
- **Correlation** (classical)

Classical correlations

Classical correlations are simply due to the composition of the ensemble:

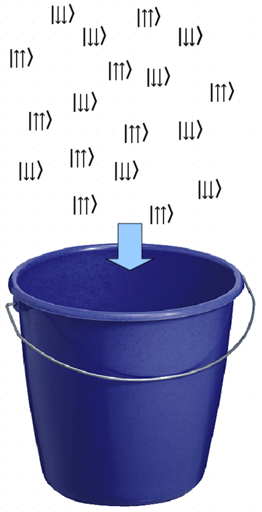
Example:

The two pure quantum states

$$|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$$
$$|\downarrow\downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$$

are mixed in the ensemble by 50-50:

$$\rho \;=\; \frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$$



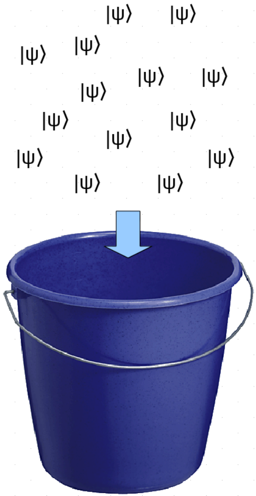
Quantum correlations (entanglement)

Quantum correlations are due to a coherent superposition of amplitudes

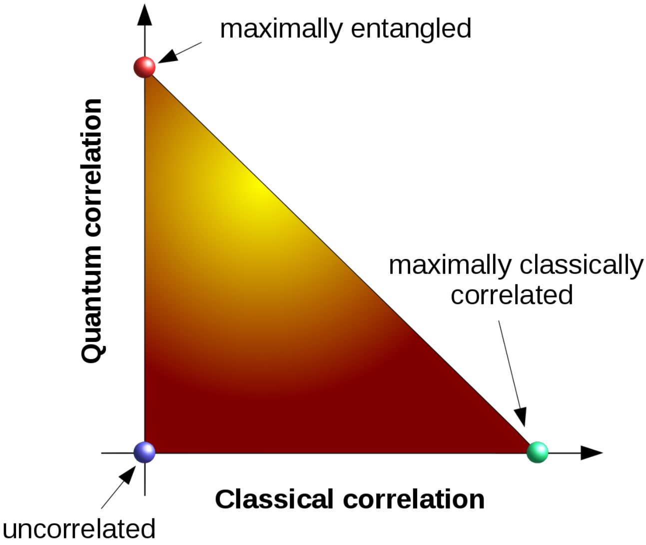
$$|\psi\rangle \;=\; \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle$$

$$\rho \;=\; |\psi\rangle\langle\psi|$$

In this case the ensemble is not mixed but it consists only of one type of pure states.



Over-simplified **cartoon** of correlation landscape



States of maximal classical and quantum corrleation

Example: 2-qubit system:

State with maximal classical correlations:

$$\rho_{class} = \frac{1}{2}\Big(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\Big) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

State with maximal quantum correlations (EPR state):

$$\rho_{quant} = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

How to distinguish classical and quantum correlations

Example: 2-qubit system:

What we would like to know: $\uparrow\uparrow$ or $\uparrow\downarrow$ or $\downarrow\uparrow$ or $\downarrow\downarrow$?

Measurement operator: $\sigma^z \otimes \sigma^z$

Measurement projectors:
$$E_{\uparrow\uparrow} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow|, \quad E_{\uparrow\downarrow} = |\uparrow\downarrow\rangle\langle\uparrow\downarrow|,$$
$$E_{\downarrow\uparrow} = |\downarrow\uparrow\rangle\langle\downarrow\uparrow|, \quad E_{\downarrow\downarrow} = |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.$$

classical	quantum
$p_{\uparrow\uparrow} = \mathrm{Tr}[\rho_{class} E_{\uparrow\uparrow}] = 1/2$	$p_{\uparrow\uparrow} = \mathrm{Tr}[\rho_{quant} E_{\uparrow\uparrow}] = 1/2$
$p_{\uparrow\downarrow} = \mathrm{Tr}[\rho_{class} E_{\uparrow\downarrow}] = 0$	$p_{\uparrow\downarrow} = \mathrm{Tr}[\rho_{quant} E_{\uparrow\downarrow}] = 0$
$p_{\downarrow\uparrow} = \mathrm{Tr}[\rho_{class} E_{\downarrow\uparrow}] = 0$	$p_{\downarrow\uparrow} = \mathrm{Tr}[\rho_{quant} E_{\downarrow\uparrow}] = 0$
$p_{\downarrow\downarrow} = \mathrm{Tr}[\rho_{class} E_{\downarrow\downarrow}] = 1/2$	$p_{\downarrow\downarrow} = \mathrm{Tr}[\rho_{quant} E_{\downarrow\downarrow}] = 1/2$

In both cases we find only $\uparrow\uparrow$ and $\downarrow\downarrow$, each with probability 1/2.

How to distinguish classical and quantum correlations

Example: 2-qubit system:

With the observable $\sigma^z \otimes \sigma^z$ we thus cannot see a difference.

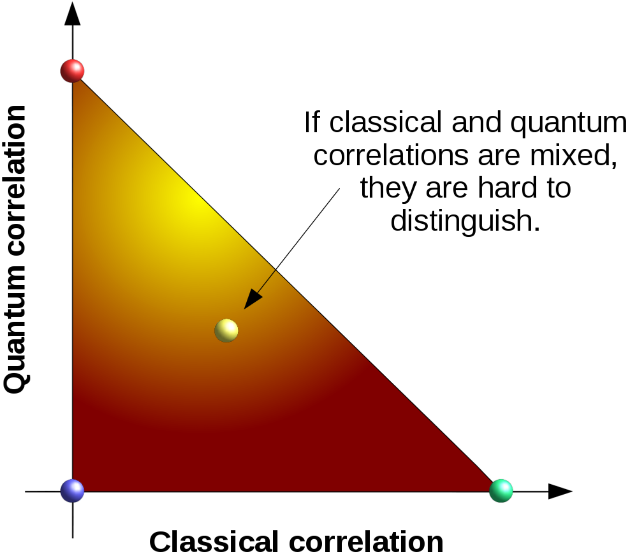
$$\begin{aligned} \langle \sigma^z \otimes \sigma^z \rangle_{class} &= \mathrm{Tr}[\rho_{class}\, \sigma^z \otimes \sigma^z] = 1 \\ \langle \sigma^z \otimes \sigma^z \rangle_{quant} &= \mathrm{Tr}[\rho_{quant}\, \sigma^z \otimes \sigma^z] = 1 \end{aligned}$$

But with the observable $\sigma^x \otimes \sigma^x$ we do see a difference.

$$\begin{aligned} \langle \sigma^x \otimes \sigma^x \rangle_{class} &= \mathrm{Tr}[\rho_{class}\, \sigma^x \otimes \sigma^x] = \mathbf{0} \\ \langle \sigma^x \otimes \sigma^x \rangle_{quant} &= \mathrm{Tr}[\rho_{quant}\, \sigma^x \otimes \sigma^x] = 1 \end{aligned}$$

In order to see the difference between classical and quantum correlations, one has to use several kinds of measurements (rotate the analyzer).

How to distinguish classical and quantum correlations



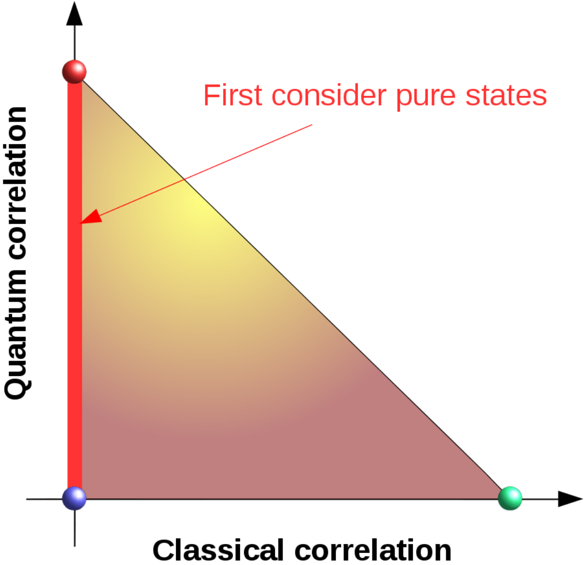
Entanglement measures

In the following we will see that...

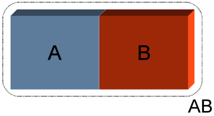
- quantifying the correlation of classical systems is easy.
- quantifying the entanglement of pure quantum states is also easy.
- separating classical correlations and quantum entanglement quantitatively in an arbitrary mixed state is extremely difficult.

$\Rightarrow$ There is a large variety of entanglement measures.
$\Rightarrow$ The whole issue of measuring entanglement is not yet settled.

Entanglement of pure quantum states



Separable $\equiv$ not entangled



- Pure states $\rho = |\psi\rangle\langle\psi|$ have no classical correlations.

- A pure state is said to be **separable** if it factorizes:

$$|\psi\rangle = |\psi_{\mathbf{A}}\rangle \otimes |\psi_{\mathbf{B}}\rangle$$

$$\rho \;=\; |\psi_{\mathbf{A}}\rangle\langle\psi_{\mathbf{A}}| \;\otimes\; |\psi_{\mathbf{B}}\rangle\langle\psi_{\mathbf{B}}|$$

- A **pure** state is said to be **entangled** if it is not separable.

Measurement on subsystems

The subsystem may be may be separated at a large distance.

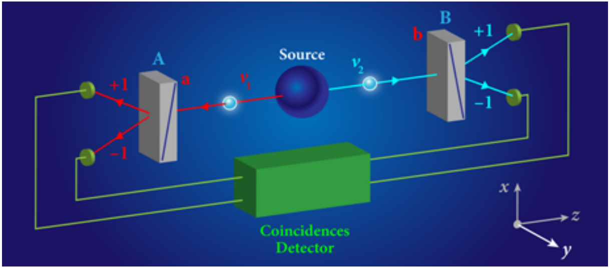
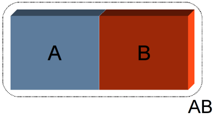
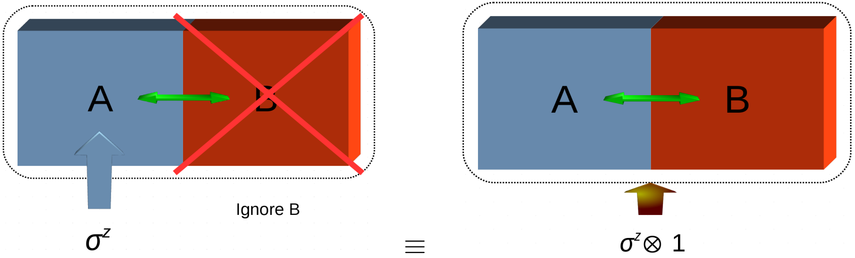


Figure: APS Alan Stonebraker



Suppose we want to measure the z-component of the spin of subsystem A. This means to perform the measurement on A while doing nothing on B.

Measurement on subsystems



Suppose we want to measure the z-component of the spin of subsystem A:

$$\left.\begin{array}{l}\text{- measure } \sigma^z \text{ on } A \\ \text{- measure nothing on } B\end{array}\right\} \quad \equiv \quad \text{measure } (\sigma^z \otimes \mathbf{1}) \text{ on } AB.$$

Partial trace

Measuring $(\sigma^z \otimes 1)$ on ρ_{AB}...

$$\langle \sigma^z \rangle = \mathrm{Tr}\Big[\rho_{AB}\,\sigma^z\Big] = \mathrm{Tr}\Big[\rho_{AB}\,(\sigma^z \otimes \mathbf{1})\Big]$$

Let $|i\rangle$ and $|j\rangle$ be orthonormal basis vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$.
Then $|ij\rangle := |i\rangle \otimes |j\rangle$ is a basis of $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\begin{aligned}
\langle \sigma^z \rangle &= \sum_{ijkl} \langle ij|\rho_{AB}\,|kl\rangle\langle kl|\,(\sigma^z \otimes 1)|ij\rangle \\
&= \sum_{ijkl} \langle ij|\rho_{AB}|kl\rangle\,\langle k|\sigma^z|i\rangle\,\langle l|j\rangle \\
&= \sum_{ik} \underbrace{\Big(\sum_j \langle ij|\rho_{AB}|kj\rangle\Big)}_{=:\,\langle i|\rho_A|k\rangle}\,\langle k|\sigma^z|i\rangle \;=\; \mathrm{Tr}\Big[\rho_A\sigma^z\Big]
\end{aligned}$$

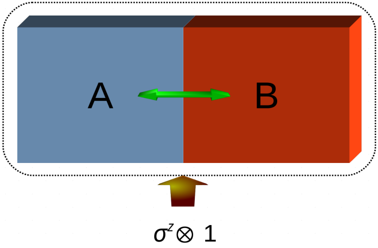
...is the same as measuring σ^z on $\rho_A = \mathrm{Tr}_B\big[\rho_{AB}\big]$.

Partial trace

Example:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \rho_{\mathbf{A}} = \mathrm{Tr}_{\mathbf{B}}[\rho] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



This is a maximally mixed state.

Pure state entanglement, seen from the perspective of a subsystem, looks like classical randomness.

Pure-state entanglement

- Pure state entanglement, seen from the perspective of a subsystem, looks like classical randomness.
- While the composite system is pure (entropy zero), the subsystems have entropy.

$\Rightarrow$ Therefore, if the total system is in a pure state, the **entropy of the subsystems** can be used to **measure the entanglement**.

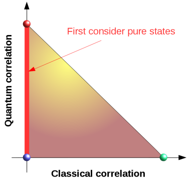
Pure-state entanglement entropy

Let AB be in a pure state $\rho_{AB} = |\psi\rangle\langle\psi|$.

$$S_\mathbf{A} = S(\rho_\mathbf{A}) = -\mathrm{Tr}[\rho_\mathbf{A} \ln \rho_\mathbf{A}]$$

$$S_\mathbf{B} = S(\rho_\mathbf{B}) = -\mathrm{Tr}[\rho_\mathbf{B} \ln \rho_\mathbf{B}]$$

One can show: $S_\mathbf{A} = S_\mathbf{B}$



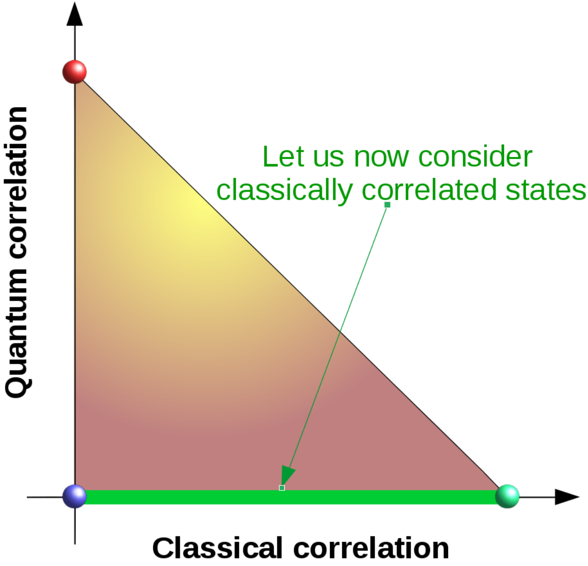
$\Rightarrow$ Undisputed unique entanglement measure for *pure* states:

$\boxed{\textbf{Entanglement entropy}: E := S_\mathbf{A} = S_\mathbf{B}}$

Example: Maximally entangled 2-qubit EPR state:

$$\rho_{quant} = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) : \qquad E = \ln 2 \simeq 1 \text{ bit}$$

Measuring classically correlated states



Separability

We have already defined separability for ***pure states****:*

- Pure states $\rho = |\psi\rangle\langle\psi|$ have no classical correlations.
- A pure state is said to be **separable** if it factorizes:

$$\rho \;=\; |\psi_{\mathbf{A}}\rangle\langle\psi_{\mathbf{A}}| \;\otimes\; |\psi_{\mathbf{B}}\rangle\langle\psi_{\mathbf{B}}|$$

- A **pure** state is said to be **entangled** if it is not separable.

Separability

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---

*Now we extend the notion of separability to **general states**:*

- Product states $\rho_{AB} = \rho_A \otimes \rho_B$ have no correlations at all, i.e., they are neither classically correlated nor entangled.
- A state is called **separable** if it can be written in the form

$$\boxed{\rho_{AB} = \sum_k q_k \, \rho_A^{(k)} \otimes \rho_B^{(k)}}$$

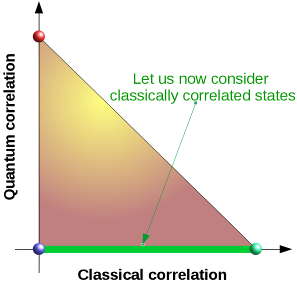
  where the $\{q_k\}$ are probability-like coefficients.
- A state is called **entangled** if it is **not separable**. $\longleftarrow$

Measuring classically correlated states

Consider a separable (= non-entangled) state along the green line in the figure.

To quantify its classical correlations, use the **mutual information**:

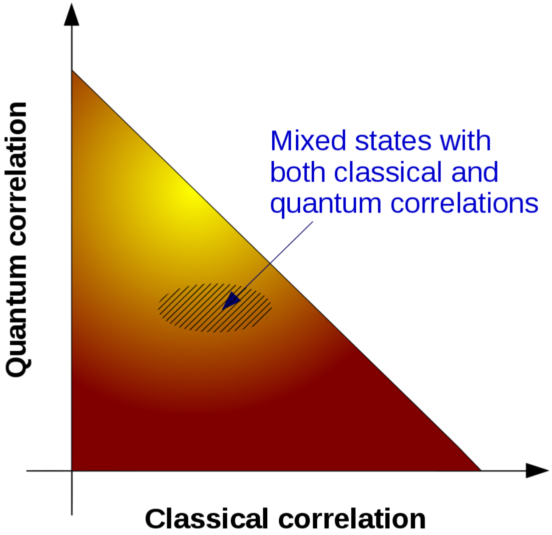
$$\boxed{I_{A:B} = S_{\mathbf{A}} + S_{\mathbf{B}} - S_{\mathbf{AB}}}$$



Example: Maximally classically correlated 2-qubit state:

$$\rho_{quant} \;=\; \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) : \qquad I_{A:B} = \ln 2 \simeq 1 \text{ bit}$$

Distinguish quantum entanglement and classical correlation



Could we use the entropy and the mutual information as coordinates?

Distinguish quantum entanglement and classical correlation

Unfortunately, the entanglement entropy responds also to classical correlations.

Likewise, the mutual information also responds to entanglement.

Example: 2-qubit system:

State	$I_{A:B}$	E
no correlations (product state)	0	0
maximal classical correlations	1	1
maximal quantum correlations	2	1

$\Rightarrow$ **More sophisticated entanglement measures needed.**

Schmidt Decomposition Theorem

The most important theorem in bipartite quantum systems.

Schmidt decomposition

Schmidt decomposition theorem:

Every pure state $|\psi\rangle$ of a bipartite system can be decomposed as

$$\boxed{|\psi\rangle = \sum_{n=1}^{r} \alpha_n \, |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}}$$

Erwin Schmidt 1876-1959

where the vectors $|n\rangle_{\mathbf{A}} \in \mathcal{H}_{\mathbf{A}}$ and $|n\rangle_{\mathbf{B}} \in \mathcal{H}_{\mathbf{B}}$ are **mutually orthonormal** with $r \leq \min(d_{\mathbf{A}}, d_{\mathbf{B}})$.

The coefficients $\alpha_n \geq 0$ are the so-called **Schmidt numbers** obeying

$$\sum_n \alpha_n^2 = 1$$

Schmidt decomposition

Note that this is much more than a basis representation.

- In an ordinary basis representation we have a double sum running independently in each subsystem:

$$|\psi\rangle = \sum_{i=1}^{d_\mathbf{A}} \sum_{j=1}^{d_\mathbf{B}} \psi_{ij} \, |i\rangle_\mathbf{A} \otimes |j\rangle_\mathbf{B}$$

- In the **Schmidt decomposition** we have only a **single sum**

$$|\psi\rangle = \sum_{n=1}^{r} \alpha_n \, |n\rangle_\mathbf{A} \otimes |n\rangle_\mathbf{B}$$

running from 1 to r where $r \leq \min(d_\mathbf{A}, d_\mathbf{B})$.

Schmidt decomposition

Interpretation of the Schmidt numbers:

$$|\psi\rangle = \sum_{n=1}^{r} \alpha_n |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

Decomposition of a pure-state density matrix:

$$\Rightarrow \rho = |\psi\rangle\langle\psi| = \sum_{n,m=1}^{r} \alpha_n \alpha_m \, |n\rangle\langle m|_{\mathbf{A}} \otimes |n\rangle\langle m|_{\mathbf{B}}$$

Compute reduced density matrices:

$$\rho_{\mathbf{A}} = \mathrm{Tr}_{\mathbf{B}}[\rho] = \sum_{n=1}^{r} \alpha_n^2 |n\rangle\langle n|_{\mathbf{A}} \,, \qquad \rho_{\mathbf{B}} = \mathrm{Tr}_{\mathbf{A}}[\rho] = \sum_{n=1}^{r} \alpha_n^2 |n\rangle\langle n|_{\mathbf{B}}$$

The α_n^2 are just the probability-eigenvalues in the reduced state.

Proof: Singular value decomposition
This also confirms that $E_{\mathrm{A}} = E_{\mathrm{B}}$

Quantum Purification

Quantum purification

is a direct consequence of the Schmidt theorem:

http://www.quantum-purification.info

Each mixed state can be represented as the reduced density matrix of a (generally entangled) pure state in a suitably extended Hilbert space.

$\Rightarrow$ Classical randomness (entropy) can always be interpreted as entanglement with something external.

Quantum Purification

- Take an arbitrary mixed state $\rho_{\mathbf{A}}$ on the Hilbert space $\mathcal{H}_{\mathbf{A}}$:

$$\rho_{\mathbf{A}} = \sum_n p_n |n\rangle\langle n|_{\mathbf{A}}$$

- Extend $\mathcal{H}_{\mathbf{A}}$ by an auxiliary Hilbert space $\mathcal{H}_{\mathbf{B}}$ of the same dimension.
- Define some orthonormal basis $|n\rangle_{\mathbf{B}}$ in $\mathcal{H}_{\mathbf{B}}$.
- Define the **pure** state

$$|\psi\rangle = \sum_n \sqrt{p_n}\, |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

$$\Rightarrow \quad |\psi\rangle\langle\psi| = \sum_{n,m} \sqrt{p_n p_m}\, |n\rangle\langle m|_{\mathbf{A}} \otimes |n\rangle\langle m|_{\mathbf{B}}$$

Quantum Purification

$$\Rightarrow \quad \rho_{\mathbf{AB}} = |\psi\rangle\langle\psi| = \sum_{n,m} \sqrt{p_n p_m}\, |n\rangle\langle m|_{\mathbf{A}} \otimes |n\rangle\langle m|_{\mathbf{B}}$$

- Take the partial trace over the auxiary space $\mathcal{H}_{\mathbf{B}}$

$$\mathrm{Tr}_{\mathbf{B}}\big[|\psi\rangle\langle\psi|\big] = \sum_k \sum_{n,m} \sqrt{p_n p_m}\, |n\rangle\langle m|_{\mathbf{A}} \underbrace{\langle k||n\rangle\langle m||k\rangle_{\mathbf{B}}}_{=\delta_{kn}\delta_{kn}}$$

$$= \sum_k p_k |k\rangle\langle k|_{\mathbf{A}} = \rho_{\mathbf{A}}$$

The reduced density matrix is just the original mixed state. □

In a suitably extended Hilbert space any mixed state can be represented as a pure state.

Entanglement Criteria

Recall definition of entanglement

- A **pure** state $|\psi\rangle$ is said to be separable if it factorizes:

$$|\psi\rangle = |\psi\rangle_{\mathbf{A}} \otimes |\psi\rangle_{\mathbf{B}}$$

$$\Rightarrow \quad \rho = |\psi\rangle\langle\psi|_{\mathbf{A}} \otimes |\psi\rangle\langle\psi|_{\mathbf{B}}$$

- A **mixed** state ρ is said to be separable if it can be expressed as a probabilistic combination of pure separable states:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \,, \quad |\psi_i\rangle \text{ separable.}$$

- One can show that a state is separable if and only if it can be written in the form

$$\rho \;=\; \sum_i p_i \; \rho_{\mathbf{A}}^{(i)} \otimes \rho_{\mathbf{B}}^{(i)} \,, \qquad 0 \le p_i \le 1, \quad \sum_i p_i = 1$$

- entangled $\equiv$ non-separable

Entanglement criteria vs. entanglement measures

Entanglement criteria are simple checks which provide a sufficient condition for the **existence** of entanglement.

- PPT criterion
- CCNR criterion
- ...

Entanglement measures are quantitative measures which tell us **how strongly** the systems are entangled.

- Entanglement distance measures
- Entanglement of formation
- Quantum discord
- ...

PPT criterion

Definition of the partial transpose $T_{\mathbf{A}}, T_{\mathbf{B}}$:

For a factorizing operator $\mathbf{C} = \mathbf{C_A} \otimes \mathbf{C_B}$ the partial transpose is defined as the transposition of one of the tensor slots:

$$\mathbf{C}^{T_\mathbf{A}} := \mathbf{C}_\mathbf{A}^T \otimes \mathbf{C_B}\,, \qquad \mathbf{C}^{T_\mathbf{B}} := \mathbf{C_A} \otimes \mathbf{C}_\mathbf{B}^T\,.$$

A non-factorizing operator can be written as a linear combination of factorizing ones. So the partial transpose is also well-defined on general operators.

$$T_\mathbf{A} \circ T_\mathbf{B} = T_\mathbf{B} \circ T_\mathbf{A} = T\,, \qquad T \circ T_\mathbf{A} = T_\mathbf{A} \circ T = T_\mathbf{B}\,.$$

PPT criterion

Observation:

Transposition is a positive operation:
If ρ is a density matrix, then ρ^T is also a valid density matrix.

Peres-Horodecki-Criterion (positive partial transpose, PPT):

If ρ is separable, then ρ^{T_A} and ρ^{T_B} are positive operators, that is, they are both physically valid density matrices.

Or the other way round:

If ρ^{T_A} or ρ^{T_B} are **not** valid density matrices, then we know that the subsystems A and B are entangled.

PPT criterion

Example:

Maximally entangled state (Bell state):

$$\rho = \frac{1}{2}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\Big)\Big(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\Big) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \qquad \rho^{T_A} = \rho^{T_B} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues of ρ : $\{0,0,0,1\}$

Eigenvalues of $\rho^{T_A} = \rho^{T_B}$: $\{0, -\mathbf{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}\}$

Interpretation of the PPT criterion

- Classical mechanics is invariant under time reversal

$$\big(q(t), p(t)\big) \quad \to \quad \big(q(-t), -p(-t)\big)$$

- Schrödinger unitary evolution is also invariant under time reversal

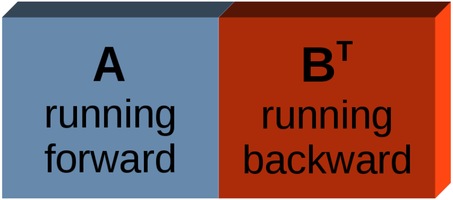
$$\psi(t), \mathbf{H} \quad \to \quad \psi(-t)^*, \mathbf{H}^*$$

which is the same as taking

$$\rho(t) \quad \to \quad \rho^*(-t) = \rho^T(-t)$$

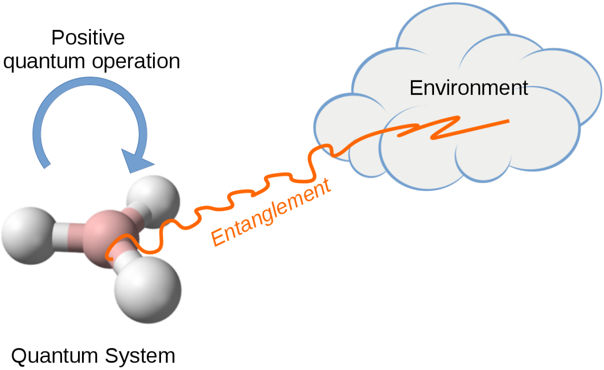
Transposition $\sim$ Time reversal

Interpretation of the PPT criterion



PPT: If this is not a physically valid scenario, then there must be entanglement between the two parts.

Are all positive operations legal?



Our quantum system could be entangled
with something external far away.

Side remark: Completely positive maps

- Completely positive maps $\Phi : \rho \to \Phi(\rho)$ are **physically realizable** positive maps.

- Not all positive maps are physically realizable.

  Example: Transposition $\rho \to \rho^T$ is positive but not physically realizable because it **could be entangled with another unknown external object**.

- Definition: Φ is called completely positive on $\mathcal{H}$ if $\Phi \otimes \mathbf{1}$ is positive on $\mathcal{H} \otimes \mathcal{H}_{aux}$ for every external Hilbert space $\mathcal{H}_{aux}$.

CCNR criterion

Computable Cross Norm or Realignment Criterion

CCNR

What we need to know:

- What is **realignment**?
- What is a **operator Schmidt decomposition?**
- How does the CCNR criterion work ?

Realignment

Operator realignment:

Let $|i\rangle_A$ und $|i\rangle_B$ be a basis of the bipartite Hilbert space $\mathcal{H}_{\mathbf{AB}} = H_{\mathbf{A}} \otimes H_{\mathbf{B}}$ and let $\mathbf{C}$ be an operator with the matrix representation

$$\mathbf{C} = \sum_{ijkl} C_{ij,kl}\, |ij\rangle\langle kl|.$$

Define the **realigned Matrix** $\mathbf{C}^R$ by

$$\mathbf{C}^R = \sum_{ijkl} C_{ij,kl} |ik\rangle\langle jl| = \sum_{ijkl} C_{ik,jl} |ij\rangle\langle kl|$$

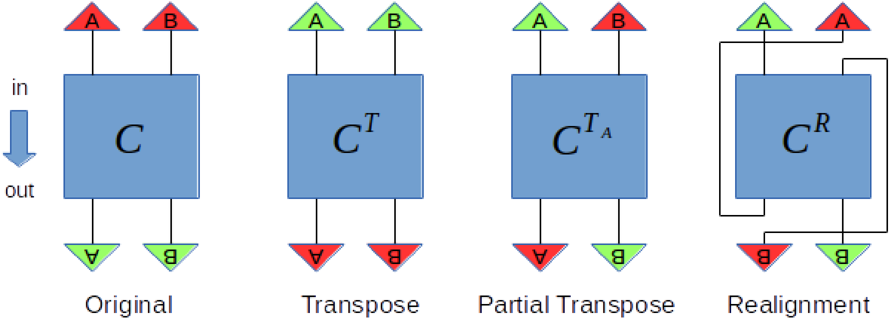
$$\boxed{C^R_{ij,kl} = C_{ik,jl}}$$

Realignment

Operation	Components	Exchanged indices
Normal transpose T	$C^{T}_{ij,kl} = C_{kl,ij}$	$(12) \leftrightarrow (34)$
Partial transpose T_A	$C^{T_A}_{ij,kl} = C_{kj,il}$	$1 \leftrightarrow 3$
Partial transpose T_B	$C^{T_A}_{ij,kl} = C_{il,kj}$	$2 \leftrightarrow 4$
Realignment R	$C^{R}_{ij,kl} = C_{ik,jl}$	$2 \leftrightarrow 3$

Realignmentn

$$\langle \Psi_{AB} | C | \Psi_{AB} \rangle$$



- C maps from the red state to the green state.
- C^R maps from subsystem **A** to subsystem **B**.

Operator Schmidt decomposition

The vector Schmidt decomposition

$$|\psi\rangle = \sum_{n=1}^{r} \alpha_n |n\rangle_{\mathbf{A}} \otimes |n\rangle_{\mathbf{B}}$$

works also for operators

$$\mathbf{C} = \sum_n \alpha_n \mathbf{C}_n^A \otimes \mathbf{C}_n^B.$$

Operator Schmidt decomposition

The vector Schmidt decomposition

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works also for operators

$$\mathbf{C} = \sum_n \alpha_n \mathbf{C}_n^A \otimes \mathbf{C}_n^B.$$

Theorem: The α_n are the singular values of $\mathbf{C}^R$
(the positive square root of the eigenvalues of $\mathbf{C}^{R\dagger}\mathbf{C}^R$)

Induced trace norm:

$$||\mathbf{C}||_s = \sum_n \alpha_n$$

CCNR criterion

Computable Cross Norm or Realignment Criterion (CCNR):

Consider a **separable pure** state:

$$\rho = |\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|_{\mathbf{A}} \otimes |\psi\rangle\langle\psi|_{\mathbf{B}}$$

$\Rightarrow$ Only a single Schmidt number $\alpha_1 = 1$ $\qquad \Rightarrow ||\rho||_s = 1$

Consider a **separable mixed** state. Then ρ is a probabilistic combination of pure separable states ρ_k:

$$||\rho||_s = ||\sum_k p_k \rho_k||_s \leq \sum_k p_k \underbrace{||\rho_k||}_{=1} = 1$$

meaning that $\sum_k \alpha_k \leq 1$. In oppsite direction, we have CCNR:

$$\sum_k \alpha_k > 1 \quad \Rightarrow \quad \text{non-separable} \quad \Leftrightarrow \quad \text{entangled}$$

Entanglement Measures

Entanglement measures

Entanglement measures

- What we expect them to do
- 1. Entanglement measures based on distance
- 2. Entanglement of formation
- 3. Quantum discord

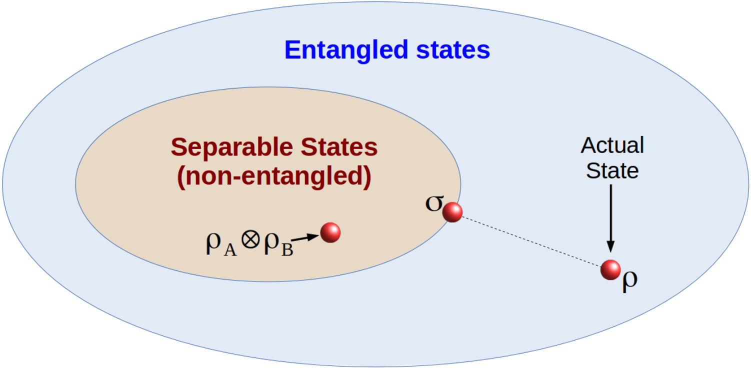
Entanglement measure – List of desired properties

What we expect an measure $E(\rho)$ to do:

1. Separable state $\Leftrightarrow$ No entanglement $\Leftrightarrow$ $E(\rho) = 0$.
2. EPR / Bell states $\Leftrightarrow$ $E(\rho)$ is maximal.
3. Pure states: $E(\rho) = S(\rho_{\mathbf{A}}) = S(\rho_{\mathbf{B}})$
4. $E(\rho)$ should be invariant under local unitary transformations.
5. $E(\rho)$ should not increase under LOCC operations.
6. Symmetry $A \leftrightarrow B$.
7. Convexity on probabilistic mixtures:

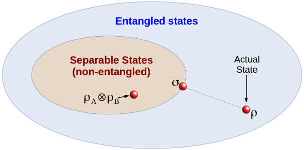
$$E\Big(\sum_k p_k \rho_k\Big) \leq \sum_k p_k E(\rho_k)$$

1. Entanglement measure based on distance



$$E_D(\rho) \;=\; \inf_{\sigma\ \text{separable}} D(\rho,\sigma)\,.$$

Entanglement measure based on distance



Example:

Relative entropy $D_R(\rho, \sigma) = \mathrm{Tr}[\rho(\ln \rho - \ln \sigma)]$
(Quantum-mechanical version of Kullback-Leibler divergence)

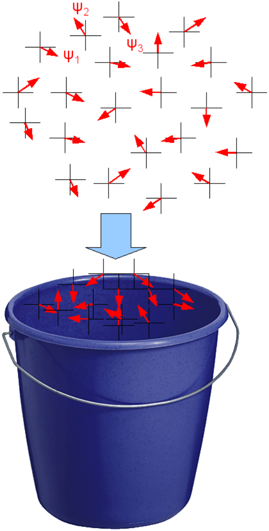
This allows us to define the:

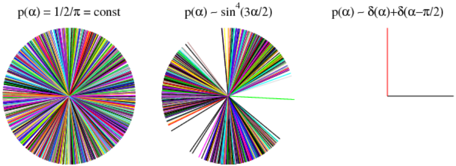
- Quantum mutual information: $S_{A:B} = D_R(\rho, \rho_A \otimes \rho_B)$
- Relative entanglement entropy: $E_R(\rho) = \inf_{\sigma \text{ separable}} D_R(\rho, \sigma)$.

$$E_r(\rho) \leq S_{A:B}$$

Entanglement of formation

- A mixed state is represented by a collection of pure states.
- Each pure state has a well-defined entanglement.
- The representation is not unique. A mixed state represents rather a class of equivalent ensembles.





Entanglement of formation

The entanglement of the representing pure states may be higher than the entanglement of the mixture.

Example: Mixture of two Bell states: $|\pm\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\Big)$

$$\Rightarrow \quad |\pm\rangle\langle\pm| \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho = \frac{1}{2}\Big(|+\rangle\langle+| + |-\rangle\langle-|\Big) \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Mixture has no correlation and no entanglement!

Entanglement of formation

Main idea:

In the equivalence class of ensembles given by the density matrix, let us find the representation of the ensemble for which the averaged entanglement of the representing pure states is minimal:

$$E_f(\rho) = \inf\left\{ \sum_i p_i \, E(|\psi_i\rangle\langle\psi_i|) \,\middle|\, \rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \right\}$$

$$= \inf\left\{ \sum_i p_i \, S_{\rho_{i,\mathbf{A}}} \,\middle|\, \rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \right\}.$$

...very hard to compute!

Entanglement of formation

The entanglement of formation is very difficult to compute.
The only exception a s 2-qubit system.
Here an exact formula has been derived:

$$E_F(\rho) \;=\; S\left[\frac{1+\sqrt{1-C^2(\rho)}}{2}\right]$$

where

$$S[x] \;=\; -x\log_2 x-(1-x)\log_2(1-x)$$
$$C(\rho)=\max(0,\lambda_1-\lambda_2-\lambda_3-\lambda_4)$$

Here λ_i are the decreasingly sorted square roots of the eigenvalues of the following 4×4 matrix:

$$\Lambda=\rho(\sigma^y\otimes\sigma^y)\rho^*(\sigma^y\otimes\sigma^y)$$

Question

Does **"entanglement"** and **"quantum correlations"**

really mean the same?

Reconsider definition of entanglement

quantum correlation = non-classical correlation
entangled state = non-separable state

Separable state:

$$\rho = \sum_i p_i \, \rho_{\mathbf{A}}^{(i)} \otimes \rho_{\mathbf{B}}^{(i)}$$

A bipartite system in a mixed state is *defined* to be entangled if the state is non-separable.

But, as we will see:

QUANTUM CORRELATIONS CAN BE PRESENT IN SEPARABLE STATES.
'NON-ENTANGLED' DOES NOT AUTOMATICALLY MEAN 'CLASSICAL'.

$\Rightarrow$ Study quantum discord

Separability vs. quantum correlation

$$|\pm\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \pm |1\rangle\Big)$$

Consider the following two-qubit state

$$\rho = \frac{1}{4}\Big(|+\rangle\langle+| \otimes |0\rangle\langle 0| \; + \; |-\rangle\langle-| \otimes |1\rangle\langle 1|$$
$$+ \; |0\rangle\langle 0| \otimes |-\rangle\langle-| \; + \; |1\rangle\langle 1| \otimes |+\rangle\langle+|\Big)$$

where $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$ are four non-orthogonal states of each qubit.

Even though ρ is **separable** (i.e. non-entangled),
we will see that it has **quantum correlations**.

Dakić et al., PRL 105, 190502 (2010)

Quantum discord

General idea:

- Quantum correlations are converted into classical correlations by measurement.
- The efficiency of this conversion depends on the choice of the measurement apparatusses on both sides.
- Let us maximize the conversion efficiency over all possible measurements.
- The discord is defined as the correlation difference before and after the measurement.

Note: It seems that the minimization over different measurements is more natural than the minimization over ficticious (non-measurable) representatives of a given ensemble.

Quantum discord

Example: Quantum correlations can be converted into classical correlations by measurement.

Consider maximally entangled Bell state:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

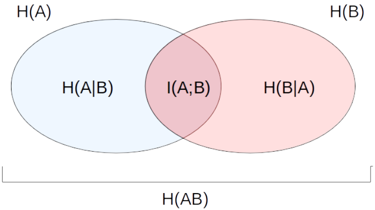
Measurement on either **A** or **B**: $E_0 = |0\rangle\langle 0|, \; E_1 = |1\rangle\langle 1|$

Measurement on **AB**: $E_{ij} = E_i \otimes E_j \; \Big(|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\Big)$

$$\rho = |\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{2}\begin{pmatrix} 1 & & & 1 \\ & & & \\ & & & \\ 1 & & & 1 \end{pmatrix} \quad \Rightarrow \quad \rho' = \sum_{ij} E_{ij}\rho E_{ij} = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & 1 \end{pmatrix}$$

entangled | classically correlated

Definition of the quantum discord



Consider two subsystems **A** and **B**.

Classical information theory:

- $I(\mathbf{A}:\mathbf{B}) = H(\mathbf{A}) + H(\mathbf{B}) - H(\mathbf{AB})$
- $J(\mathbf{A}:\mathbf{B}) = H(\mathbf{B}) - H(\mathbf{B}|\mathbf{A})$

with the Shannon entropy $H(X) = -\sum_i p_X^{(i)} \ln p_X^{(i)}$.

Thanks to Bayes rule these expressions are identical, i.e. we have two equivalent descriptions of the mutual information.

But in quantum theory they are different!

Definition of the quantum discord

- $I(\mathbf{A}:\mathbf{B}) = S(\mathbf{A}) + S(\mathbf{B}) - S(\mathbf{A},\mathbf{B})$
  well defined, quantifies the total correlation (classical+quantum)
- $J(\mathbf{A}:\mathbf{B}) = S(\mathbf{B}) - S(\mathbf{B}|\mathbf{A})$
  depends on the chosen measurement in $\mathbf{A}$!

Definition of the quantum discord

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- $J(\mathbf{A}:\mathbf{B}) = S(\mathbf{B}) - S(\mathbf{B}|\mathbf{A})$
  depends on the chosen measurement in **A**!

If we maximize J over all possible measurements on **A**, it is expected to quantify the classical correlations between the systems. Thus one defines the quantum discord as the difference:

$$\mathcal{D}_A(\rho) = I(\mathbf{A}:\mathbf{B}) - \max_{\{\Pi_j^A\}} J_{\{\Pi_j^A\}}(\mathbf{A}:\mathbf{B})$$

$$= S(\rho_A) - S(\rho) + \min_{\{\Pi_j^A\}} S(\rho_{B|\{\Pi_j^A\}})$$

Definition of the quantum discord

Total amount of correlation:

$$\mathcal{I}(A:B) = S(\rho_{\mathbf{A}}) + S(\rho_{\mathbf{B}}) - S(\rho)$$

Classical correlation:

$$\mathcal{J}(A:B) = S(\rho_{\mathbf{B}}) - \min_{\{\Pi_j\}} \sum_j p_j S(\rho_{B|j})$$

Quantum discord:

$$\mathcal{D}(A:B) = \mathcal{I}(A:B) - \mathcal{J}(A:B)$$

Note that $\mathcal{D}(\mathbf{A}:\mathbf{B}) \neq \mathcal{D}(\mathbf{B}:\mathbf{A})$ and $0 \leq \mathcal{D}(\mathbf{A}:\mathbf{B}) \leq S(\mathbf{A})$.

Test of the quantum discord

Consider the 2-qubit **Werner states:**

$$|\Psi\rangle := \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

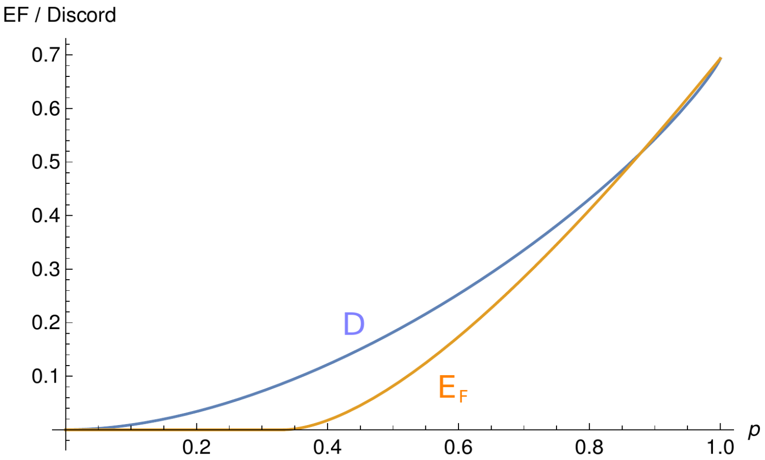
$$\rho_z = (1-z)\frac{\mathbf{1}}{4} + z|\Psi\rangle\langle\Psi|,$$

where $0 \leq z \leq 1$.

One can show:

- ρ_z is separable for $z \leq \frac{1}{3}$.
- ρ_z is non-separable for $z > \frac{1}{3}$.

Entanglement of Formation vs. Quantum Discord in a Werner state



SUMMARY

- There are two types of correlations, namely classical and quantum-mechanical correlations.
- States are defined as entangled if they are not separable.
- There is no unique entanglement measure.
- The entanglement of formation is the standard choice, but hard to compute.
- Quantum correlations may even be present in non-entangled states.
- The quantum dicord is probably a better measure for quantum correlations.

Outline

Complexity

Definition of complexity

Complexity measures the difficulty to perform a certain quantum operation.

Roughly speaking complexity is something like the number of quantum gates needed to realize a certain operation on a quantum computer.

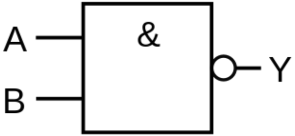
Please note:

- Quantum complexity is something very recent.
- Various definitions exist.
- The whole issue is not yet settled.

Definition of complexity

Recall classical complexity.

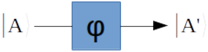
Any finite Boolean operation can be realized by composing a finite number of NAND gates:



The classical complexity is defined as the **minimal number of NAND gates** needed to perform the task.

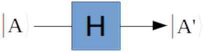
Definition of complexity

Any finite unitary operation can be built by composing a finite number of universal quantum gates:



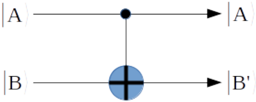
Phase shift gate

$$\begin{pmatrix} 1 & 0 \\ 0 & i\varphi \end{pmatrix}$$



Hadamard gate

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



CNOT gate

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

Quantum Complexity $\simeq$ minimal number of quantum gates

Definition of complexity

There are in principle two ways to define quantum complexity:

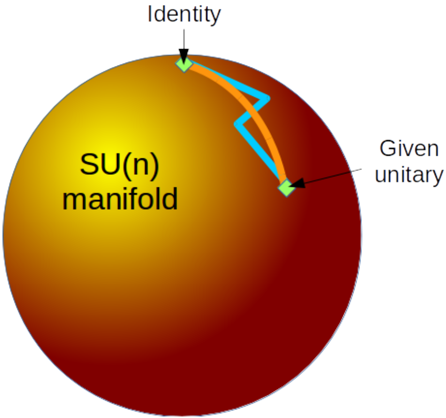
- **operation-based:** Minimal number of quantum gates needed to build a given unitary transformation.
- **state-based:** Minimal number of quantum gates needed to transform a given **reference state** into another **target state**.

Note: A quantum state alone does not have a complexity. The complexity of a quantum state is always defined **relative to a certain reference state**.

$\Rightarrow$ First have a look at the **operation-based** case.

Operation-based complexity

Think of the given unitary transformation as a point on the "Bloch surface" ($SU(n)$-manifold):



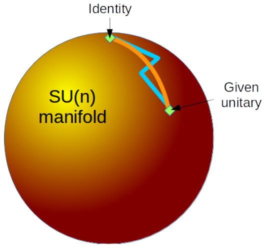
Operation-based complexity

The $SU(n)$ manifold comes with a
a "natural" distance measure (metric)

$$\mathrm{d}s^2 = \mathrm{Tr}[\mathrm{d}U^\dagger \mathrm{d}U].$$

"Straight" lines (geodesics) in this metric
are generated by the **modular Hamiltonian**

$$U(\tau) = e^{-iH\tau}$$



- The tangent space around the identity is the Lie algebra $su(n)$.
- The modular Hamiltonian H is an element of the Lie algebra and can be interpreted as the tangent vector of the geodesic line in the starting point.

Operation-based complexity

Represent H in the **Pauli basis:**

- One-qubit generator $\in su(2)$:

$$H = \lambda_0 \mathbf{1} + \lambda_1 \sigma^x + \lambda_2 \sigma^y + \lambda_3 \sigma^z \ \in \ \text{span}\left\{\mathbf{1}, \sigma^x, \sigma^y, \sigma^z\right\}$$

- Two-qubit generator $\in su(4)$:

$$\begin{aligned} H \quad \in \quad & \text{span}\{\mathbf{1}\otimes\mathbf{1},\ \ \mathbf{1}\otimes\sigma^x,\ \ \mathbf{1}\otimes\sigma^y,\ \ \mathbf{1}\otimes\sigma^z, \\ & \sigma^x\otimes\mathbf{1},\ \ \sigma^x\otimes\sigma^x,\ \ \sigma^x\otimes\sigma^y,\ \ \sigma^x\otimes\sigma^z, \\ & \sigma^y\otimes\mathbf{1},\ \ \sigma^y\otimes\sigma^x,\ \ \sigma^y\otimes\sigma^y,\ \ \sigma^y\otimes\sigma^z, \\ & \sigma^z\otimes\mathbf{1},\ \ \sigma^z\otimes\sigma^x,\ \ \sigma^z\otimes\sigma^y,\ \ \sigma^z\otimes\sigma^z,\ \ \} \end{aligned}$$

- Three-qubit generator $\in su(8)$:

64 possible combinations

Operation-based complexity

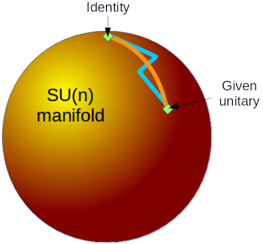
Consider now a fixed number of qubits,
e.g. generators for 4 qubits $\in su(16)$:

- 0-qubit operation: $\mathbf{1}_{16} = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2$
- 1-qubit operation: e.g. $\sigma^x \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2$
- 2-qubit operation: e.g. $\mathbf{1}_2 \otimes \sigma^y \otimes \sigma^z \otimes \mathbf{1}_2$
- 3-qubit operation: e.g. $\sigma^x \otimes \sigma^y \otimes \mathbf{1}_2 \otimes \sigma^x$
- 4-qubit operation: e.g. $\sigma^x \otimes \sigma^y \otimes \sigma^z \otimes \sigma^y$

Idea: **Complexity $\approx$ number of Pauli matrices**

Operation-based complexity

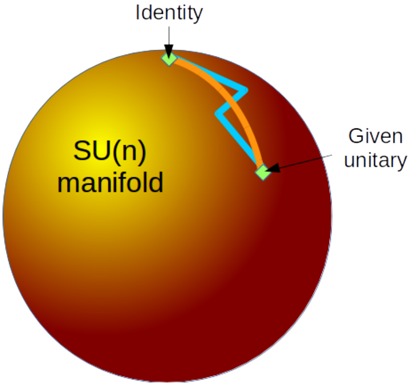
- Each operation corresponds to a certain direction in tangent space.
- Define a new metric which is proportional to the complexity.
- The new geodesics are different from the straight line as they are trying to avoid directions of high complexity.



Operation-based complexity

A possible choice could be:

Generator	Weight
Identity (phase shifts)	1
1-qubit operation	1
2-qubit operation	1
$\geq$3-qubit operation	r



For $r = 1$ one recovers the natural metric.
For $r \to \infty$ the geodesic line consists only of $\leq$ 2-qubit operations.

Elements of Holography

Black Holes

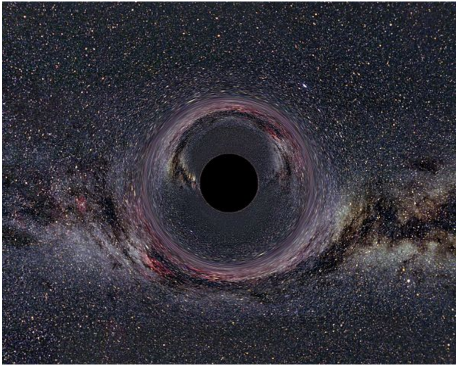


Figure by Corvin Zahn, Wikimedia

Bekenstein-Hawking (1973):

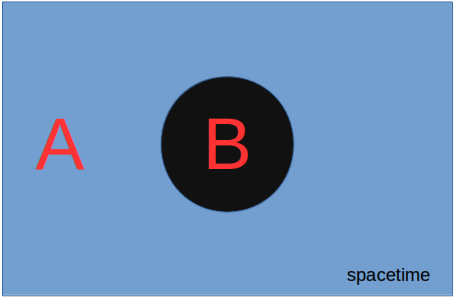
Black holes carry entropy proportional to their surface

$$S = A/4\ell_P^2$$

where $\ell_P = \sqrt{G\hbar/c^3}$ is the Planck length.

Black Holes

We cannot interact with the interior of a black hole, but we are entangled with the interior of a black hole.



$\Rightarrow$ We have to trace out the interior:

$$\rho_A = \mathrm{Tr}_B \, \rho_{AB}$$

Black Holes

Bekenstein-Hawking: The information of a black hole is proportional to its surface, not to its volume!

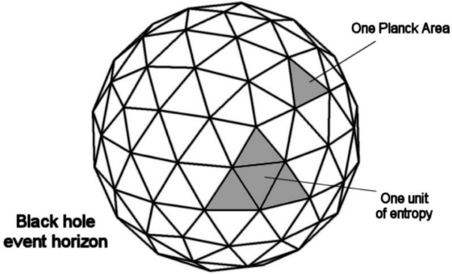


Figure by Bekenstein, Scholarmedia

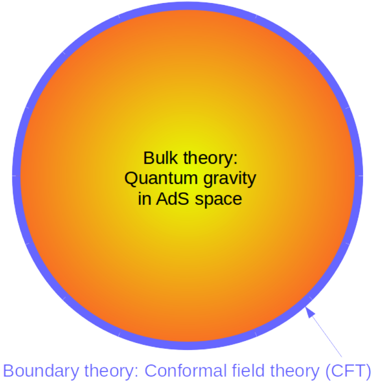
Holography

The information of a black hole
is encoded on its surface,
like a holographic picture.

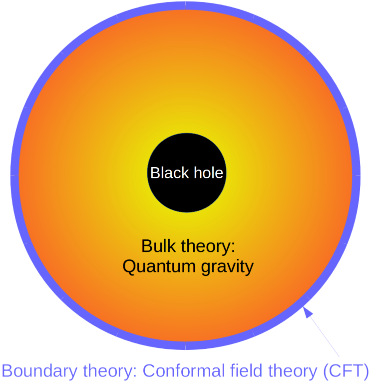
$\Rightarrow$ **Holographic conjecture:**
Quantum gravity has much less
degrees of freedom than
Planck cells in the bulk.

Holography

Maldacena (1997): A certain type of quantum gravity (super strings) on a certain type of geometry (Anti-de-Sitter space) is dynamically equivalent (dual) to a conformal field theory (CFT) on the boundary:



$T = 0$

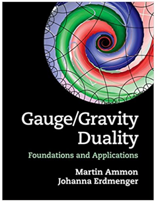


$T > 0$

Holographic Dictionary

Objects that are conjectured to be in correspondence:

Bulk	Boundary
Superstrings in AdS_{d+1}	d-dimensional CFT
fields	operators
weak coupling	strong coupling
mass	scaling dimension
geodesics	entanglement
black hole	temperature
...	...



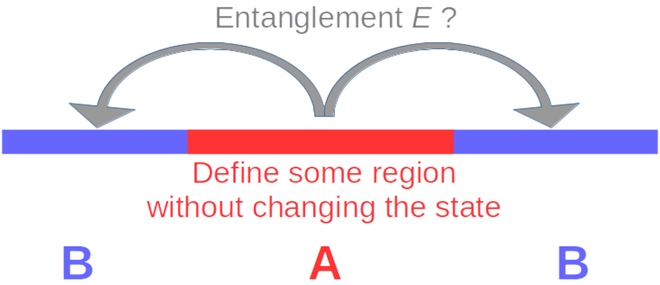
Holographic entanglement

Consider for simplicity a one-dimensional conformal field theory (CFT) at zero temperature:

Boundary theory: Conformal field theory (CFT)

pure state $|\psi\rangle\langle\psi|$

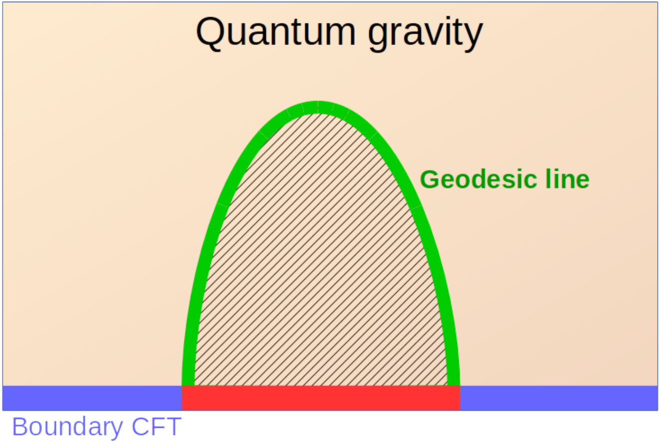
Holographic entanglement



$$E = S(\rho_{\mathbf{A}}) = S(\rho_{\mathbf{B}})$$

Ryu-Takayanagi surface

Claim by Ryu and Takanayagi [PRL 2006]:



Entanglement $\simeq$ Length of geodesic

Ryu-Takayanagi surface

Analogous in higher dimensions:

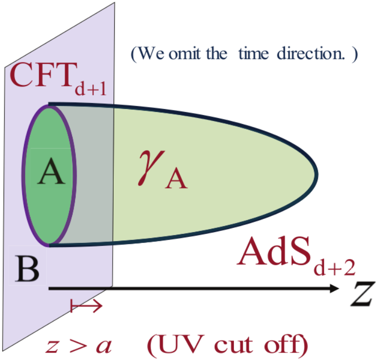


Figure: [taken from Ryu/Takanayagi]

How can we explain this qualitatively?

Tensor Networks

From product to matrix product states

Product states:

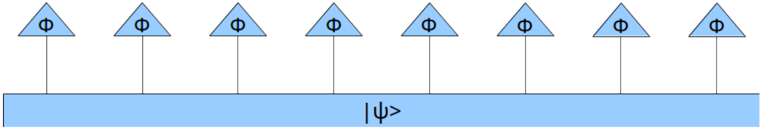
Consider a one-dimensional chain of qubits in $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$.
The periodic chain is said to be in a (pure) **product state** if:

$$|\psi\rangle = |\phi\rangle \otimes |\phi\rangle \otimes \ldots \otimes |\phi\rangle\,, \qquad |\phi\rangle = \phi_0|0\rangle + \phi_1|1\rangle.$$

In components:

$$\psi_{i_1 i_2 \ldots i_N} = \phi_{i_1}\phi_{i_1} \cdots \phi_{i_N}.$$

Product states have no entanglement.



From product to matrix product states

Matrix product states:

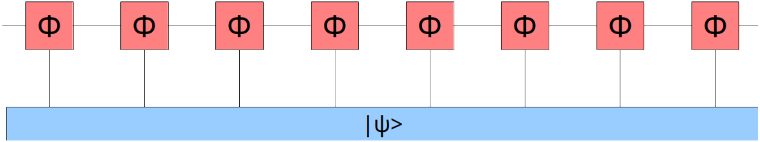
Replace the numbers ϕ_0, ϕ_1 by matrices Φ_0, Φ_1 in an auxiliary space V and finally take the trace:

$$|\psi\rangle = \mathrm{Tr}_V \Big[|\Phi\rangle \otimes |\Phi\rangle \otimes \ldots \otimes |\Phi\rangle \Big] , \qquad |\Phi\rangle = \Phi_0 |0\rangle + \Phi_1 |1\rangle .$$

In components:

$$\psi_{i_1 i_2 \ldots i_N} = \mathrm{Tr}_V \Big[\Phi_{i_1} \Phi_{i_1} \cdots \Phi_{i_N} \Big] = \sum_{\mu_1, \mu_2, \ldots, \mu_N} \Phi_{i_1, \mu_1 \mu_2} \Phi_{i_1, \mu_2 \mu_3} \cdots \Phi_{i_N, \mu_N \mu_1} .$$

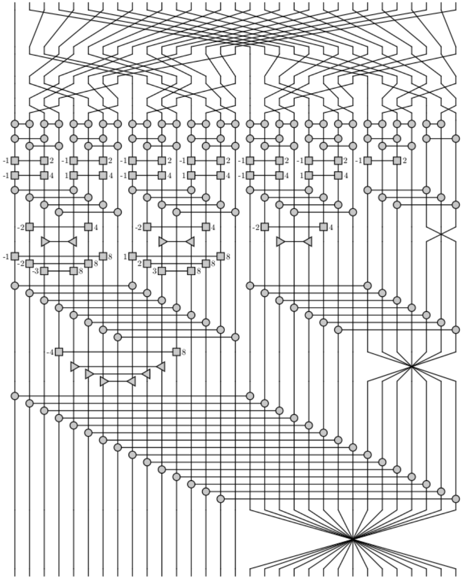
Matrix product states exhibit short-range (exponential) entanglement.



Tensor Networks

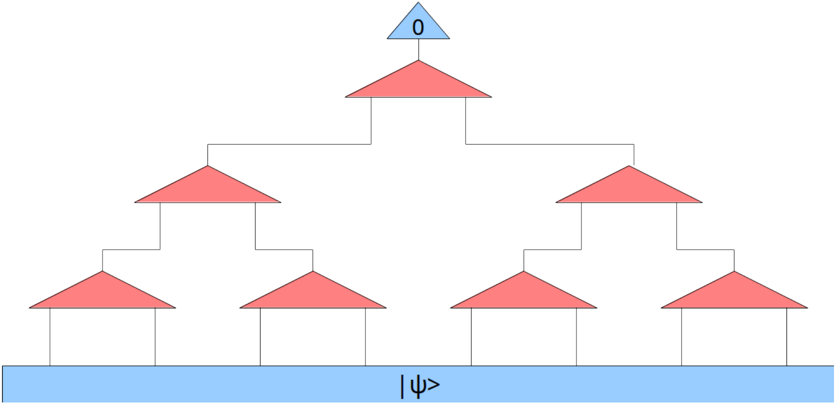
Tensor networks are like matrix product states, but more complicated.

Our recent work
on quantum sine transforms: ⇒



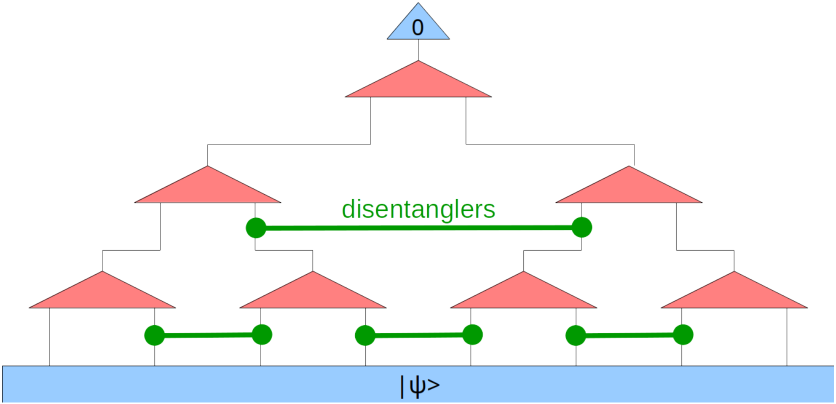
MERA Tensor Networks

Multiscale Entanglement Renormalization Ansatz (MERA):



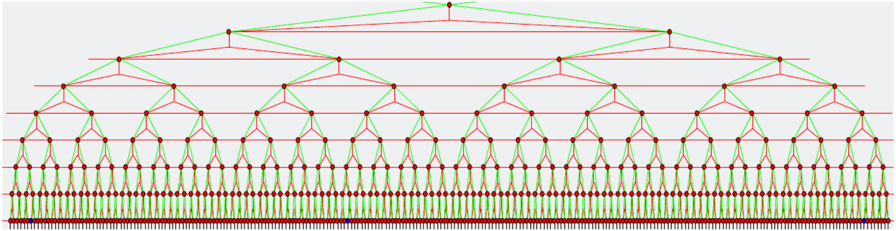
MERA Tensor Networks

Multiscale Entanglement Renormalization Ansatz (MERA):



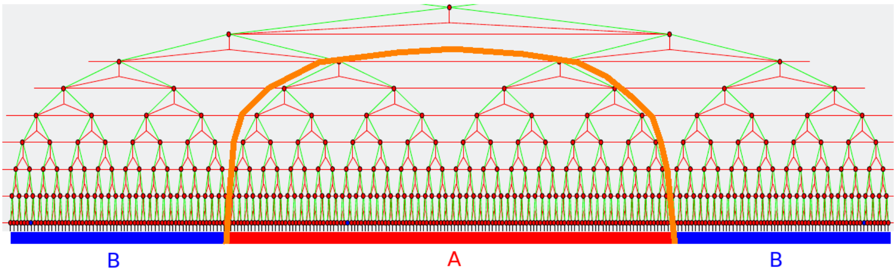
[Vidal/Evenbly et al]

Holography interpreted as a MERA



- The vertical direction can be viewed as a renormalization direction.

Entanglement and complexity of a MERA



- The line cutting the minimal number of bonds defines the RT line.
- The number of bonds cut by the RT line is the entanglement.
- The number of gates inside the RT region is the complexity.

Thank you !