Quantum Information-theoretic approaches to Thermodynamics

David Jennings

UNIVERSITY OF OXFORD

THE ROYAL SOCIETY

KIAS KOREA INSTITUTE FOR ADVANCED STUDY

Workshop on Quantum Information & Thermodynamics

Overview

1. Motivations and criteria.
2. Resource theories and properties.
3. Resource formulation of thermodynamics.
4. Majorization — order & disorder.
5. Information & energy.
6. Gibbs-rescaling & work bits.

The Thermodynamic Limit

- *"Thermodynamics means the thermodynamic limit."*

The Thermodynamic Limit

- *"Thermodynamics means the thermodynamic limit."*
- (Except it doesn't)

The Thermodynamic Limit

- *“Thermodynamics means the thermodynamic limit.”*
- (Except it doesn’t)

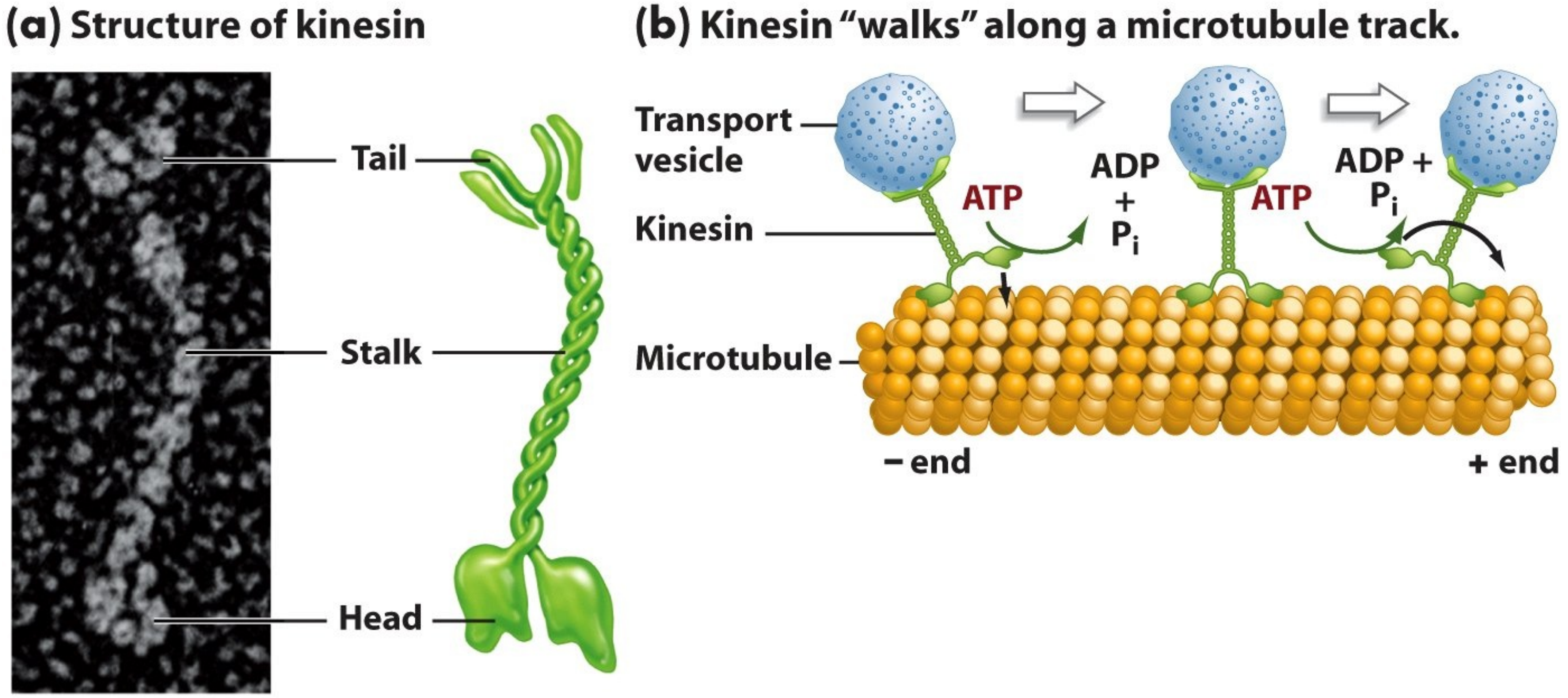
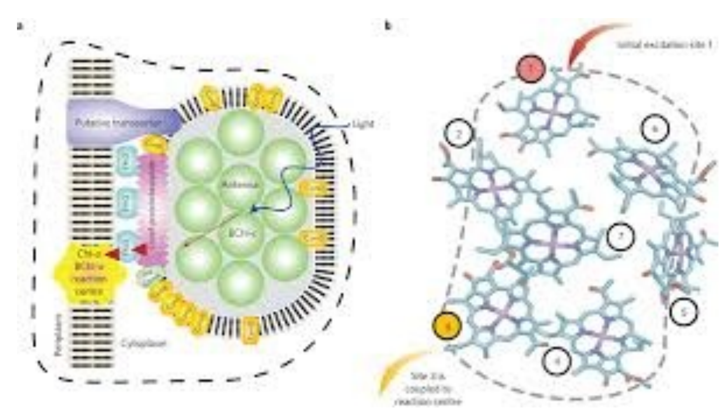
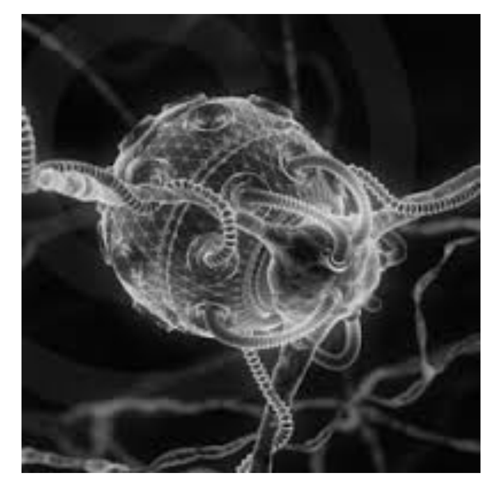


Figure 7-37 Biological Science, 2/e

The central question:

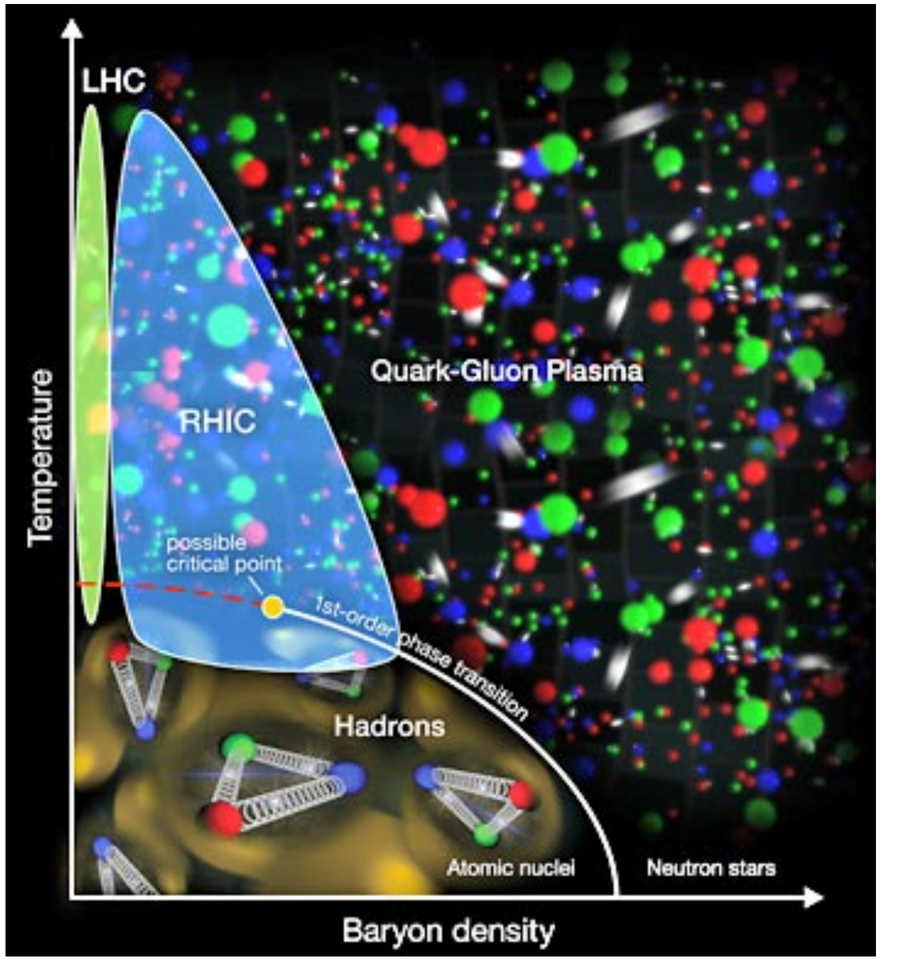
Q: What thermodynamic principles apply at micro/nano/pico/… scales and include intrinsically quantum properties?

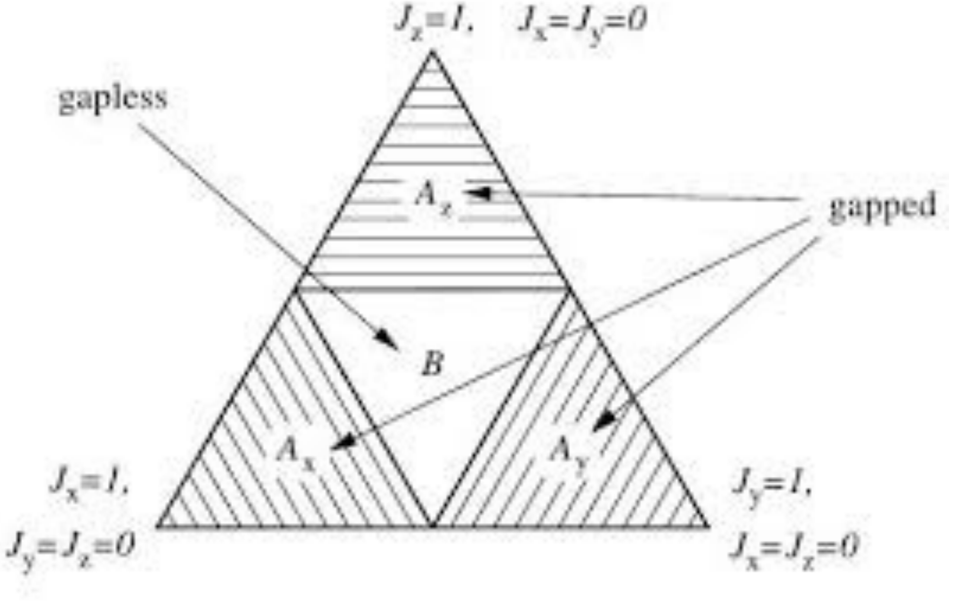




Quantum & Thermodynamic

- Horizons in QFT.
- Fermi gases, neutron stars.
- Quantum phase transitions.
- Topological, anyonic systems.



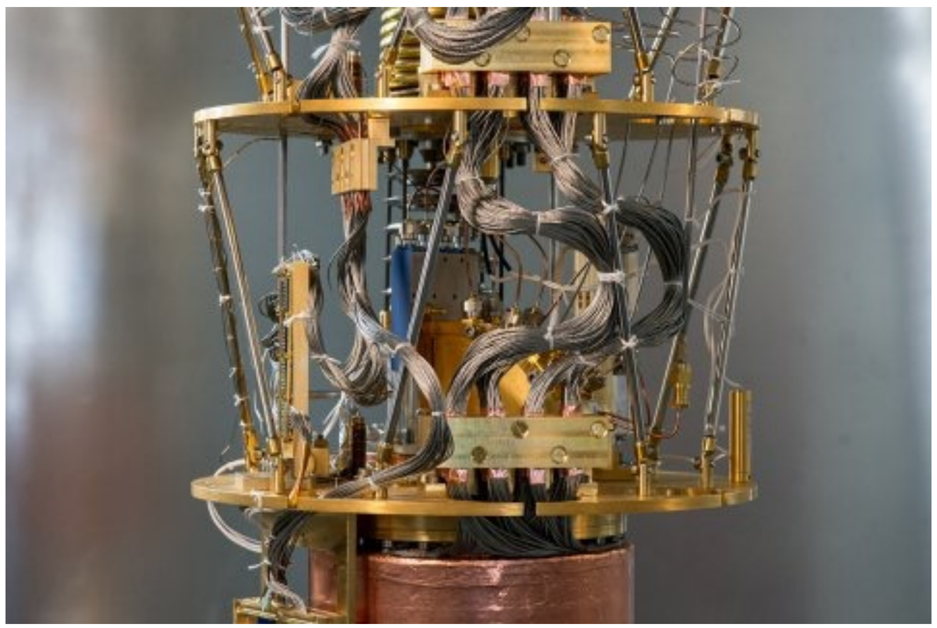


What do we mean by quantum thermodynamics?

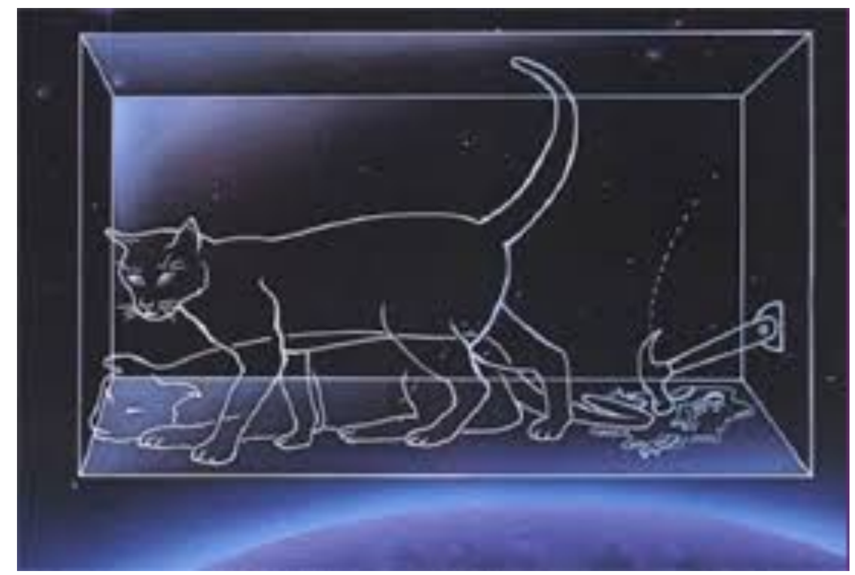
" To be sure of hitting the target, shoot first and call whatever you hit the target. "

Quantum Thermodynamics*

Emerging Tech



Foundational aspects



What is quantum thermodynamics?
How does it differ from classical theory?
Applications to quantum information science?
Structural connections with entanglement?

Criteria

1. **Non-asymptotic & non-equilibrium.**
2. **Not isomorphic to stochastic dynamics.**
3. **Non-trivial quantum phenomena.**
4. **“Ports well” with quantum information.**

Recent QI approaches

L. del Rio et al, Nature 474 (2011)

I. Marvian, R. Spekkens, Nature Comm 5 (2014)

Toyabe et al, Nature Physics, (2010)

M. Horodecki, Oppenheim, Nature Comm 4 (2013)

F. Brandao et al, Phys. Rev. Lett. 111 (2014)

F. Brandao et al, Nature Phys. (2014)

J. Aberg, Nature Comm (2013)

V. Narasimhachar, G. Gour Nature Comm (2015)

F. Brandao, M. Plenio, Nature Physics (2010)

and many, many others….

1. Lostaglio, DJ, Rudolph, Nature Communications (2015)
2. Lostaglio, Korzekwa, DJ, Rudolph, Physical Review X (2015)
3. Korzekwa, Lostaglio, Oppenheim, DJ, New Journal of Physics (2015)
4. Sparaciari, DJ, Oppenheim, Nat Communications (2017)
5. Gour, DJ, Buscemi, Duan, Marvian arXiv (2017)

Quantum Thermodynamics

Two problems:

Quantum?

Thermodynamics?

Quantum Thermodynamics

Two problems:

Quantum? → *"Not describable by a stochastic statistical-mechanics model"*

Thermodynamics? → ???

Thermodynamics

Caratheodory (1909) — *"One can derive the entire theory without assuming the existence of a physical quantity that deviates from ordinary mechanical quantities, namely,* ***heat****."*

Lieb & Yngvason (1999) — *If a unique* ***entropy*** *exists then: (a) the thermodynamics has scale-invariance, and (b) any system can be arbitrarily sub-divided.*

[1] Caratheodory, Examination of the foundations of thermodynamics (1909)
[2] Lieb, Yngvason, Phys. Rept. (1999)

Thermodynamics

Caratheodory (1909) — *"One can derive the entire theory without assuming the existence of a physical quantity that deviates from ordinary mechanical quantities, namely,* ***heat****."*

Lieb & Yngvason (1999) — *If a unique* ***entropy*** *exists then: (a) the thermodynamics has scale-invariance, and (b) any system can be arbitrarily sub-divided.*

Avoid building a framework on problematic concepts!

[1] Caratheodory, Examination of the foundations of thermodynamics (1909)
[2] Lieb, Yngvason, Phys. Rept. (1999)

QI framework for QT

Minimal assumptions:

1. *Energy conserved microscopically.*
2. *An equilibrium state exists.*
3. *Quantum coherence has thermodynamic cost.*

$$\rho \longrightarrow \sigma$$
if and only if
$$S_{\boldsymbol{\eta}}(\rho) \leq S_{\boldsymbol{\eta}}(\sigma)$$

(classical)

- thermo-majorization.
- no unique ordered energy or "heat".
- Many *derived* entropies.

(quantum)

- Free energies insufficient.
- Superselection rules.
- Quantum clocks are central.

[1] Gour, DJ, Buscemi, Duan, Marvian arXiv:1708.04302 (2017)

Core concepts

QI: Rethinking properties

“ Fundamental ***classical*** *properties:* ***total*** *orders*
Fundamental ***quantum*** *properties:* ***partial*** *orders ”*

Examples

(**mass**) “Protons have more mass than electrons”
(**charge**) “A positron has more charge than a photon”
(**speed**) “Light is faster than everything”
(**temperature**) “The sun is hotter than the vacuum”
(**entanglement**) “A Bell state is more entangled than a product state”
(**shape**) “A knife is more pointy than a ball”
(**temporal ordering**) “Paper was invented before the internet”

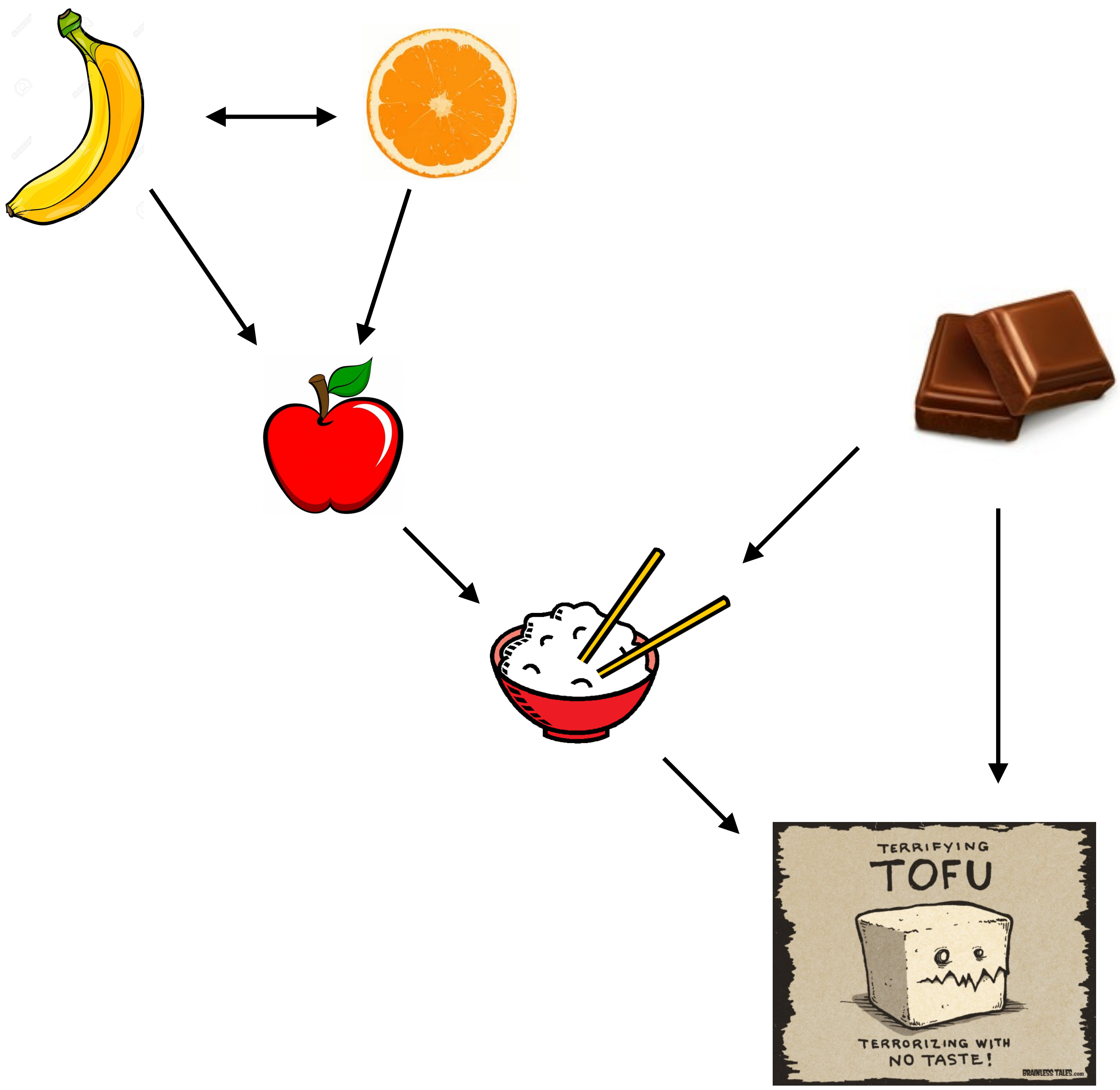
Taste

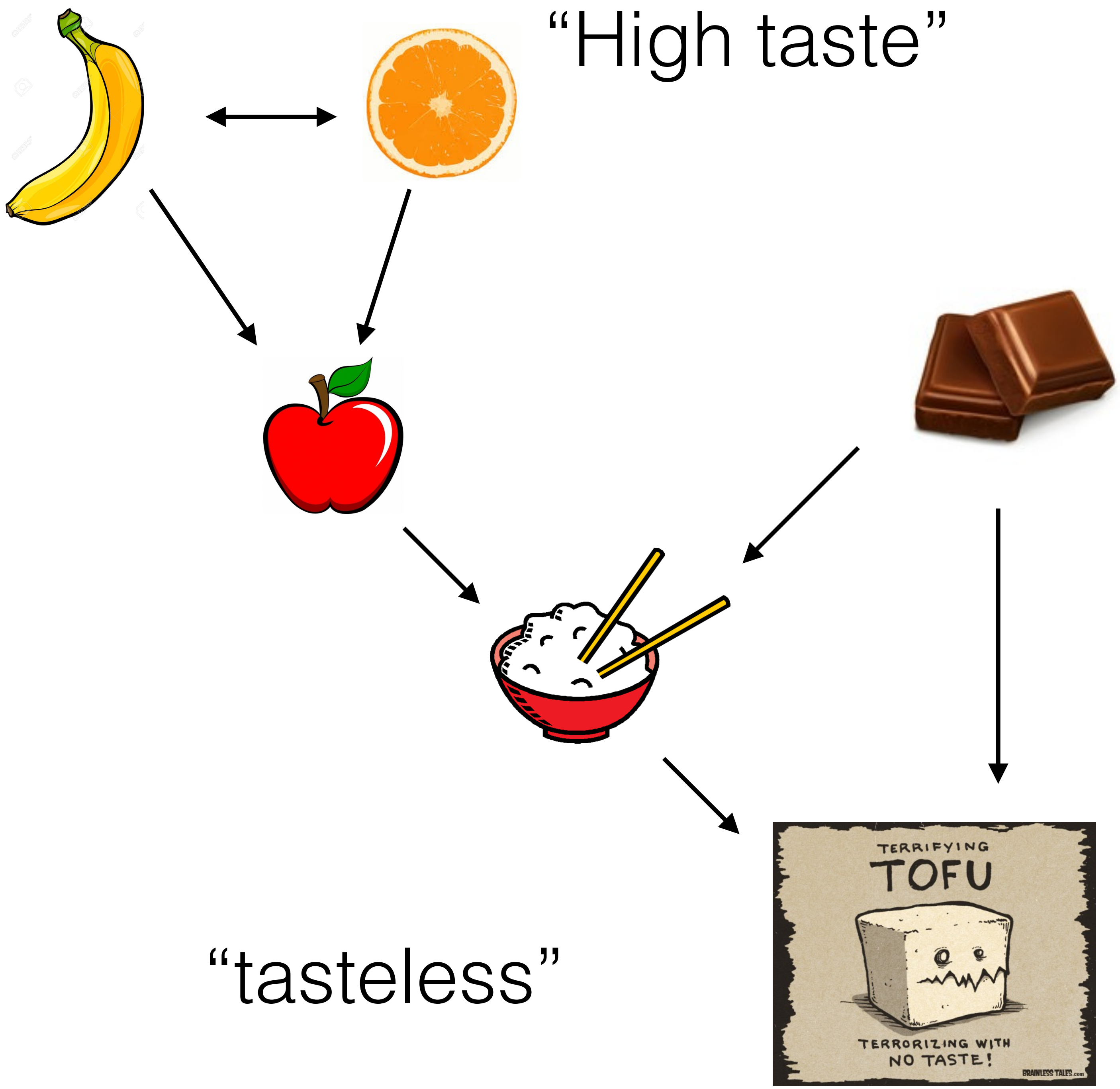
"An orange is more tasty than an apple"

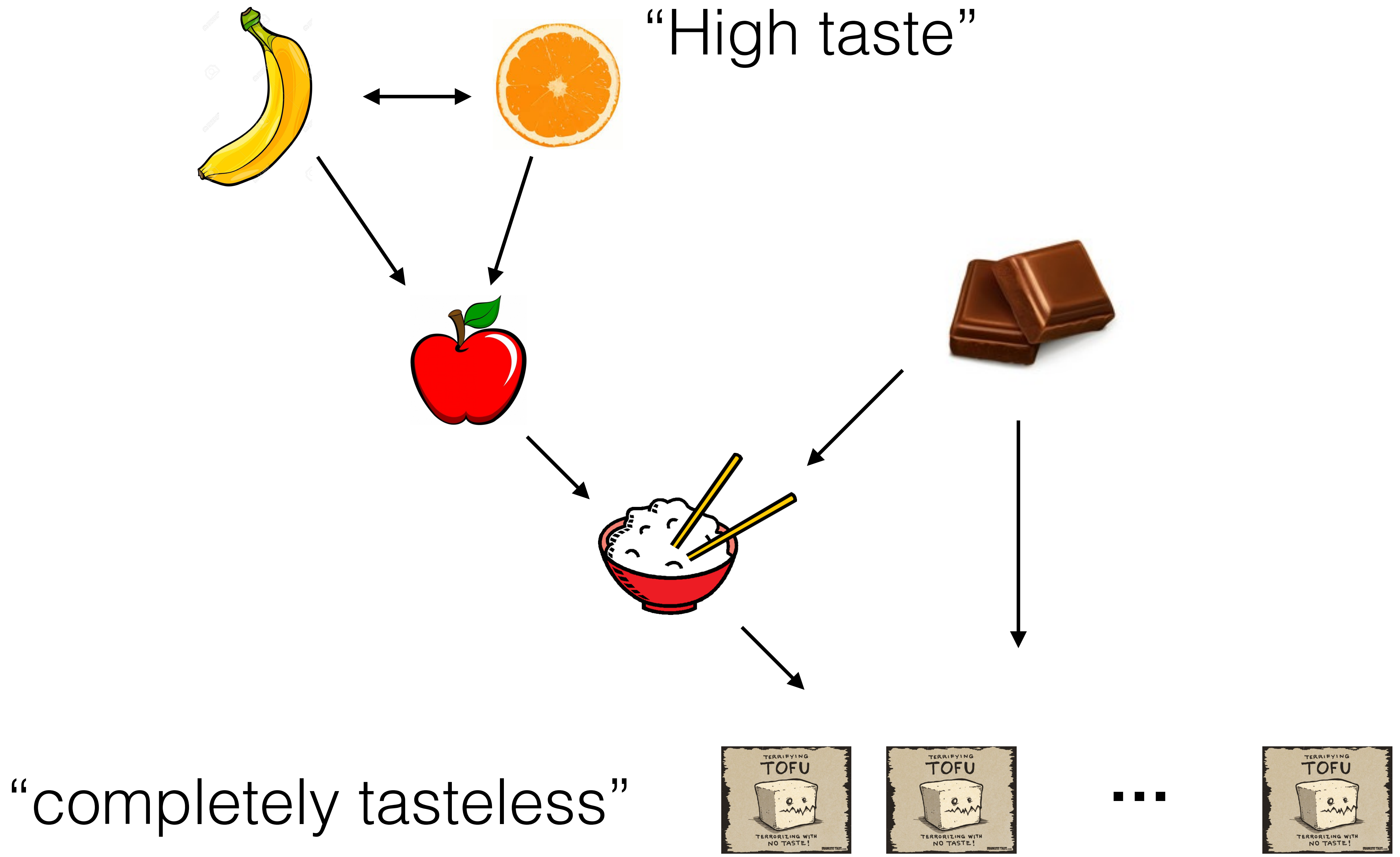
Taste

- “An orange is more tasty than an apple”
- “An apple is more tasty than rice”
- “Rice is more tasty than tofu”
- “An orange is more tasty than a banana”
- “Chocolate is more tasty than rice or tofu”
- “A banana is more tasty than an orange”

Taste — a partial order




"High taste"
"tasteless"
TERRIFYING
TOFU
TERRORIZING WITH NO TASTE!


“High taste”
“completely tasteless”
...

Properties

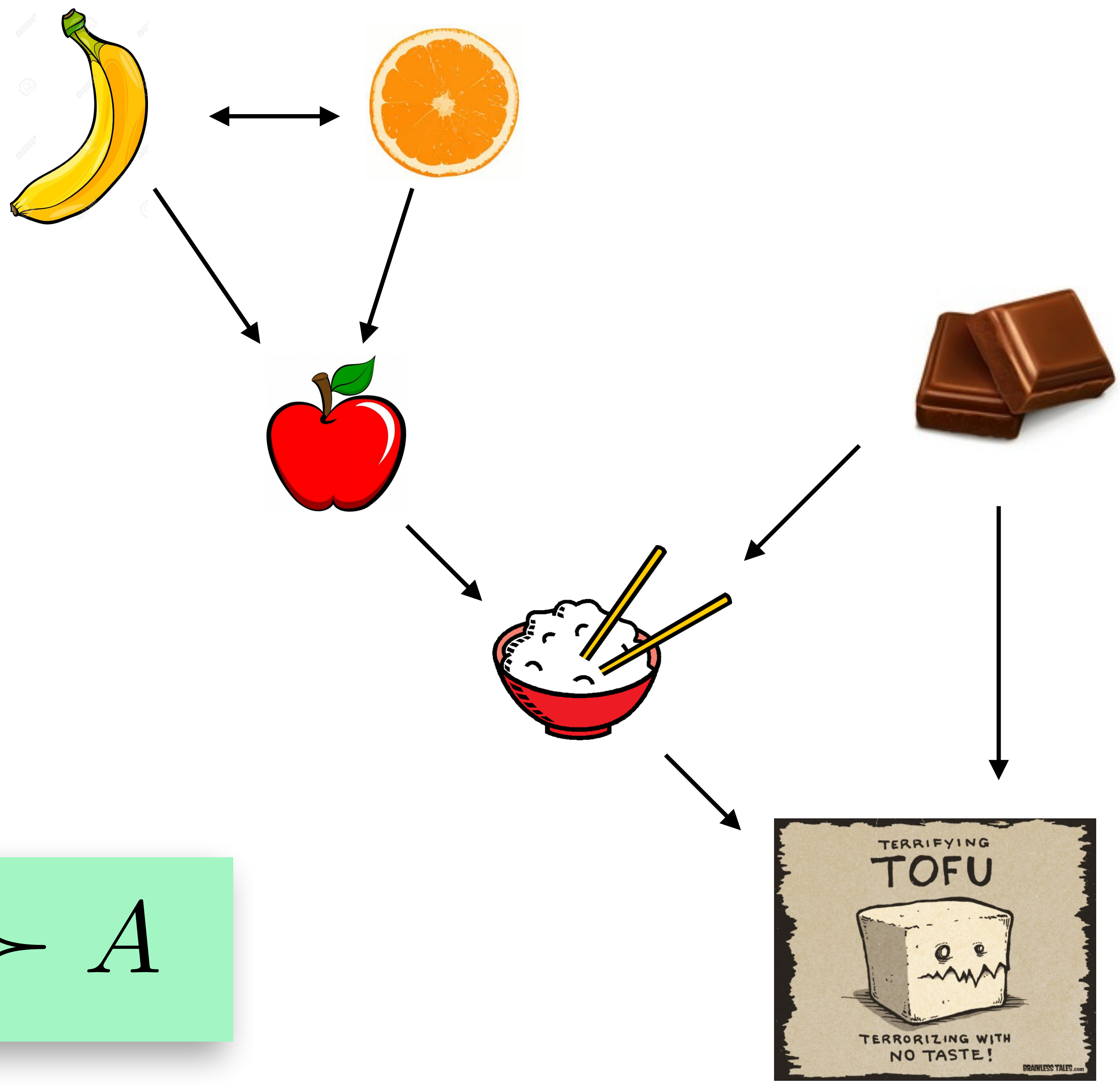
A "property" $\mathcal{P}$ is (in general) a partial order $\succ$ on objects.

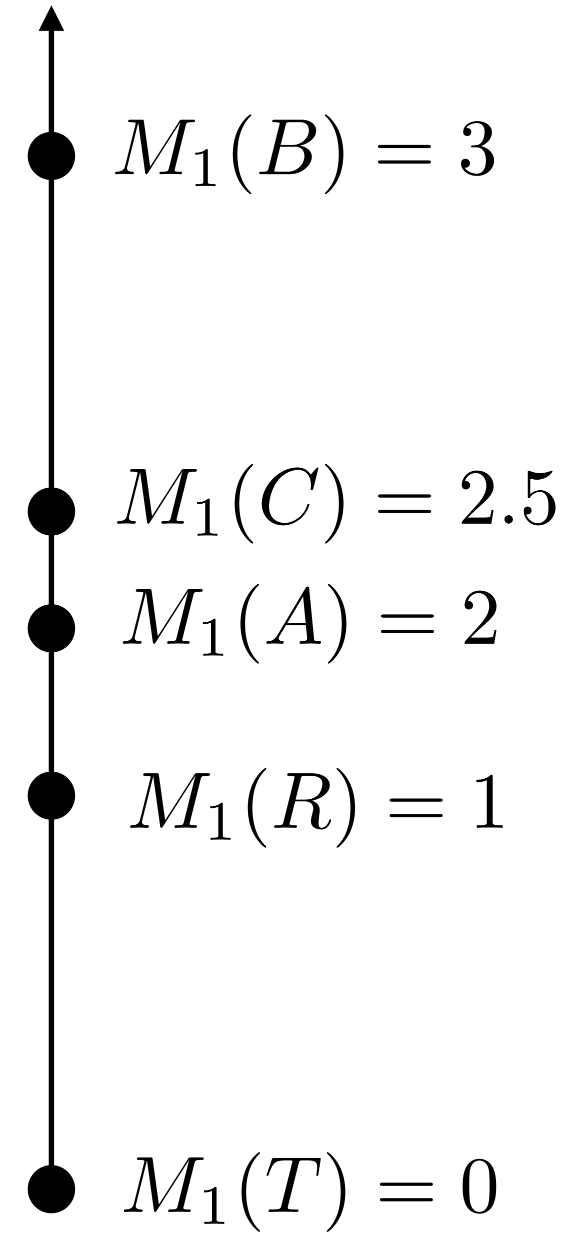
If $X_1 \succ X_2$

then any **measure** M of the property $\mathcal{P}$ must obey:

$$M(X_1) \geq M(X_2)$$

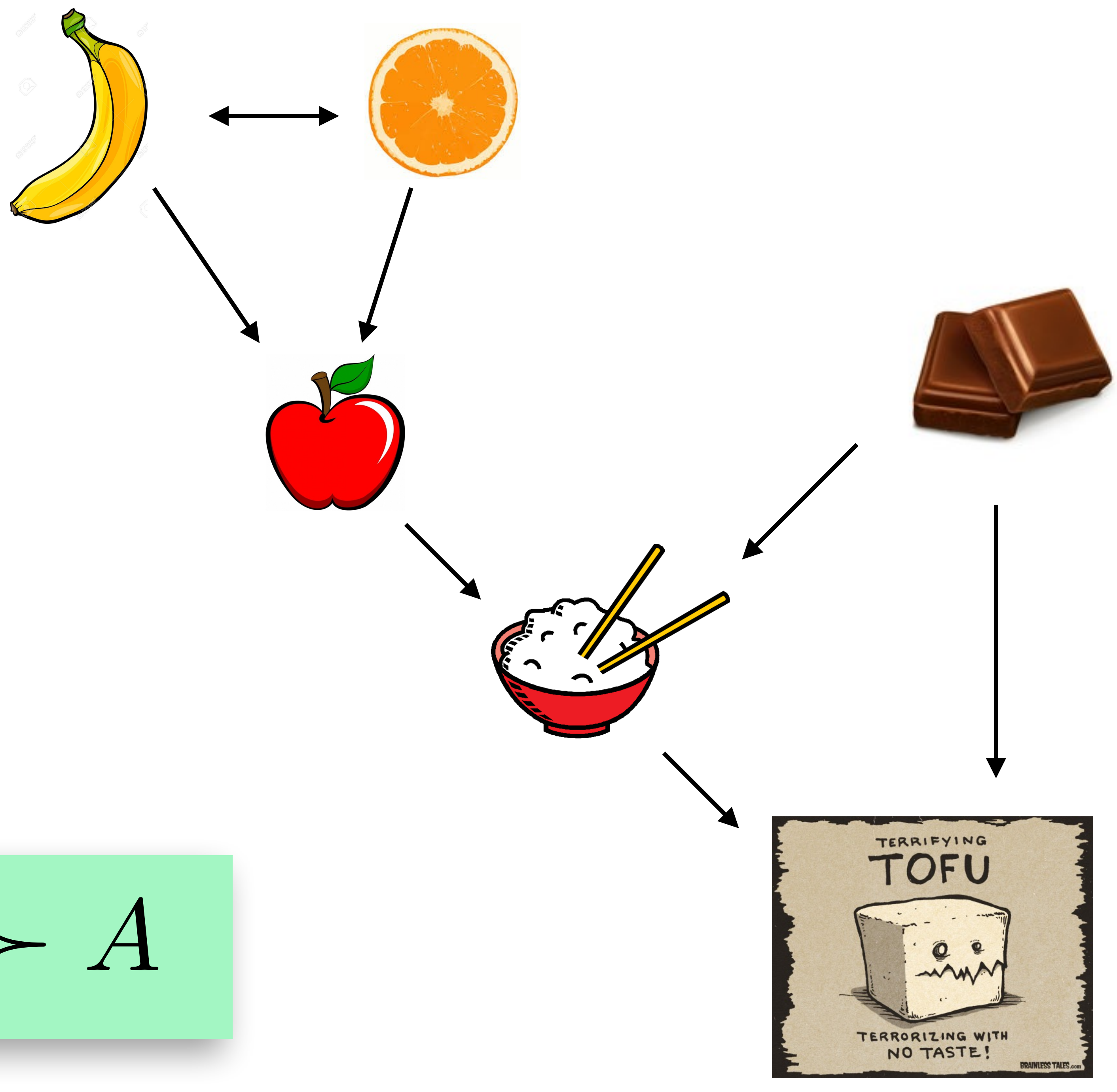
Measures of taste





$$B \succ A$$

Measures of taste



$M_2(B) = 7$

$M_2(A) = 6$

$M_2(C) = 4$

$M_2(R) = 3.5$

$M_2(T) = 1$

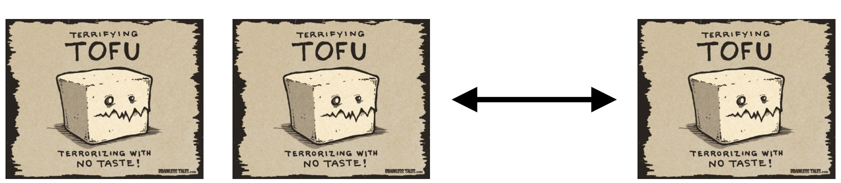
$$B \succ A$$

Measures of taste

- A minimum of 2 measures are needed for this property.
- We must have M(B)=M(O) for any measure M
- If we want an **additive** measure then M(T) = 0.

$$M(X \otimes Y) = M(X) + M(Y)$$ “additive”

T “completely tasteless”:



$$M(T^{\otimes 2}) = M(T)$$

$$M(T^{\otimes n}) = M(T) \text{ for all } n \in \mathbb{N}$$

$$\Rightarrow nM(T) = M(T) \text{ for all } n \in \mathbb{N}$$

$$\Rightarrow M(T) = 0$$

Examples

total orders

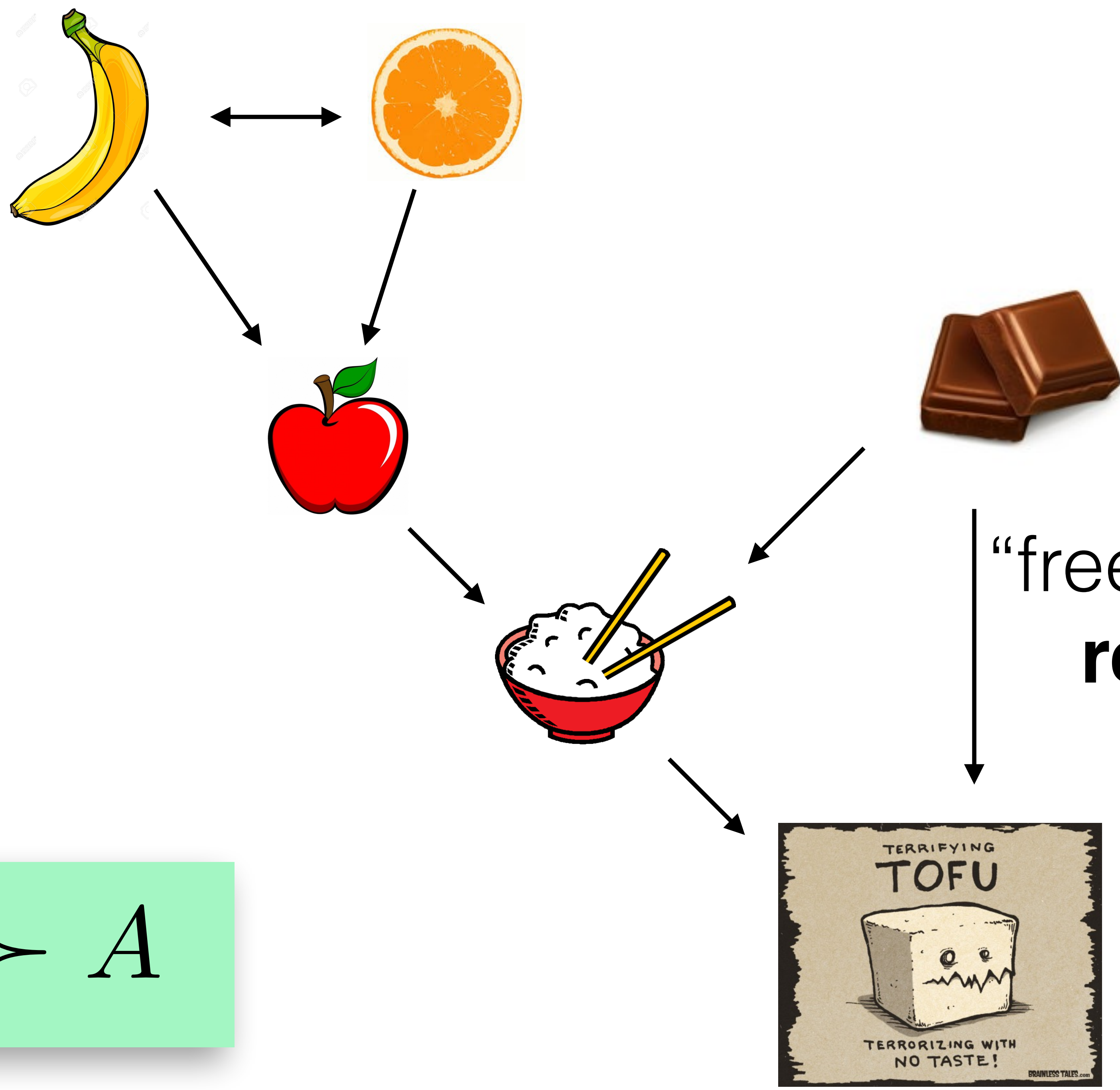
(**mass**) "Protons have more mass than electrons"
(**charge**) "A positron has more charge than a photon"
(**speed**) "Light is faster than everything"
(**temperature**) "The sun is hotter than the vacuum"
(**entanglement**) "A Bell state is more entangled than a product state"
(**shape**) "A knife is more pointy than a ball"
(**temporal ordering**) "Paper was invented before the internet"

partial orders

Overview

1. **Motivations and criteria.**
2. Resource theories and properties.
3. Resource formulation of thermodynamics.
4. Majorization — order & disorder.
5. Information & energy.
6. Gibbs-rescaling & work bits.

Properties = Resource theories



"free operations" can only **reduce** the resource

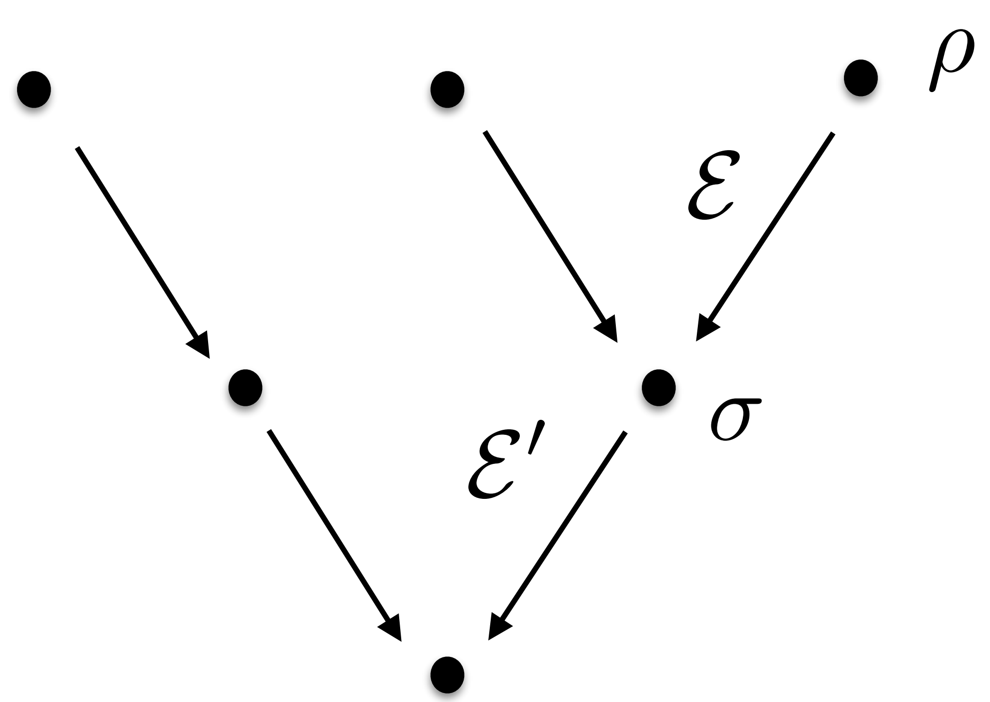
"free states" - zero resources

$$B \succ A$$

Quantum resource theories

Recipe:

Define class of “free states” that are “cheap”.
Define a class of “free operations” that are “easy”.
Any state not free is a “resource state”.
Any operation not free is “costly”.



Quantum resource theories

Free operations define a **partial order** on all states:

$$\rho \succ \sigma \text{ means that } \sigma = \mathcal{E}(\rho) \text{ for some } \mathcal{E} \text{ a free operation}$$

All measures of the resource are monotones for the ordering

$$\rho \succ \sigma \Rightarrow M(\rho) \geq M(\sigma)$$

Quantum resource theories

Free operations define a **partial order** on all states:

$$\rho \succ \sigma \text{ means that } \sigma = \mathcal{E}(\rho) \text{ for some } \mathcal{E} \text{ a free operation}$$

All measures of the resource are monotones for the ordering

$$\rho \succ \sigma \Rightarrow M(\rho) \geq M(\sigma)$$

Some basic questions:

- When does $\rho \to \sigma$?
- Complete set of measures?
- Macroscopic limits?

Overview

1. **Motivations and criteria.**
2. **Resource theories and properties.**
3. Resource formulation of thermodynamics.
4. Majorization — order & disorder.
5. Information & energy.
6. Gibbs-rescaling & work bits.

“Ordered Energy” ?

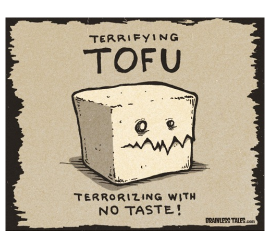
1. *Energy conserved microscopically.*
2. *An equilibrium state exists.*

constraint on the “free operations”.

“thermodynamic tofu” — the “free states”

??

??



(thermodynamic equilibrium)

The set of free states

The free states

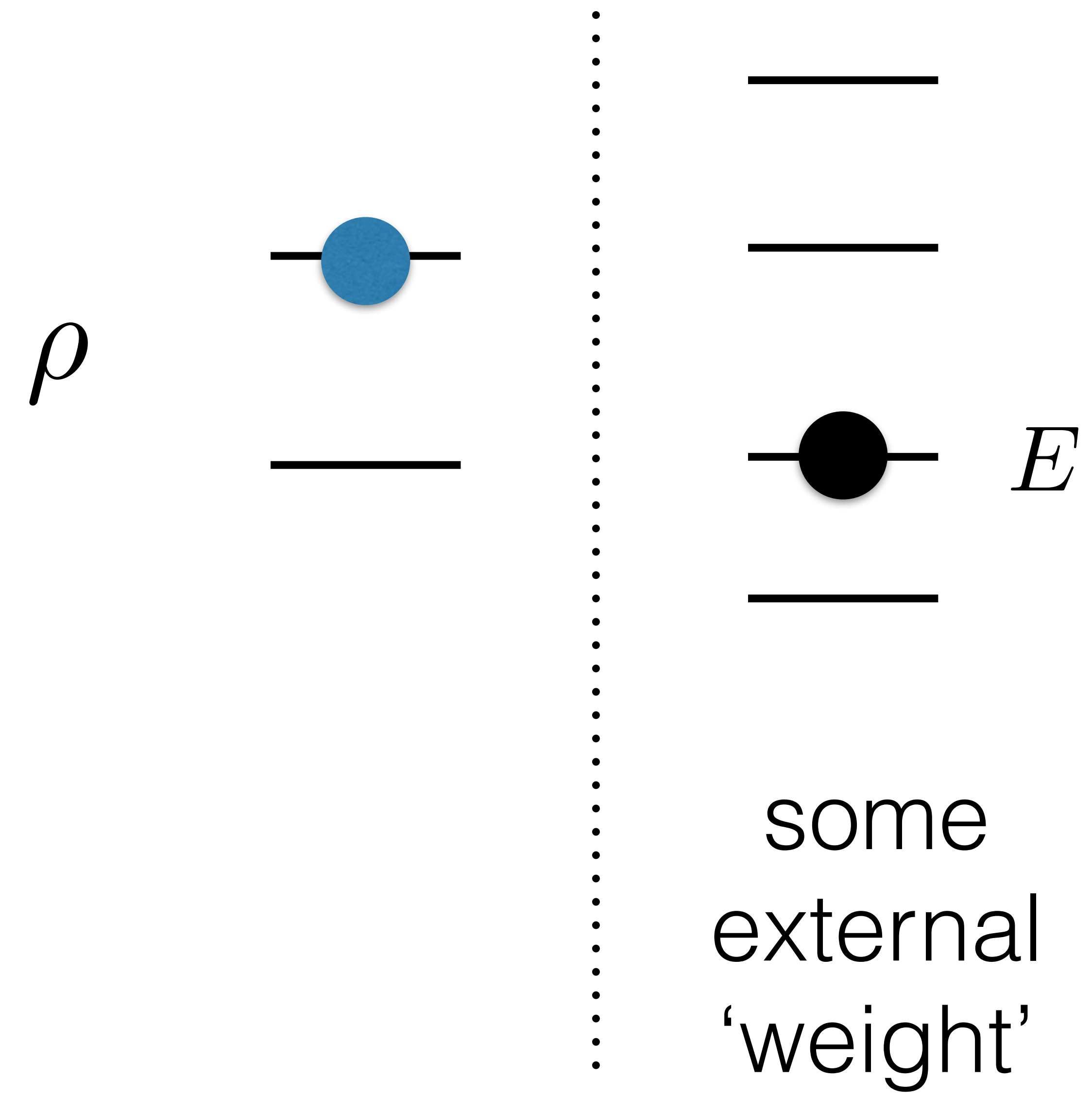
Definition:

A state ρ is **passive** if

$$\mathrm{tr}[(\mathrm{U}\rho\mathrm{U}^\dagger - \rho)\mathrm{H}] \geq 0$$

for every unitary U

H = Hamiltonian



The free states

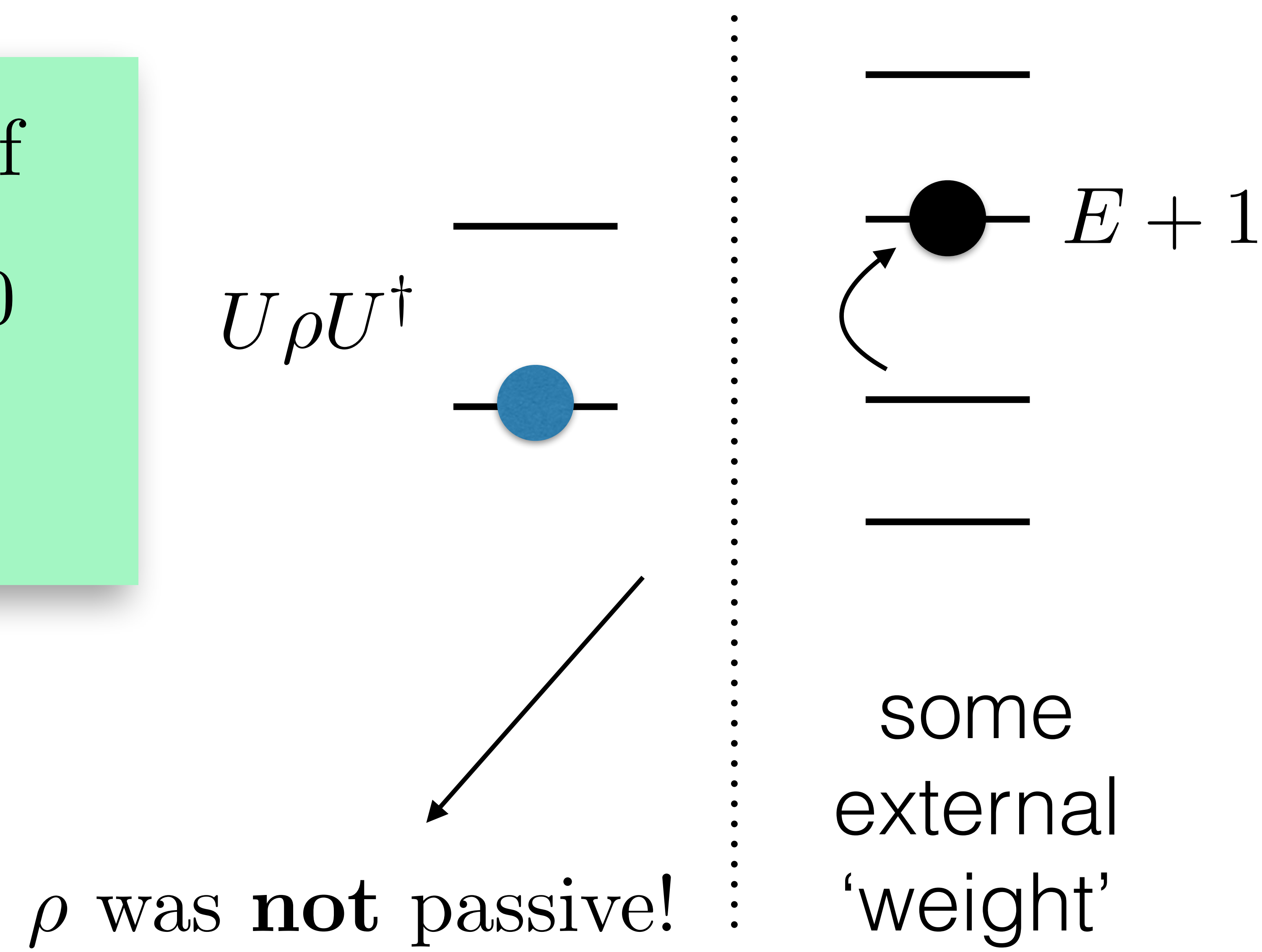
Definition:

A state ρ is **passive** if

$$\mathrm{tr}[(\mathrm{U}\rho\mathrm{U}^\dagger - \rho)\mathrm{H}] \geq 0$$

for every unitary U

H = Hamiltonian



Passive states

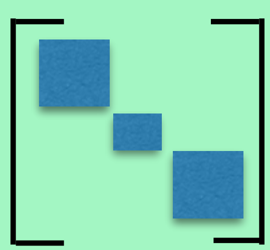
Theorem:

A state ρ is passive if and only if

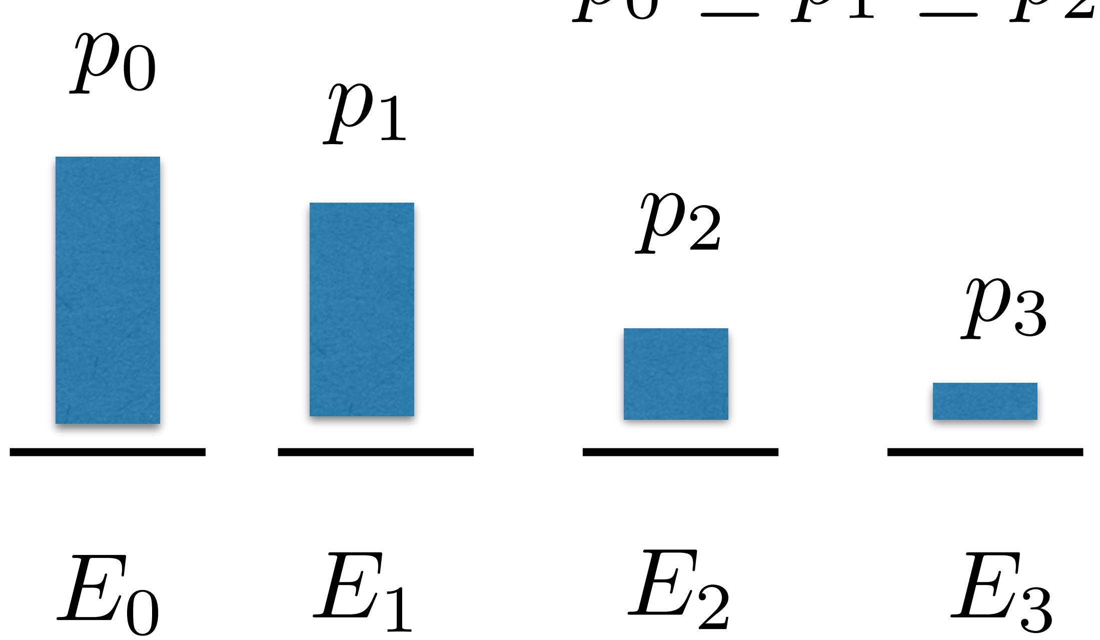
$[\rho, H] = 0$ and has no 'population inversions'

$$p_0 \geq p_1 \geq p_2 \geq p_3$$

ρ



block-diagonal

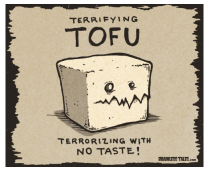


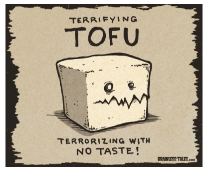
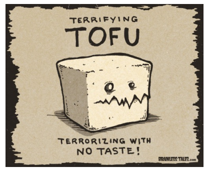
[1] Pusz, Woronowicz, Communications in Mathematical Physics, 58, (1978).

Complete passivity

Definition:

A state ρ is **completely passive** if $\rho^{\otimes n}$ is passive for all $n \geq 1$



... "completely tasteless"

Theorem:

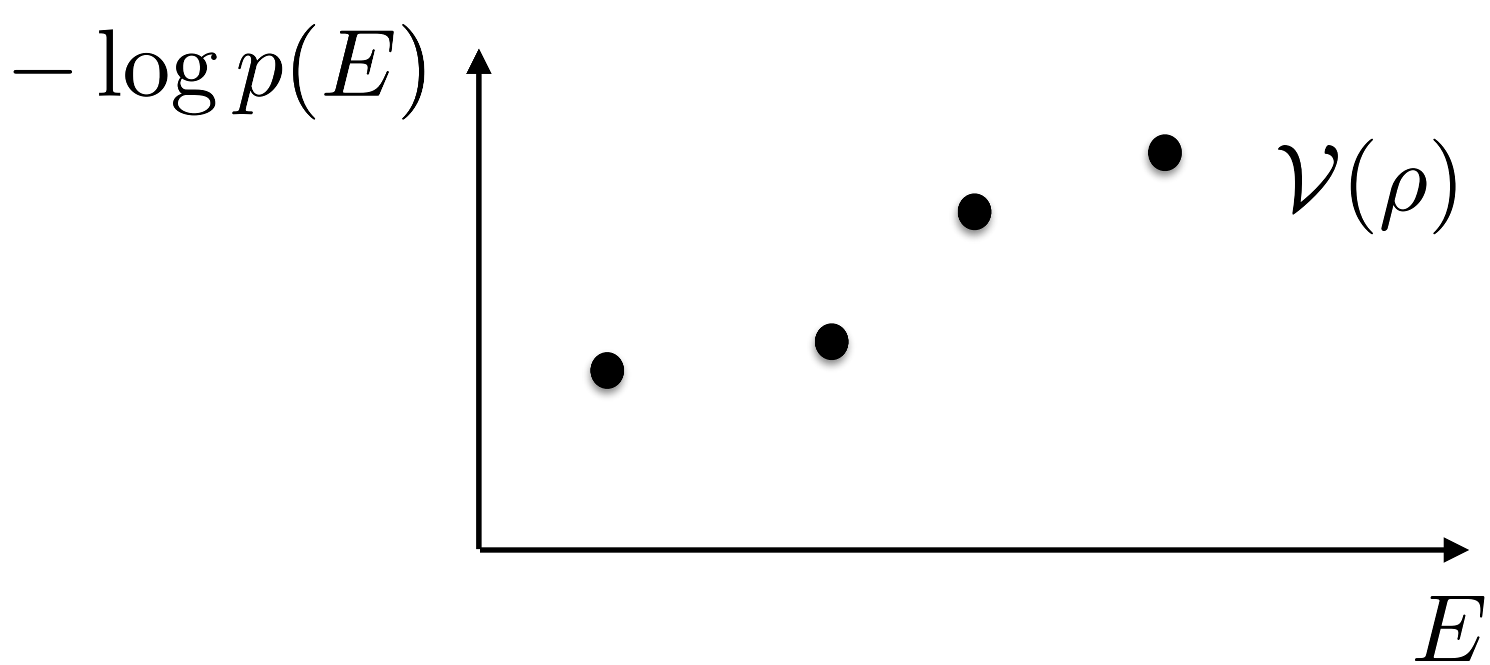
A state ρ is completely passive if and only if

$$\rho = \frac{1}{Z} e^{-\beta H} \text{ for some } \beta \geq 0$$

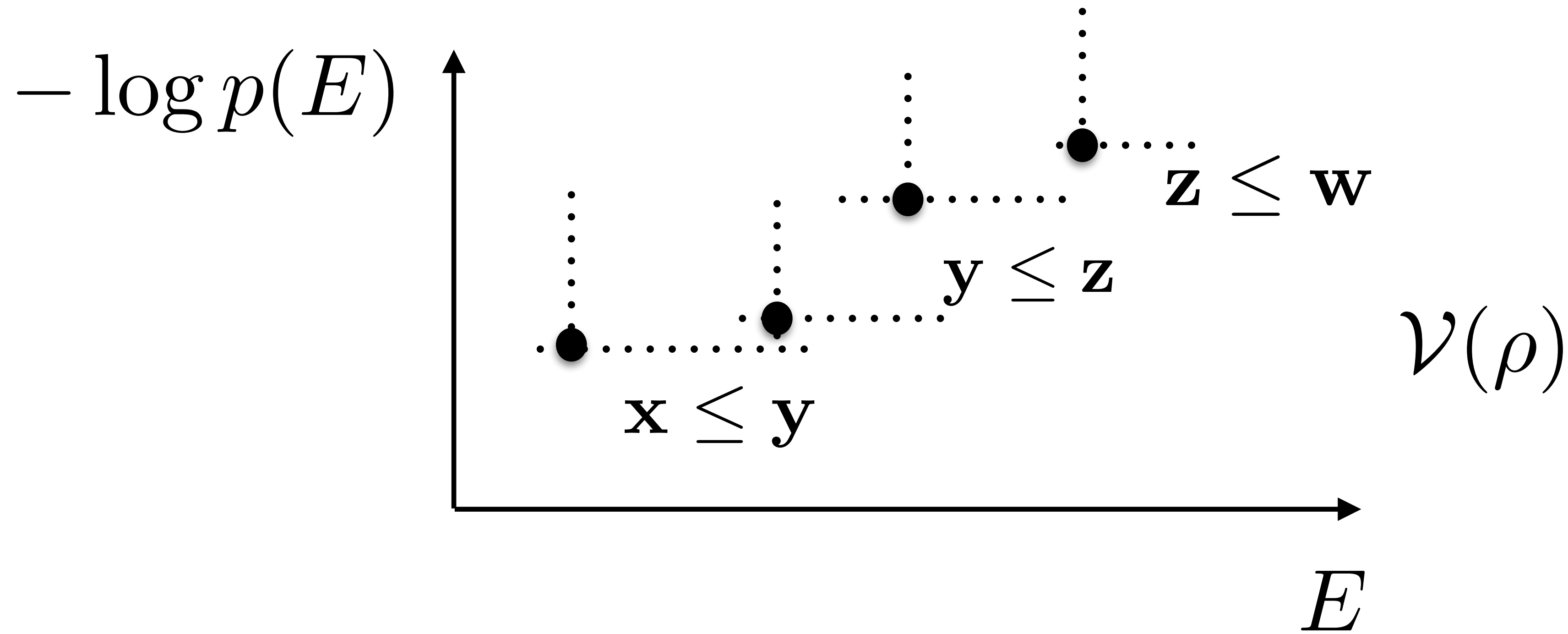
A quick proof

Given $\rho = \mathrm{diag}(\mathrm{p}_0, \mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{d})$

Define: $\mathcal{V}(\rho) := \{(E_0, -\log p_0), (E_1, -\log p_1), \ldots, (E_d, -\log p_d)\}$



(1) Define ordering on vectors



Show that ρ is passive if and only if

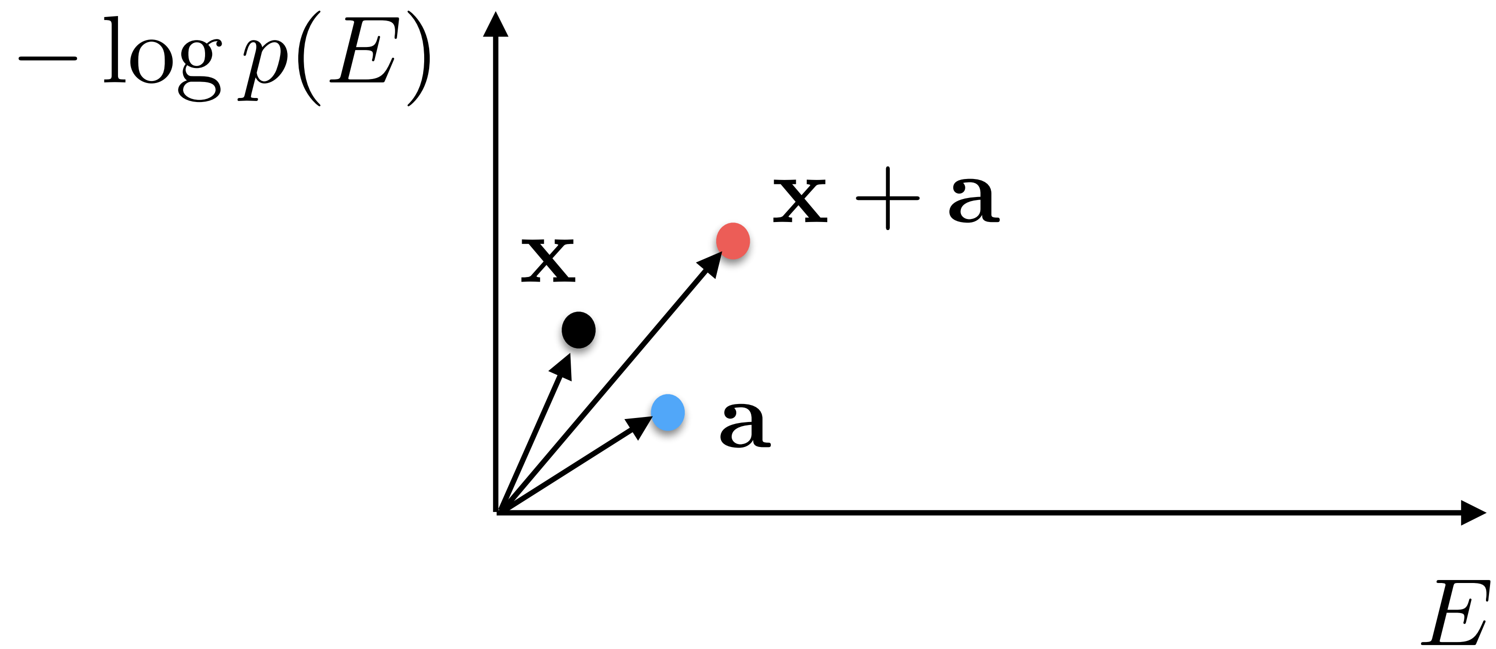
Elements of $\mathcal{V}(\rho)$ are ordered $\mathbf{x} \leq \mathbf{y} \leq \mathbf{z} \leq \mathbf{w} \cdots$

$$\mathbf{x} = (x_1, x_2)$$

$$\mathbf{y} = (y_1, y_2)$$

$\mathbf{x} \leq \mathbf{y}$ means that $[x_1 \leq y_1 \text{ and } x_2 \leq y_2]$

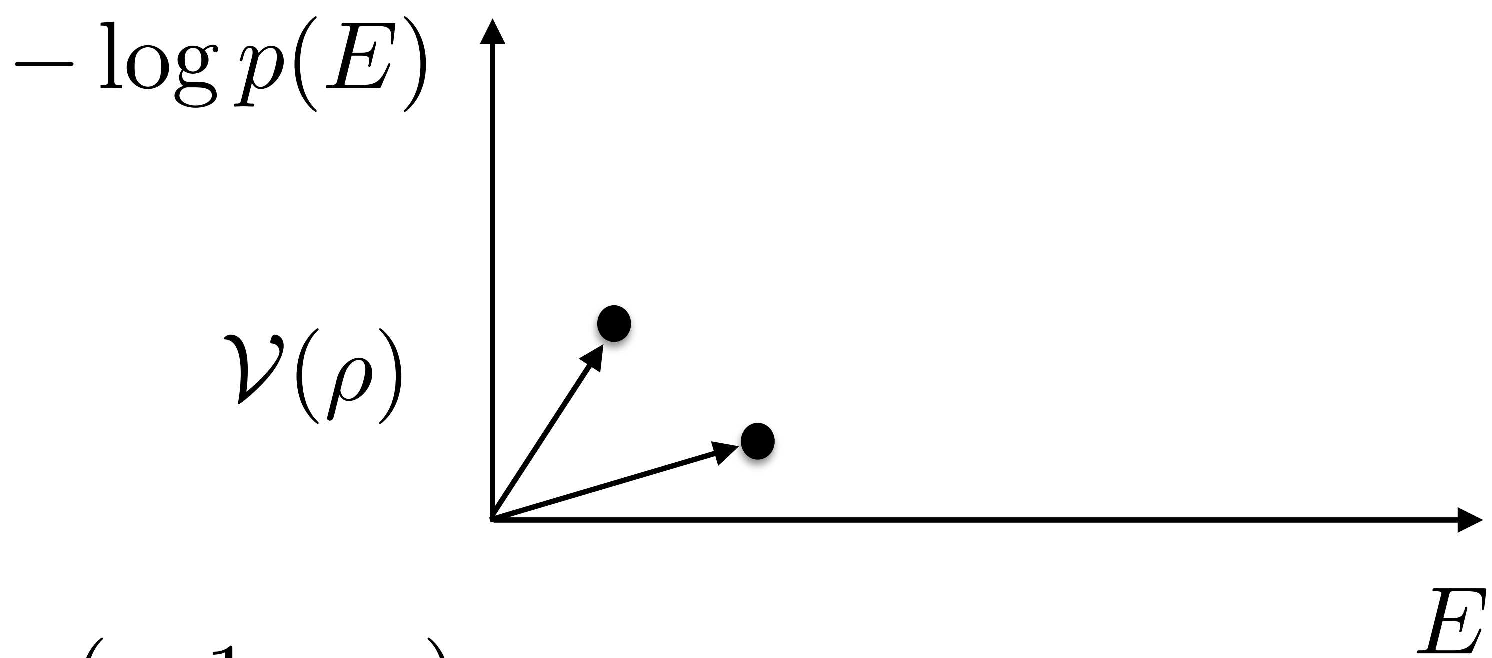
(2) Composing systems



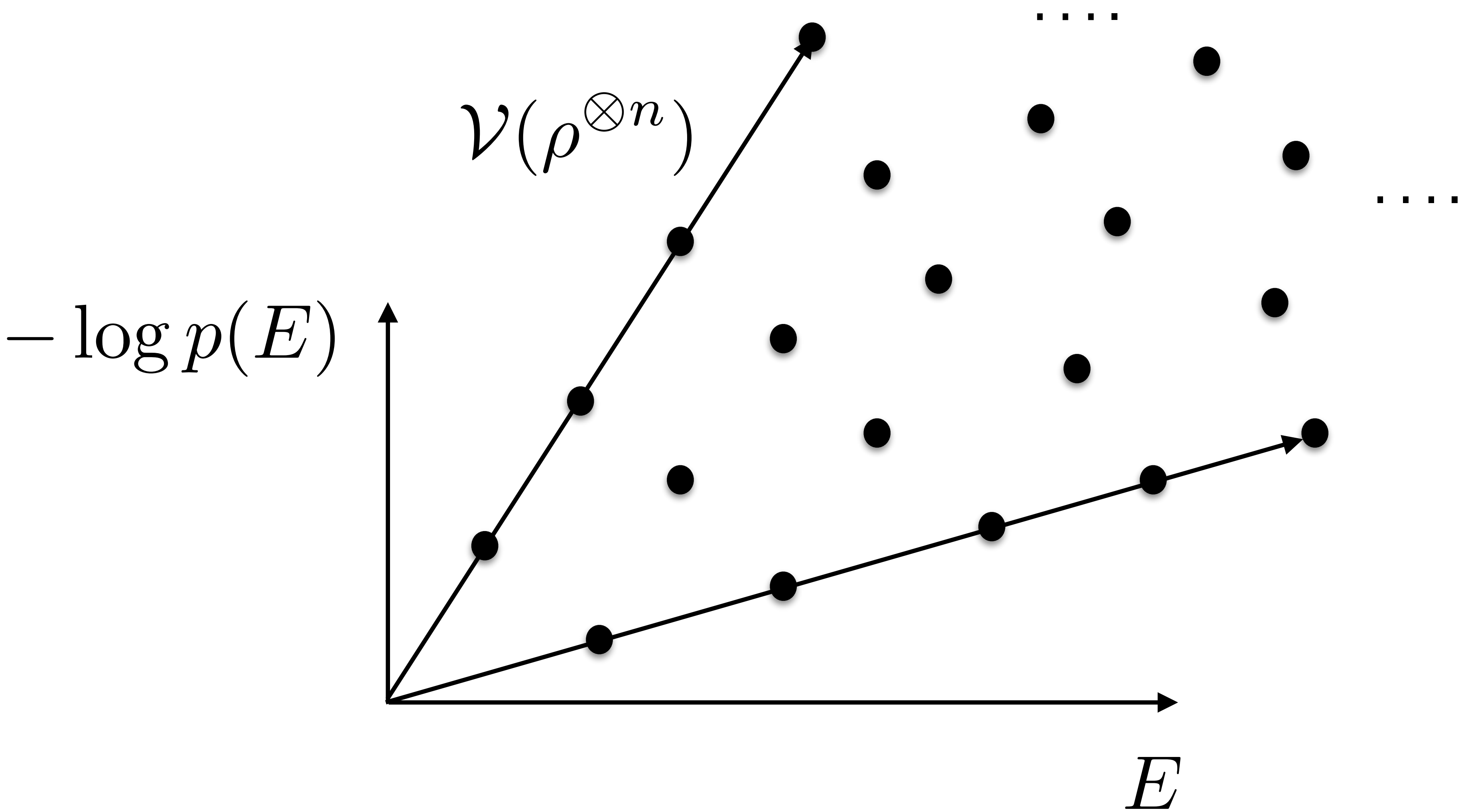
Show:

$$\text{Then } \mathcal{V}(\rho\otimes\sigma)=\mathcal{V}(\rho)\oplus\mathcal{V}(\sigma) \qquad \text{for any } \rho,\sigma$$

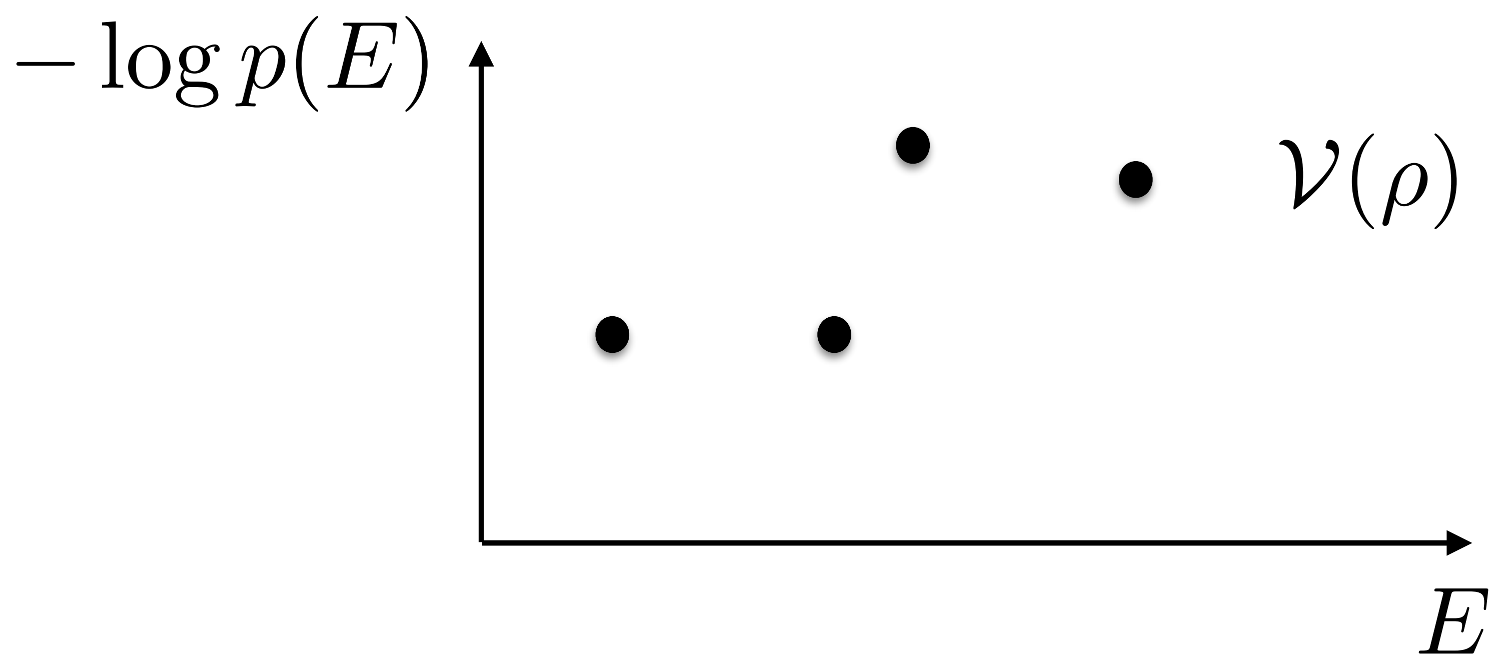
$$\{\mathbf{x},\mathbf{y}\}\oplus\{\mathbf{a},\mathbf{b}\}=\{\mathbf{x}+\mathbf{a},\mathbf{x}+\mathbf{b},\mathbf{y}+\mathbf{a},\mathbf{y}+\mathbf{b}\} \quad \text{etc}\ldots$$



$$\rho = \mathrm{diag}(\mathrm{p}, 1 - \mathrm{p})$$



$\mathcal{V}(\rho^{\otimes n})$
$-\log p(E)$
E
....
....

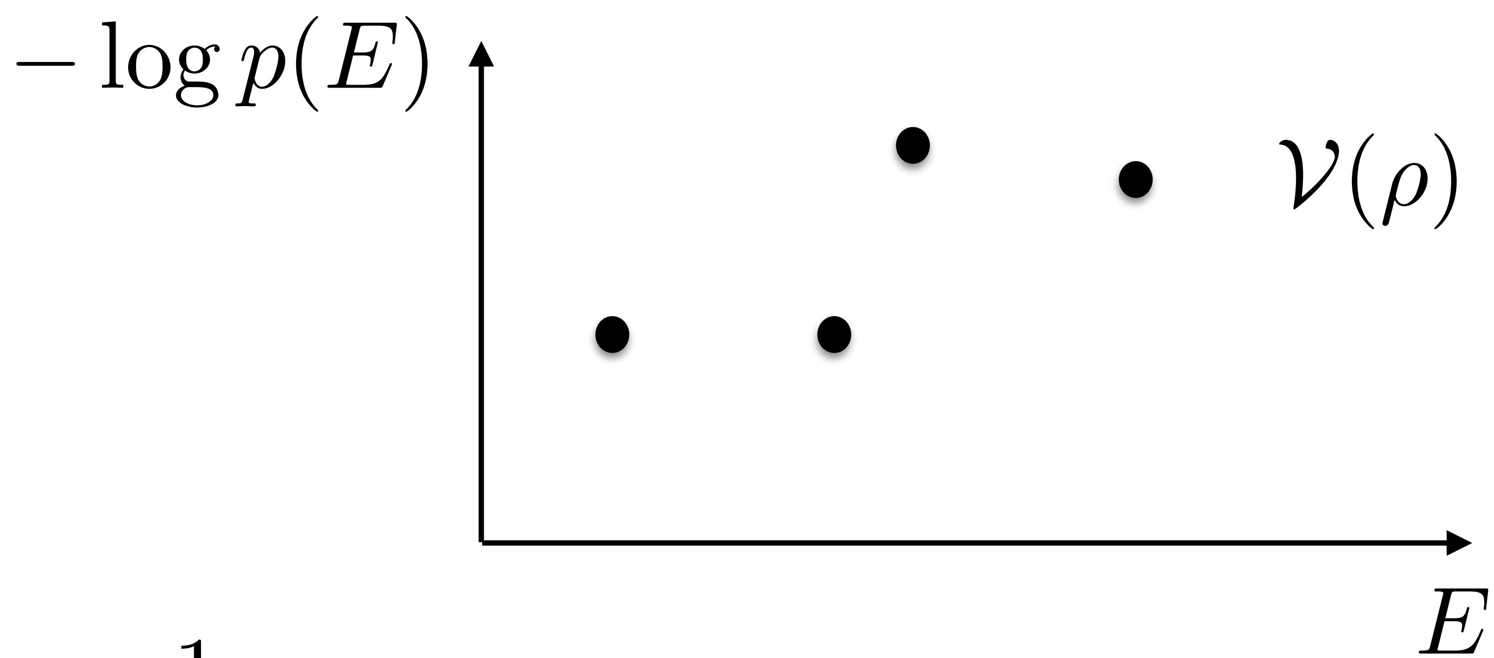


Show that

If $\mathcal{V}(\rho) = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots\}$ then

$$\mathcal{V}(\rho^{\otimes N}) = \{\sum_k n_k \mathbf{v}_k : \sum_k n_k = N\}$$

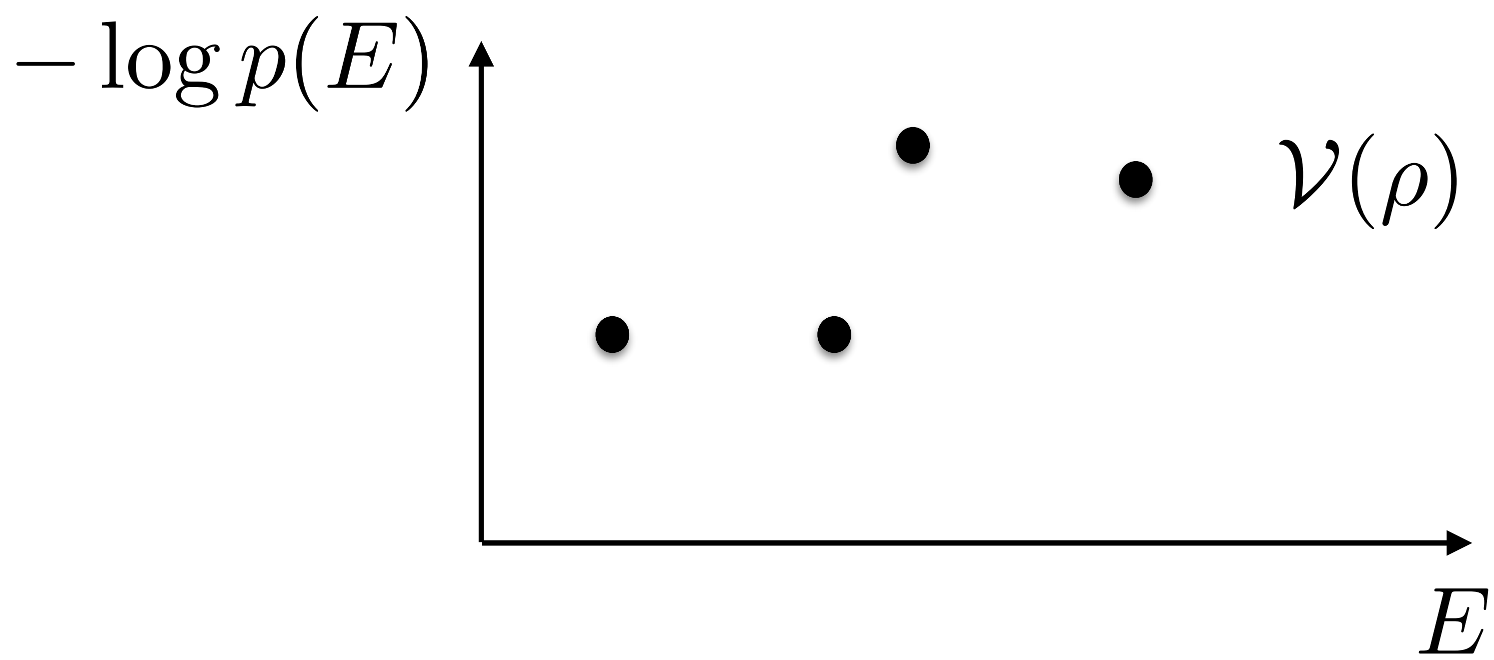
$$n_k \in \{0, 1, 2, \dots\}$$



$$\frac{1}{N}\mathcal{V}(\rho^{\otimes N}) = \frac{1}{N}\{\sum_k n_k \mathbf{v}_k : \sum_k n_k = N\}$$

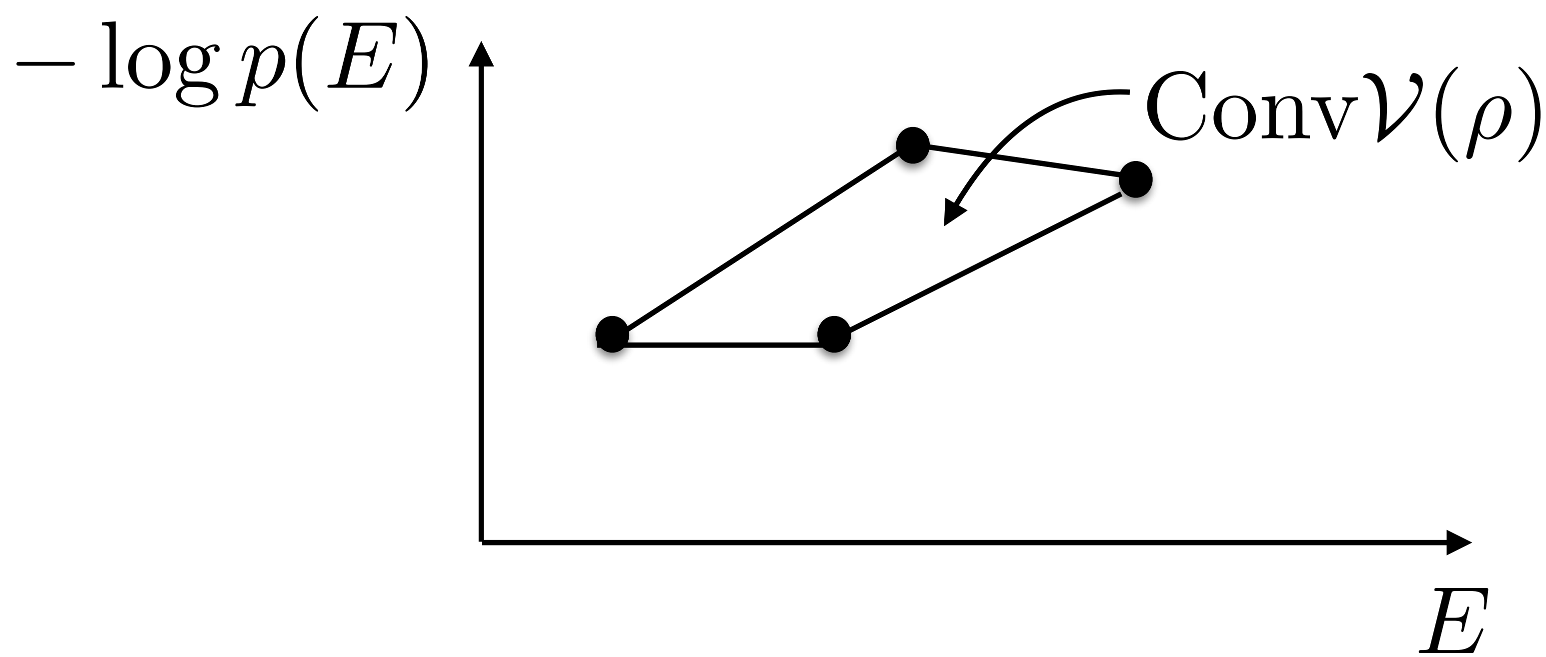
Therefore:

$$\lim_{N\to\infty}\frac{1}{N}\mathcal{V}(\rho^{\otimes N}) = \{\sum_k p_k \mathbf{v}_k : \sum_k p_k = 1\}$$



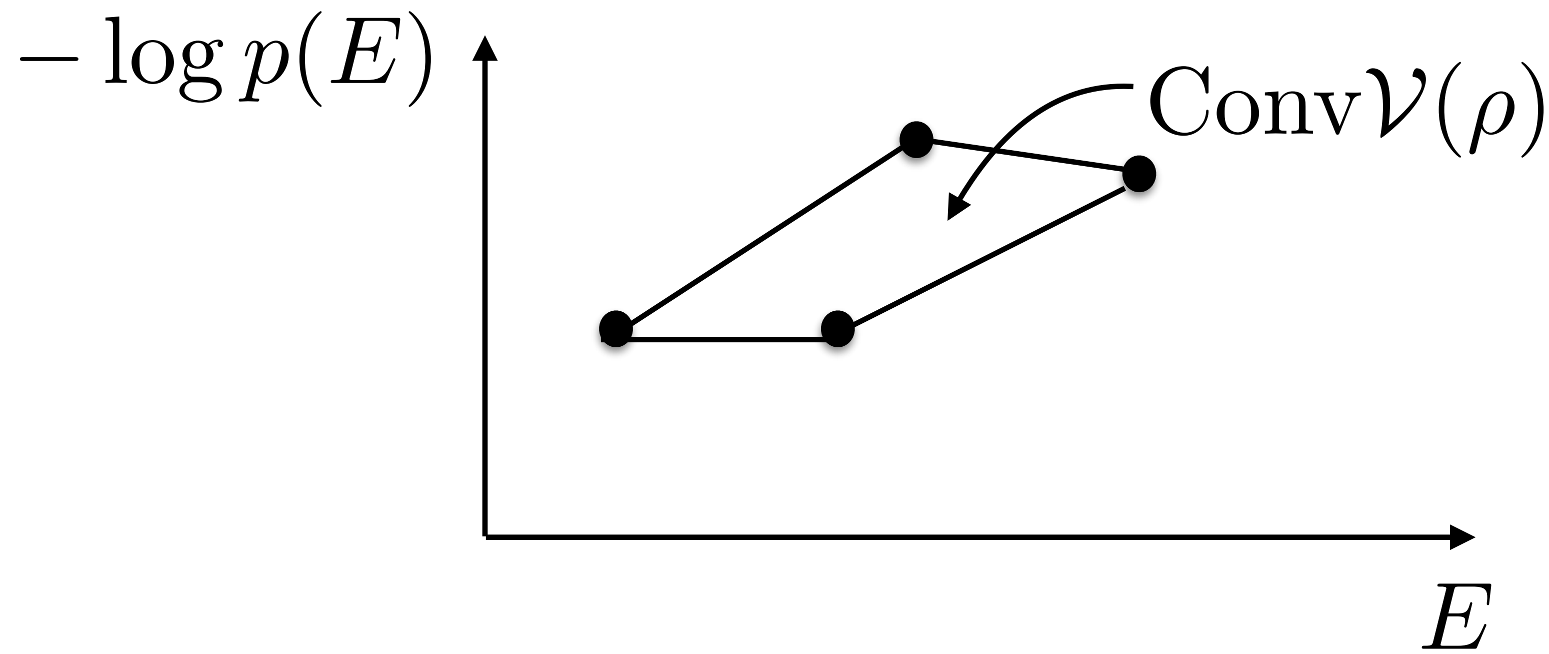
$$\lim_{N\to\infty} \frac{1}{N}\mathcal{V}(\rho^{\otimes N}) = \{\sum_k p_k \mathbf{v}_k : \sum_k p_k = 1\}$$

$$= \mathrm{Conv}\mathcal{V}(\rho) = \text{Convex Hull of } \mathcal{V}(\rho)$$



$$\lim_{N\to\infty}\frac{1}{N}\mathcal{V}(\rho^{\otimes N}) = \{\sum_k p_k \mathbf{v}_k : \sum_k p_k = 1\}$$

$$= \mathrm{Conv}\mathcal{V}(\rho) = \text{Convex Hull of } \mathcal{V}(\rho)$$



Now show:

$$\rho \text{ is completely passive } \Leftrightarrow \rho^{\otimes n} \text{ is passive } \forall n$$

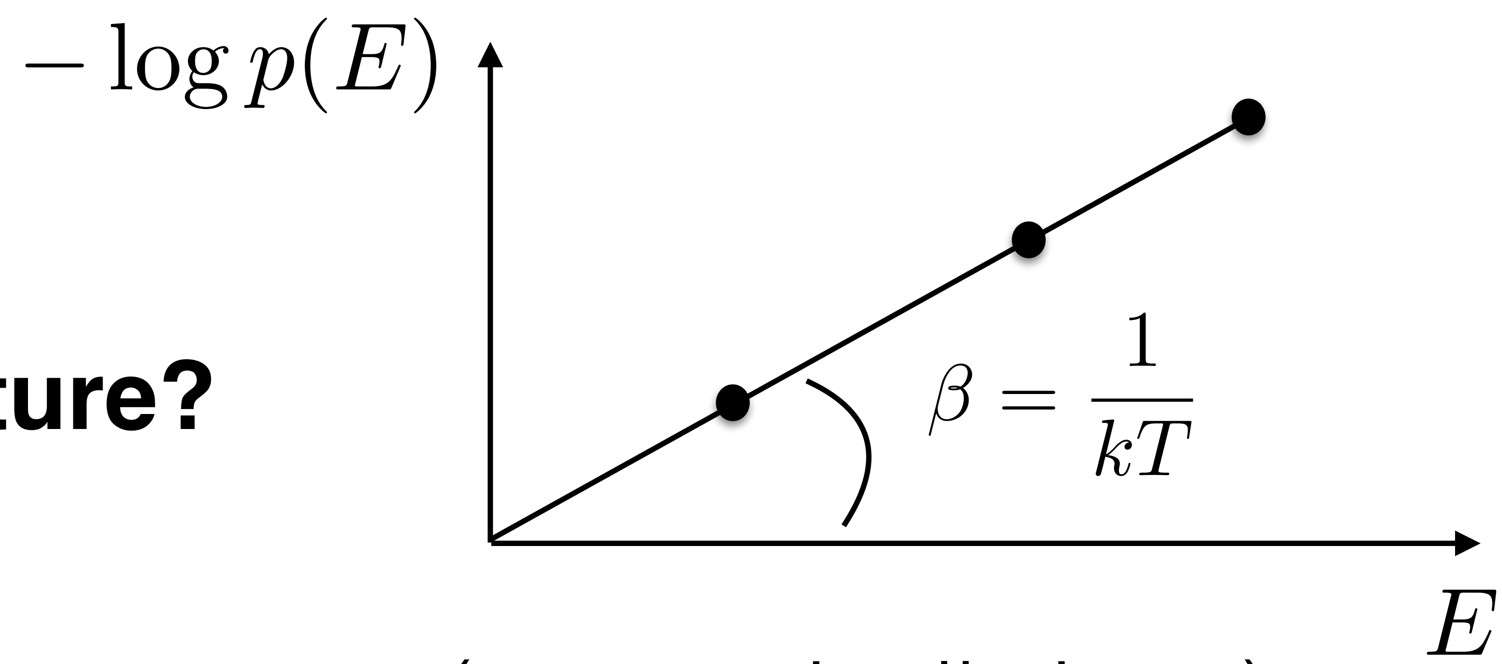
$$\Leftrightarrow \mathcal{V}(\rho^{\otimes n}) \text{ is totally ordered under } \leq \quad \forall n$$

$$\Leftrightarrow Conv(\mathcal{V}(\rho)) \text{ is totally ordered under } \leq$$

$$\Leftrightarrow \mathcal{V}(\rho) \text{ is a line segment}$$

$$\Leftrightarrow \rho \text{ is Gibbs state} \quad \blacksquare$$

Is it temperature?

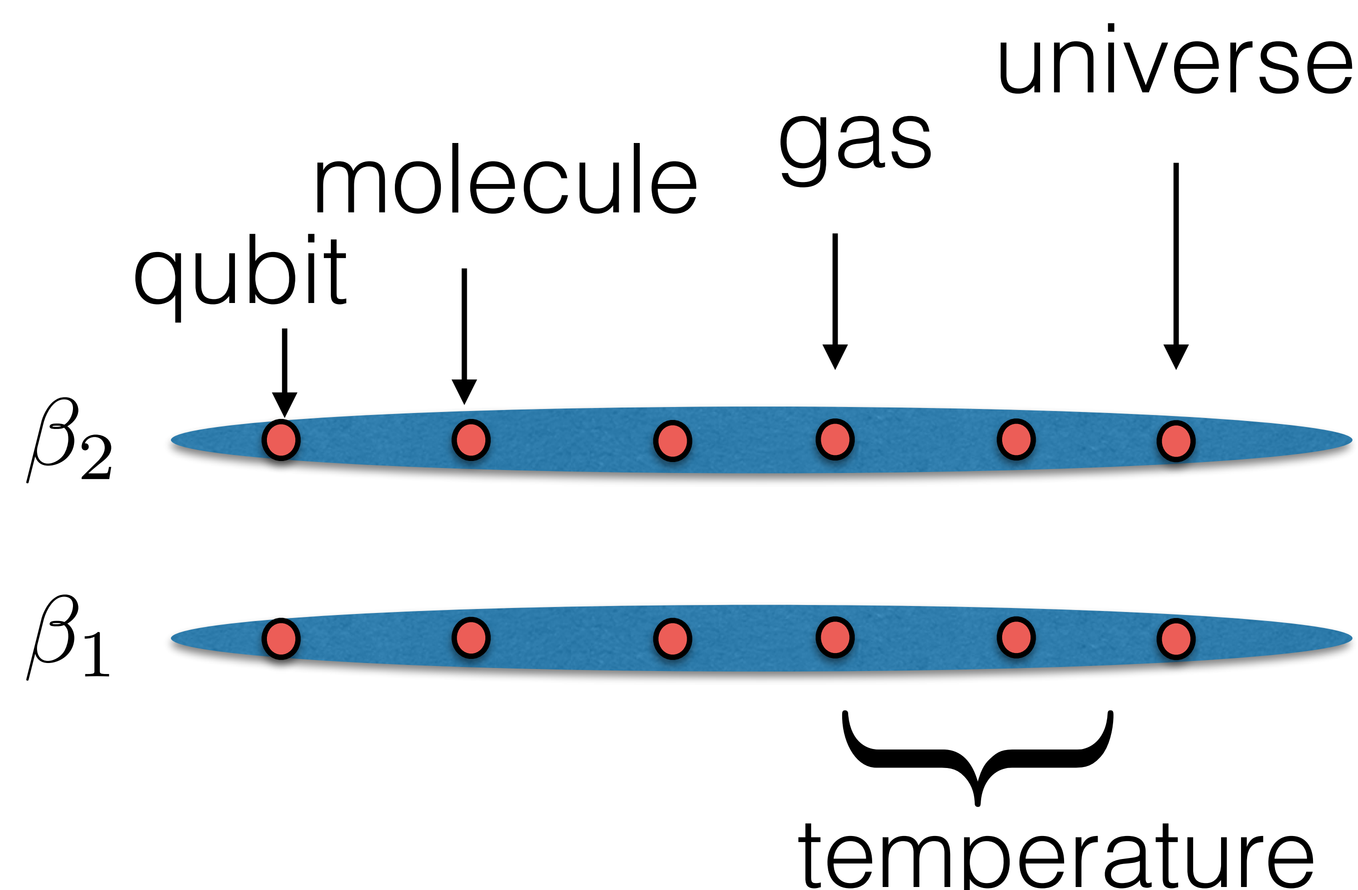


ρ completely positive (energetically inert)

$\mathcal{V}(\rho^{\otimes n})$ points all along a line

slope = "temperature"

$\rho \otimes \sigma$ energetically inert



Question — average energy & Gibbs states

Do MaxEnt subject to:

$$\langle H \rangle = U = \text{constant} \rightarrow \rho = \frac{1}{Z} e^{-\beta H}$$

$$\langle H^2 \rangle = C = \text{constant} \rightarrow \rho = \frac{1}{Z'} e^{-\lambda H^2}$$

different states!

But if we replace $H \rightarrow H^2$ in definition

*(or any other positive power of H)

$$\mathrm{tr}[(\mathrm{U}\rho\mathrm{U}^\dagger - \rho)\mathrm{H}] \geq 0 \ \ \forall U$$

we get the *same* set of "passive" states.

Q: ***So why does passivity route coincide with MaxEnt?***

The free operations

Thermal operations

(A1) Energy conserved microscopically

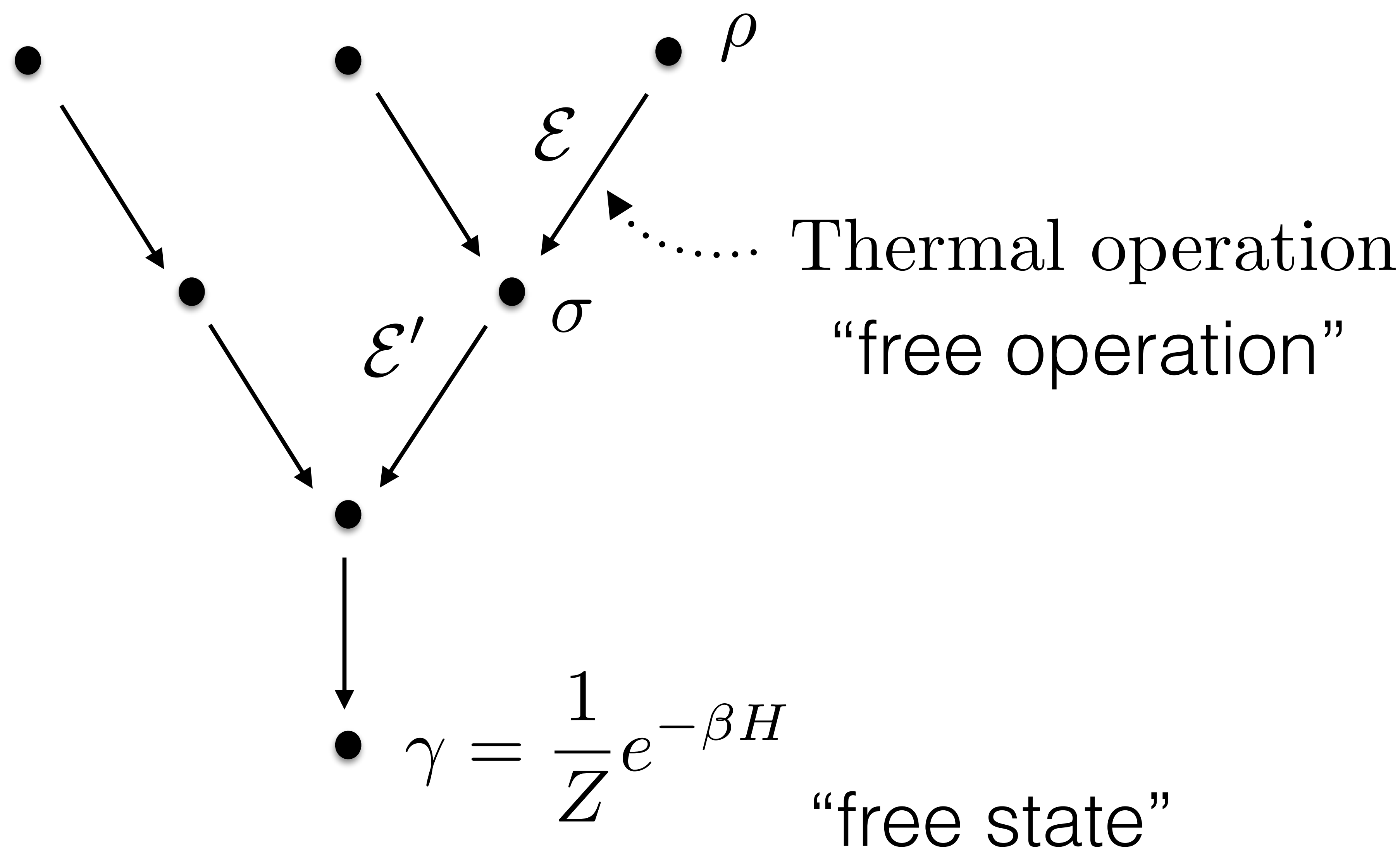
$$[V, H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B] = 0$$

(A2) Equilibrium "free" state: $\gamma_B = \frac{1}{Z} e^{-\beta H_B}$ $\quad \beta = \frac{1}{kT}$

"Free operations"=Thermal operations

$$\mathcal{E}(\rho_A) = \mathrm{tr_B V}(\rho_\mathrm{A} \otimes \gamma_\mathrm{B})\mathrm{V}^\dagger$$

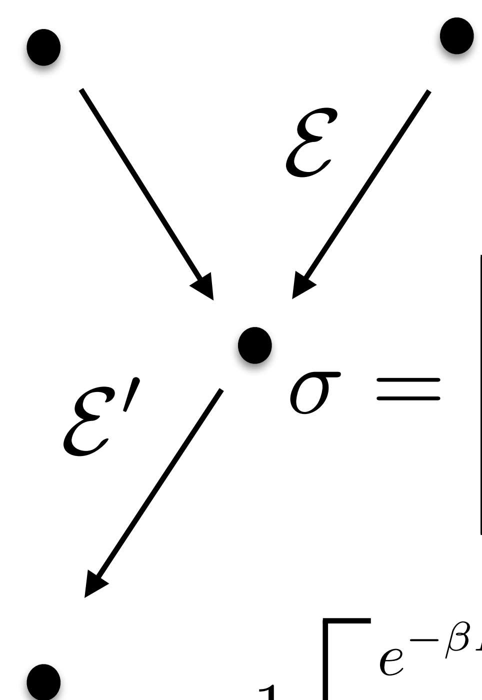
Partial order — “athermality"



Classical component

A "classical" component of thermodynamics

Restrict to states diagonal in energy.



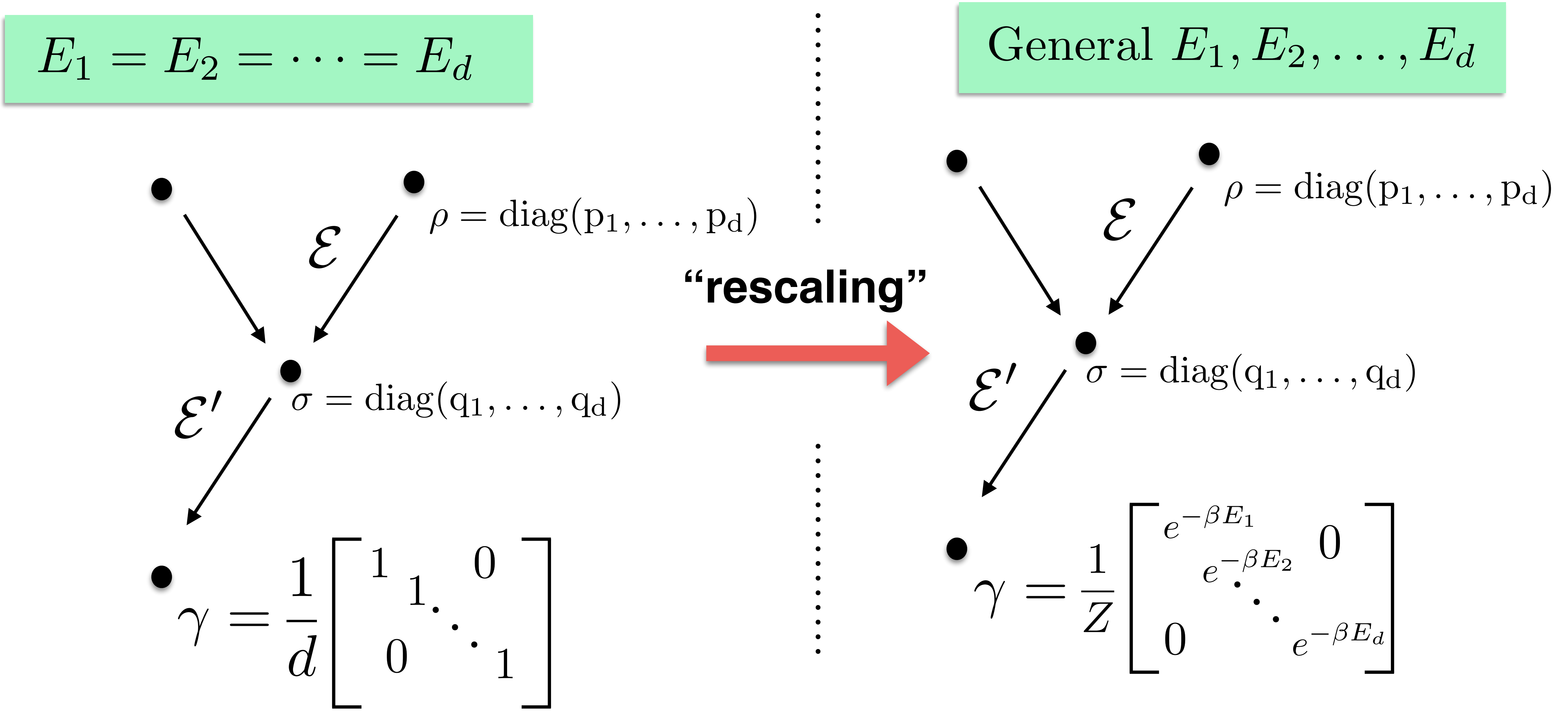
$$\rho = \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_d \end{bmatrix}$$

$$\sigma = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_d \end{bmatrix}$$

$$\gamma = \frac{1}{Z} \begin{bmatrix} e^{-\beta E_1} & & & 0 \\ & e^{-\beta E_2} & & \\ & & \ddots & \\ 0 & & & e^{-\beta E_d} \end{bmatrix}$$

Core structure

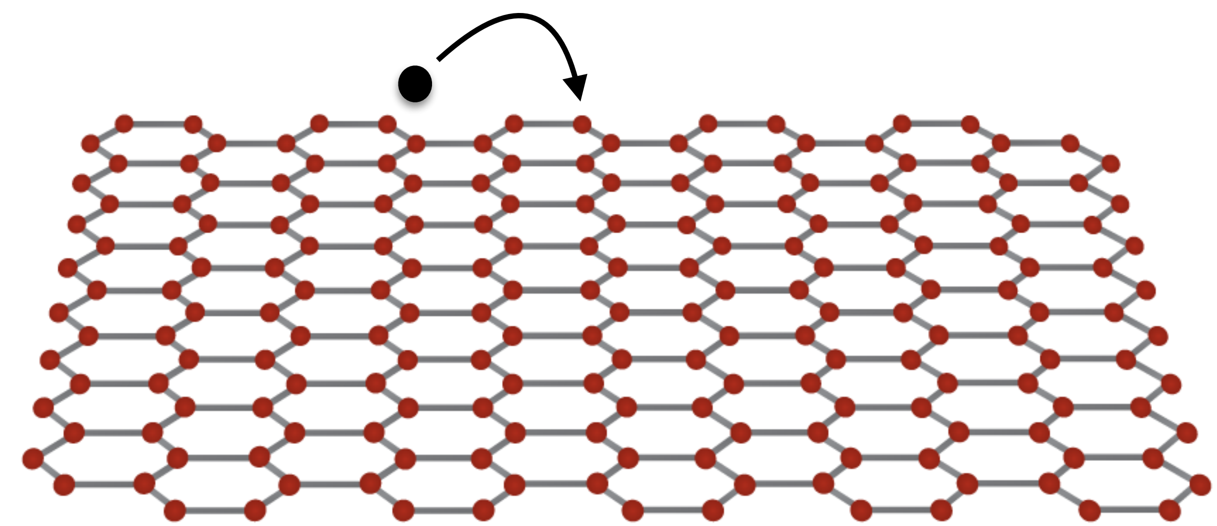
The general thermodynamic structure turns out to be fully captured by the **degenerate** case:



Degenerate case structure

First assume $H_A = 0$

$$\Rightarrow \gamma_A = \frac{1}{d}\mathbb{I}$$



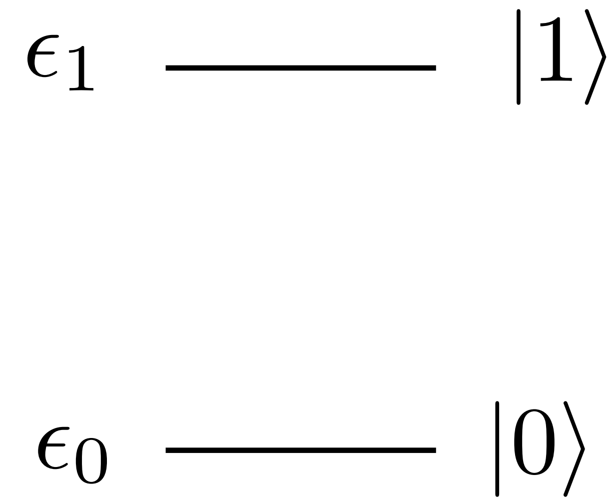
(all sites equivalent)

What states are accessible from a fixed ρ ?

$$\sigma = \mathcal{E}(\rho) = \mathrm{tr_B V}(\rho_\mathrm{A} \otimes \gamma_\mathrm{B})\mathrm{V}^\dagger$$

$$[V, H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B] = 0$$

Example



$$H = \epsilon_0|0\rangle\langle 0| + \epsilon_1|1\rangle\langle 1|$$

$$\gamma = \frac{1}{Z}(e^{-\beta\epsilon_0}|0\rangle\langle 0| + e^{-\beta\epsilon_1}|1\rangle\langle 1|)$$

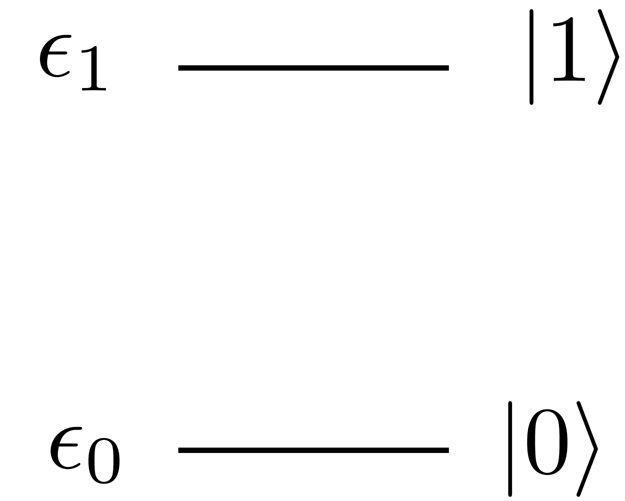
$$= \frac{1}{Z}\begin{bmatrix} e^{-\beta\epsilon_0} & 0 \\ 0 & e^{-\beta\epsilon_1} \end{bmatrix}$$

$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1)$$

$$\rho = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

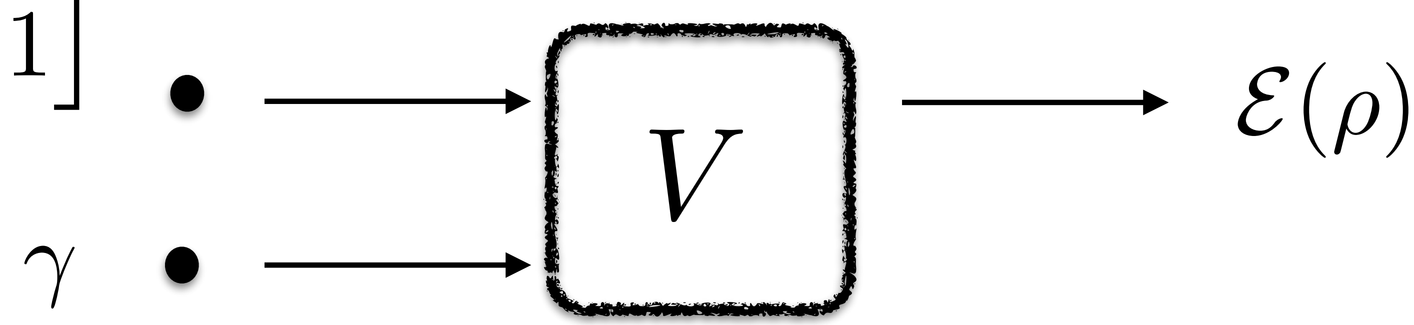
$$\mathbf{p} = (0, 1)$$

Example



$$H = \epsilon_0 |0\rangle\langle 0| + \epsilon_1 |1\rangle\langle 1|$$

$$\rho = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



$$V|00\rangle = |00\rangle \qquad V|01\rangle = \cos t|01\rangle + \sin t|10\rangle$$

$$V|11\rangle = |11\rangle \qquad V|10\rangle = -\sin t|01\rangle + \cos t|10\rangle$$

Example

ϵ_1 ——— $|1\rangle$

ϵ_0 ——— $|0\rangle$

$$\rho \otimes \gamma = \begin{bmatrix} p_0\gamma_0 & 0 & 0 & 0 \\ 0 & p_0\gamma_1 & 0 & 0 \\ 0 & 0 & p_1\gamma_0 & 0 \\ 0 & 0 & 0 & p_1\gamma_1 \end{bmatrix}$$

Example

$$V(\rho \otimes \gamma)V^\dagger = \begin{bmatrix} p_0\gamma_0 & 0 & 0 & 0 \\ 0 & p_0\gamma_1 \cos^2 t + p_1\gamma_0 \sin^2 t & * & 0 \\ 0 & * & p_1\gamma_0 \cos^2 t + p_0\gamma_1 \sin^2 t & 0 \\ 0 & 0 & 0 & p_1\gamma_1 \end{bmatrix}$$

Example

$$V(\rho \otimes \gamma)V^\dagger = \begin{bmatrix} p_0\gamma_0 & 0 & 0 & 0 \\ 0 & p_0\gamma_1 \cos^2 t + p_1\gamma_0 \sin^2 t & * & 0 \\ 0 & * & p_1\gamma_0 \cos^2 t + p_0\gamma_1 \sin^2 t & 0 \\ 0 & 0 & 0 & p_1\gamma_1 \end{bmatrix}$$

$$\mathrm{tr_B}[\mathrm{V}(\rho \otimes \gamma)\mathrm{V}^\dagger] = \begin{bmatrix} p_0\gamma_0 + p_0\gamma_1 \cos^2 t + p_1\gamma_0 \sin^2 t & 0 \\ 0 & p_1\gamma_0 \cos^2 t + p_0\gamma_1 \sin^2 t + p_1\gamma_1 \end{bmatrix}$$

$$= \begin{bmatrix} A_{00}p_0 + A_{01}p_1 & 0 \\ 0 & A_{10}p_0 + A_{11}p_1 \end{bmatrix}$$

Noisy operations

$$\rho = \mathrm{diag}(\mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{d}) \xrightarrow{\mathcal{E}} \sigma = \mathrm{diag}(\mathrm{q}_1, \ldots, \mathrm{q}_\mathrm{d})$$

$$\mathbf{q} = A\mathbf{p}$$

$$A = (A_{ij}) = \text{bistochastic matrix}$$

$$A_{ij} \geq 0$$

$$\sum_i A_{ij} = \sum_j A_{ij} = 1$$

Overview

1. **Motivations and criteria.**
2. **Resource theories and properties.**
3. **Resource formulation of thermodynamics.**
4. Majorization — order & disorder.
5. Information & energy.
6. Gibbs-rescaling & work bits.

Majorization

Majorization theory $\mathbf{p} \succ_m \mathbf{q}$

$$\Leftrightarrow \; \mathbf{q} = A\mathbf{p} \qquad\qquad A \text{ bistochastic}$$

$$\Leftrightarrow \quad L_{\mathbf{p}}(x) \geq L_{\mathbf{q}}(x) \;\; \forall x$$

$$L_{\mathbf{p}}(x) = \text{ The } \textbf{Lorenz curve} \text{ of } \mathbf{p}$$

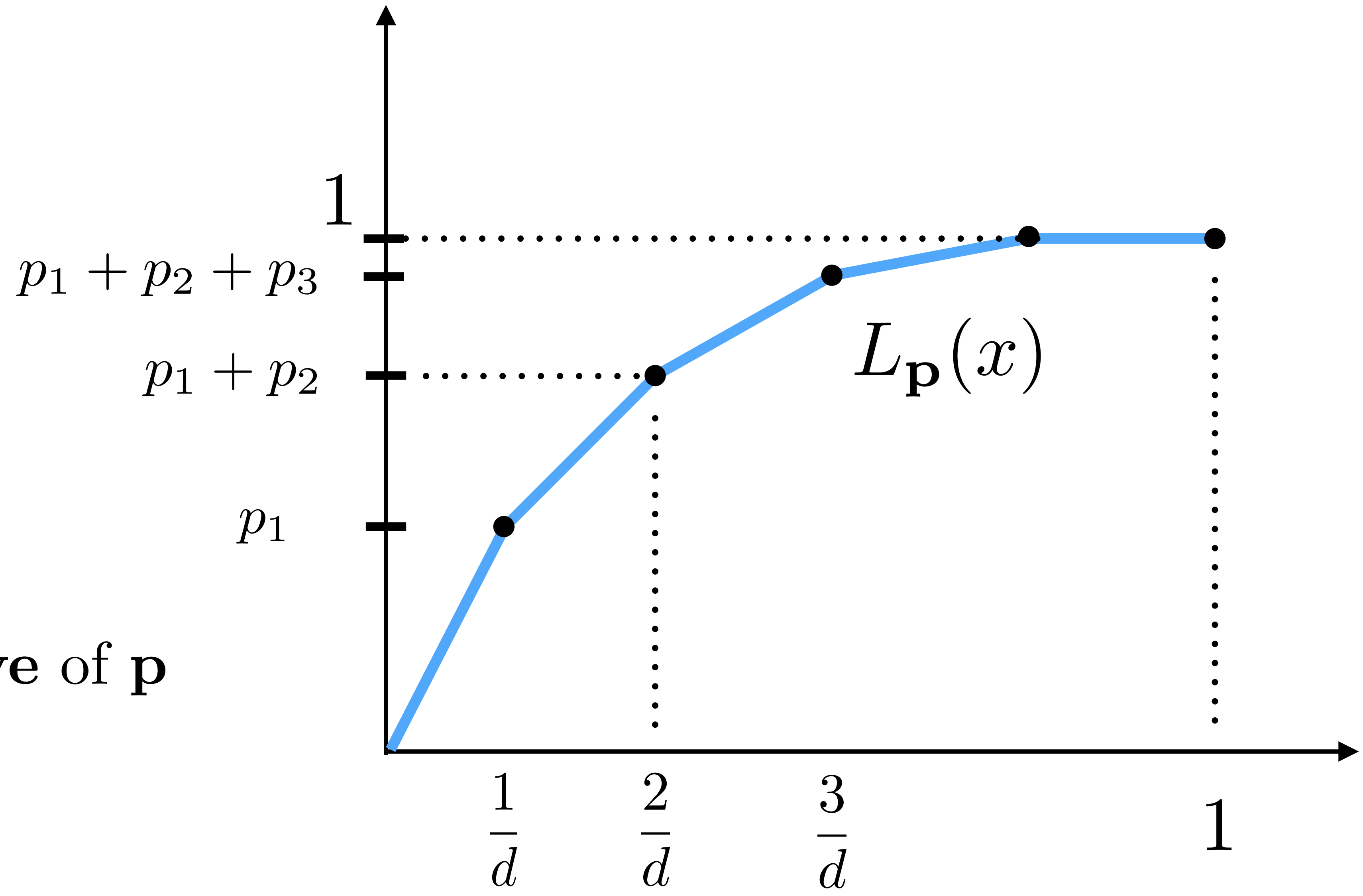
Lorenz curves

$$\mathbf{p} = (p_1, p_2, \ldots, p_d)$$

ordered so that:

$$p_1 \geq p_2 \cdots \geq p_d$$

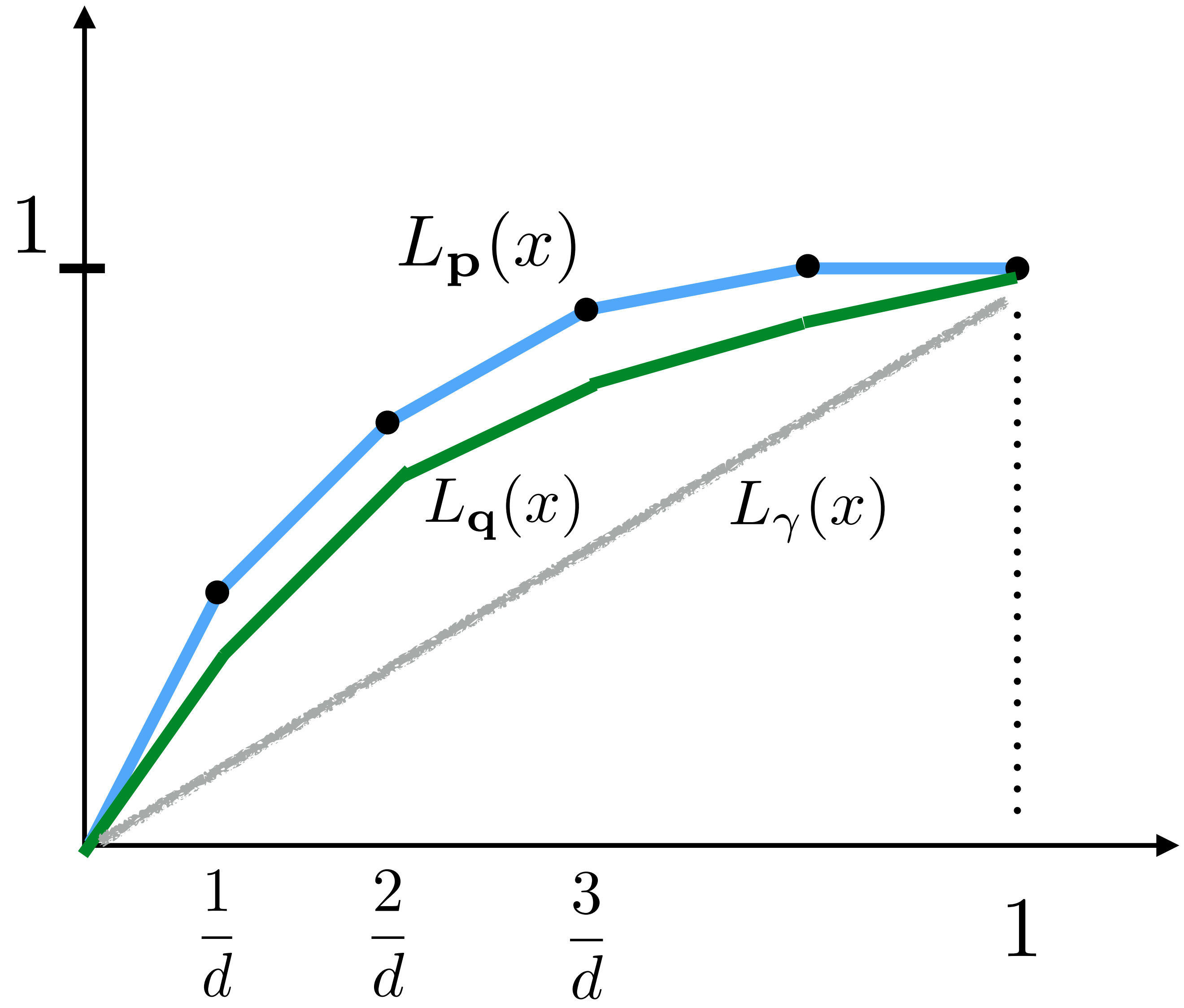
$L_{\mathbf{p}}(x) =$ The **Lorenz curve** of $\mathbf{p}$



Majorization

$$\mathbf{p} \succ_m \mathbf{q}$$

$$\Leftrightarrow \quad L_{\mathbf{p}}(x) \geq L_{\mathbf{q}}(x) \;\; \forall x$$



Example

$$\mathbf{p} = (2/5, 2/5, 1/10, 1/10)$$

$$\mathbf{q} = (1/2, 1/4, 1/4, 0)$$

$$p_1 = 2/5$$

$$q_1 = 1/2$$

$$p_1 + p_2 = 4/5$$

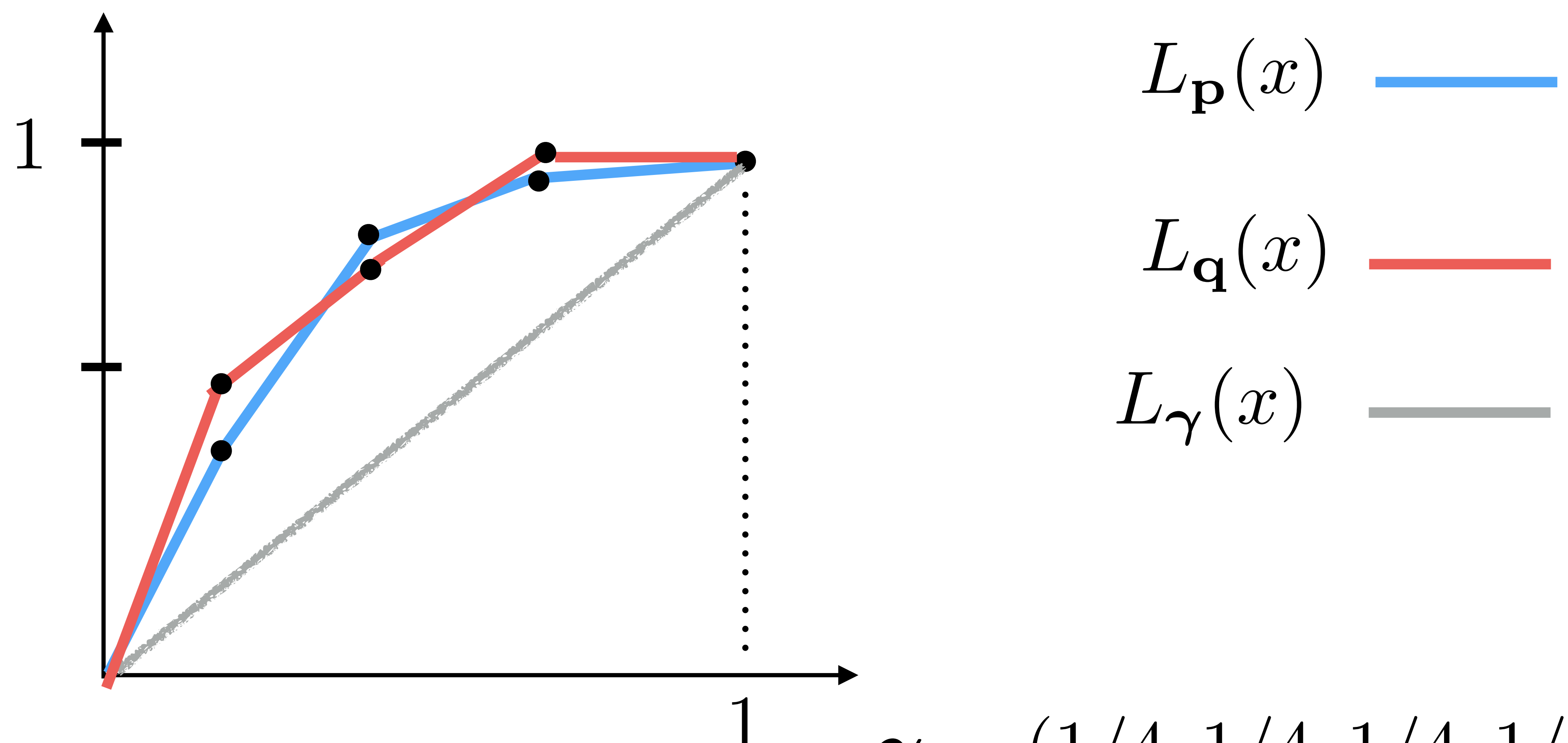
$$q_1 + q_2 = 3/4$$

$$p_1 + p_2 + p_3 = 9/10$$

$$q_1 + q_2 + q_3 = 1$$

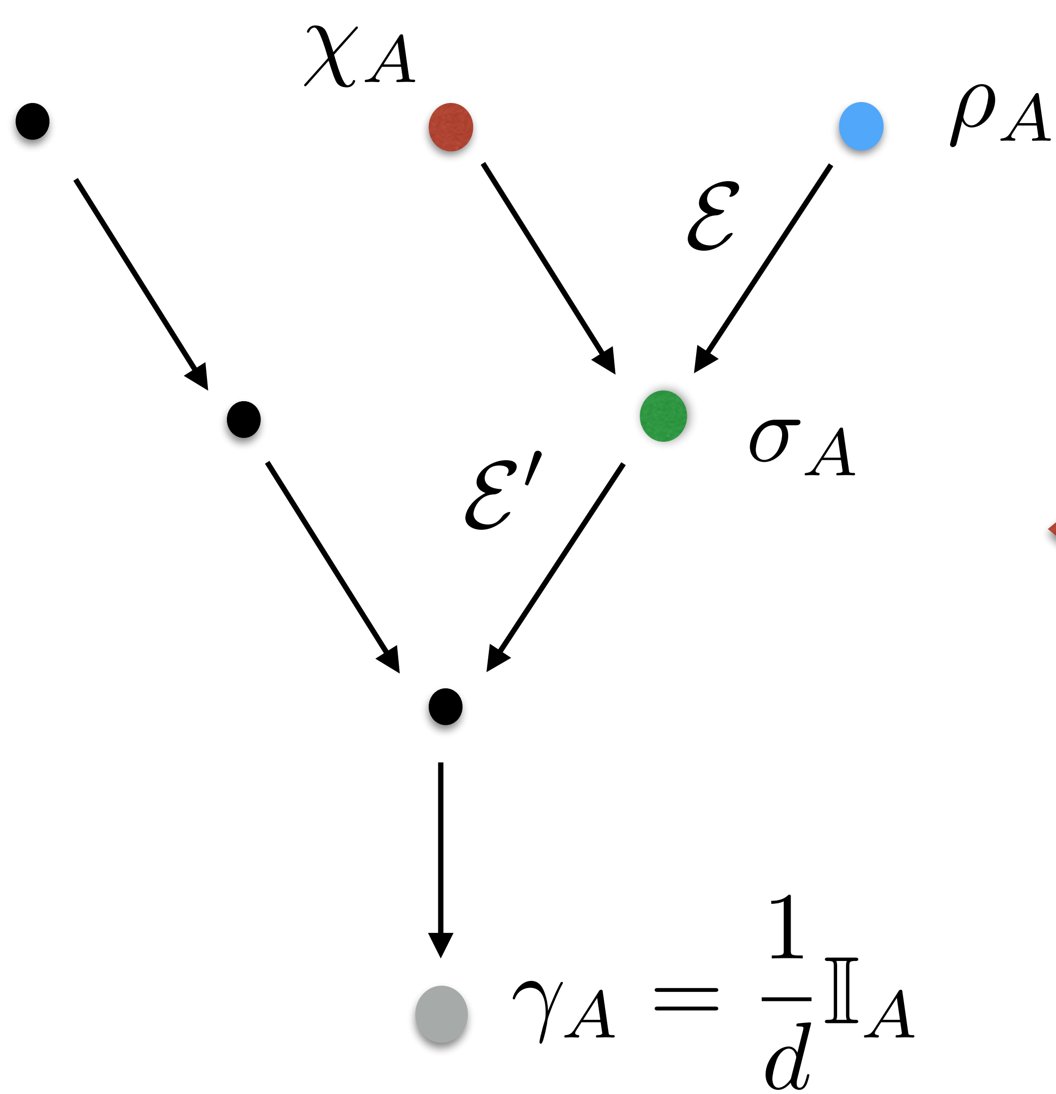
Example

$\mathbf{p} = (2/5, 2/5, 1/10, 1/10)$ $\mathbf{q} = (1/2, 1/4, 1/4, 0)$

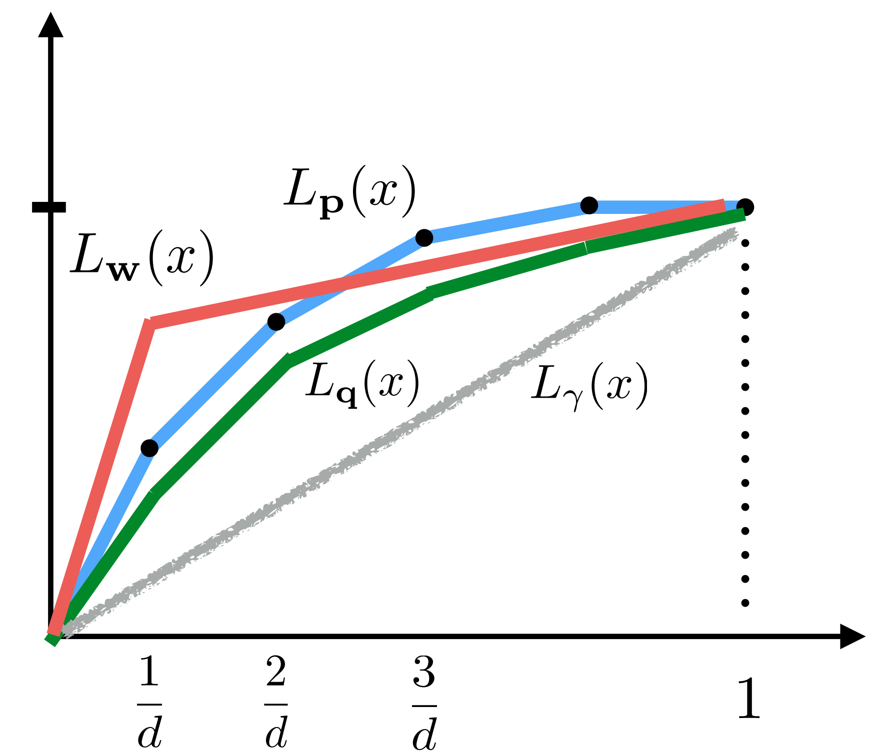


$\boldsymbol{\gamma} = (1/4, 1/4, 1/4, 1/4)$

Classical ordering of states



"ordering of concave functions"!



$\mathbf{w}$ and $\mathbf{p}$ are **incomparable**

χ_A ρ_A

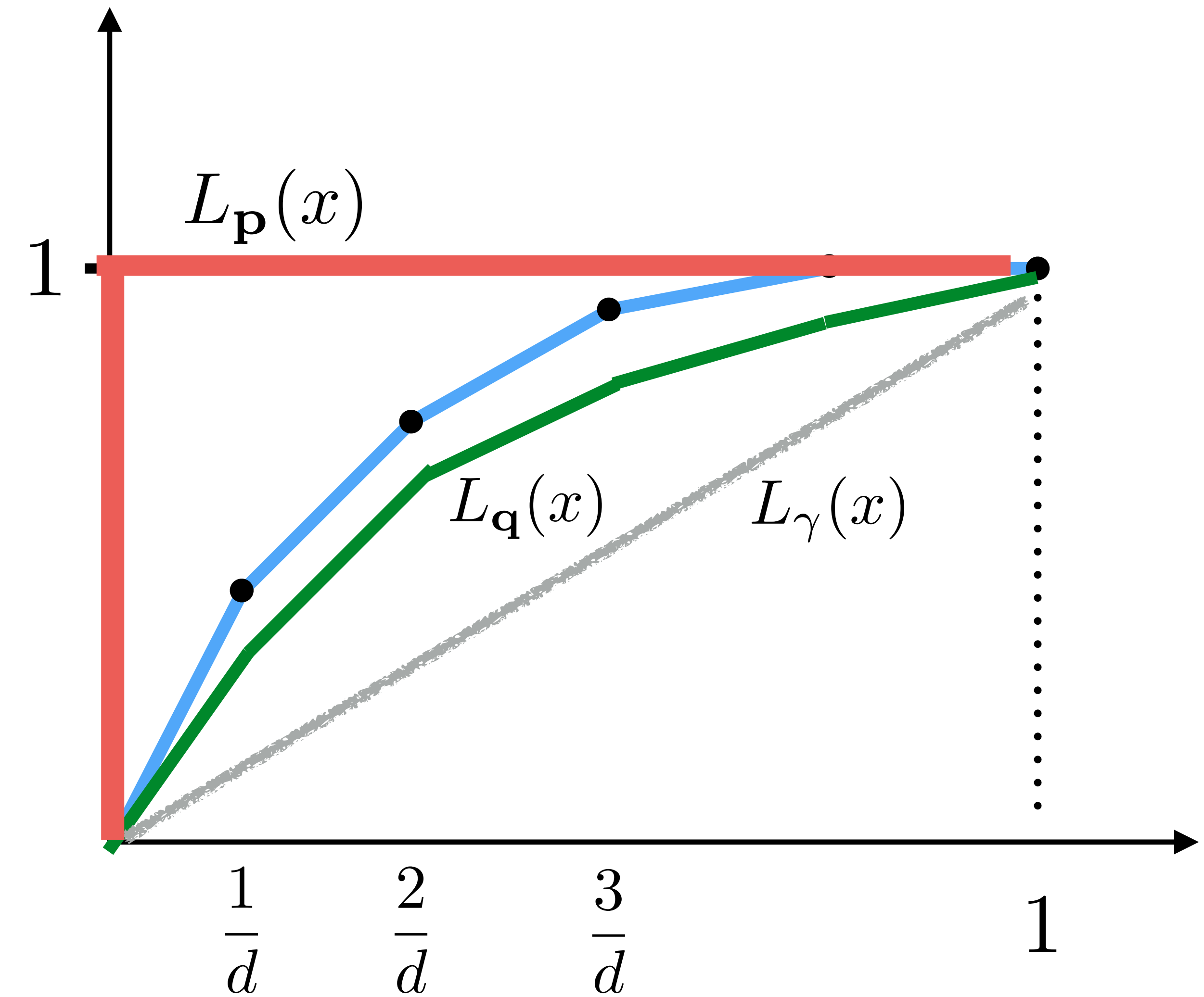
Ordered & disordered Energy

perfectly ordered:

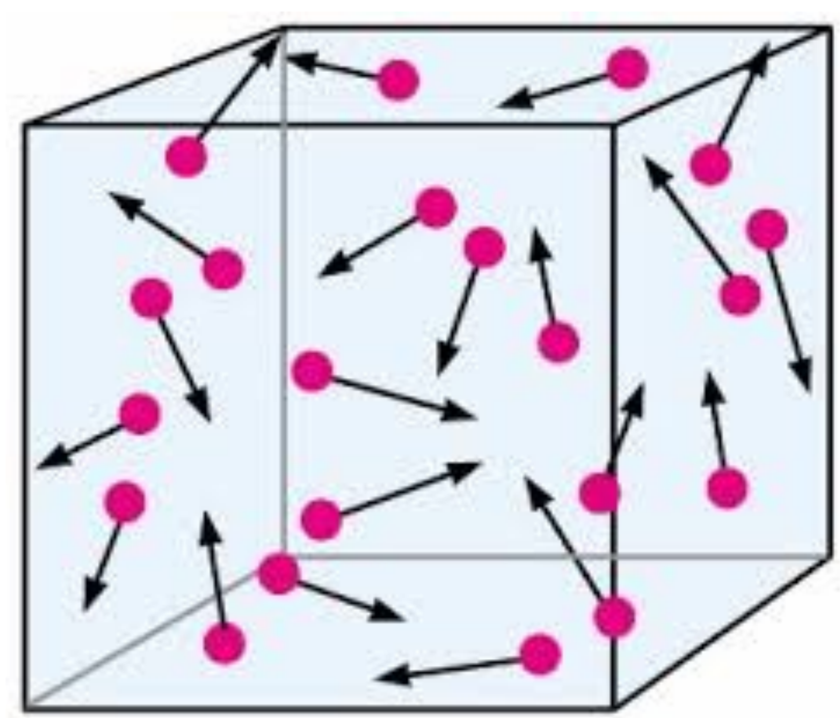
$$\mathbf{p} = (1, 0, 0 \ldots, 0)$$

perfectly disordered:

$$\boldsymbol{\gamma} = (\frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d})$$



Macroscopic regime



$n \gg 1$

extrinsic

$$\rho_{total} \approx \rho^{\otimes n}$$

$\rho =$ encodes the **intrinsic** variables

Macroscopic limits — typicality

Toss a million coins:

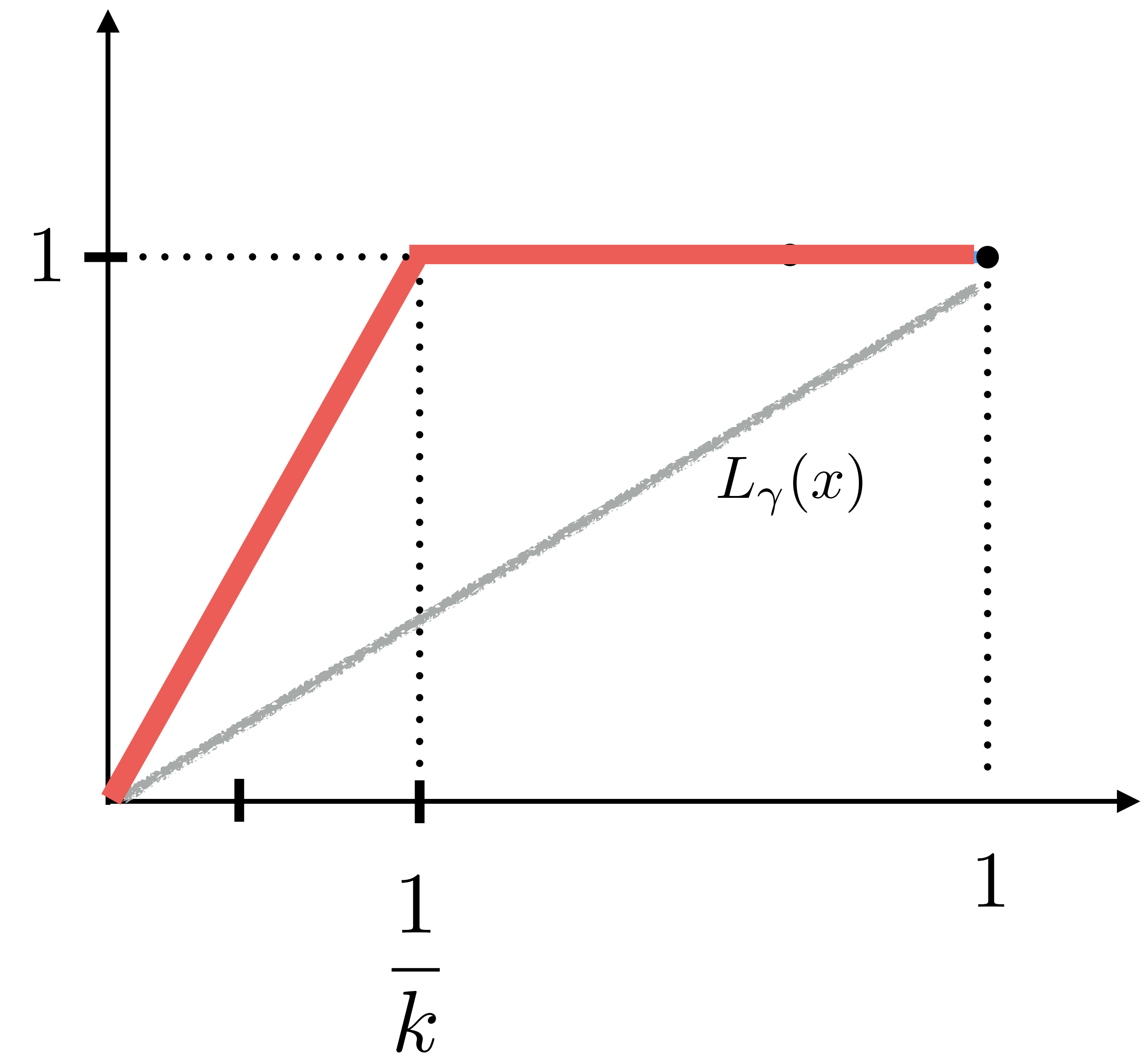
~half up half down, ordering **uniformly random**

$$\rho^{\otimes n} = (1-\epsilon)\frac{1}{k}\mathbf{I}_{typical} + \epsilon\sigma_{other} \qquad\qquad \epsilon \to 0$$

$$\rho^{\otimes n} \approx \frac{1}{k}\mathbf{I}_{typical} = \frac{1}{k}\mathrm{diag}(0,0,0,1,1,1,\ldots,1,0,0)$$

k-sharp states:

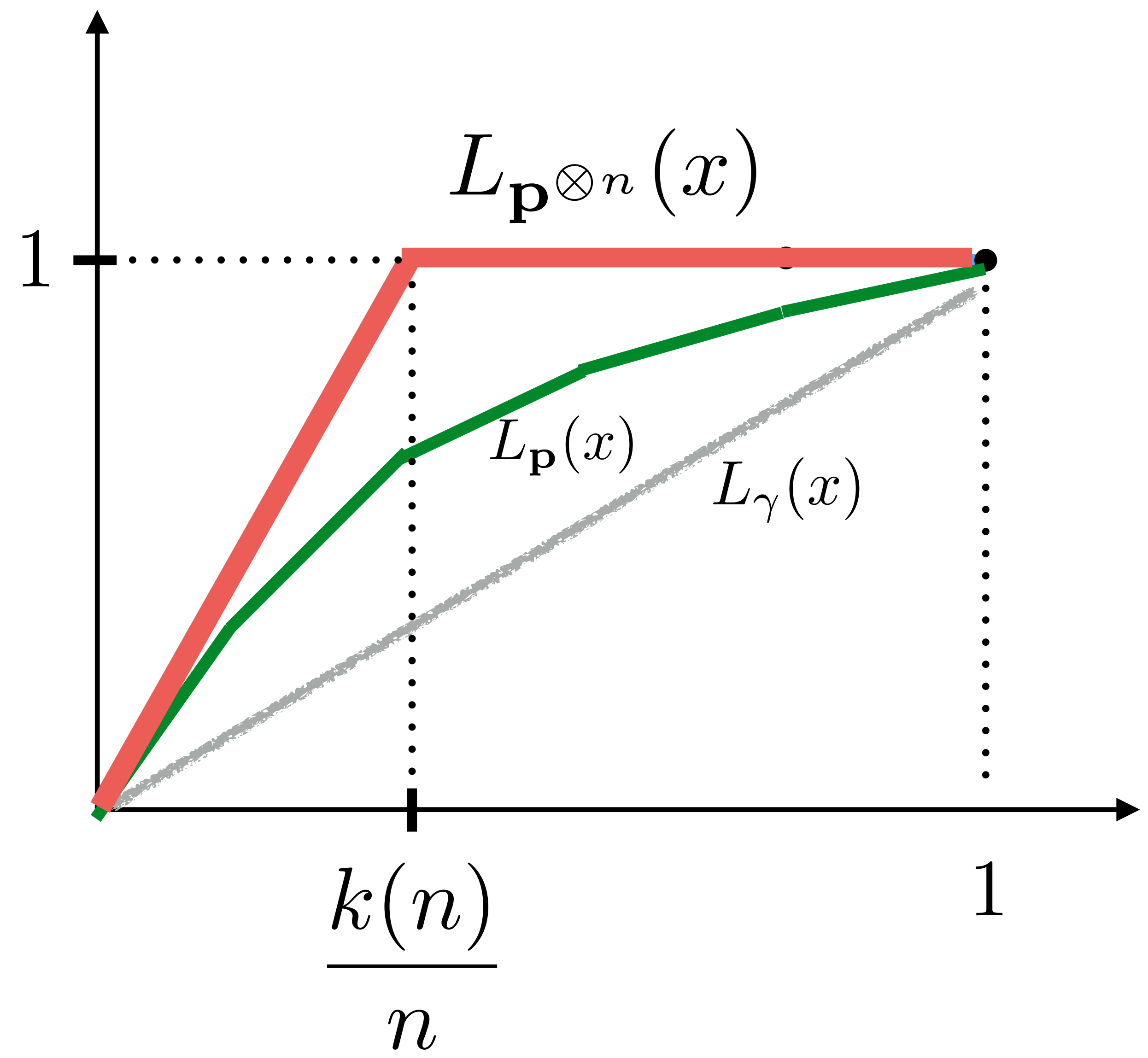
$$\mathbf{s}_k = \frac{1}{k}(1, 1, \ldots, 1, 0, 0, 0 \ldots, 0)$$



Macroscopic limit

$\rho^{\otimes n} \quad n \gg 1$

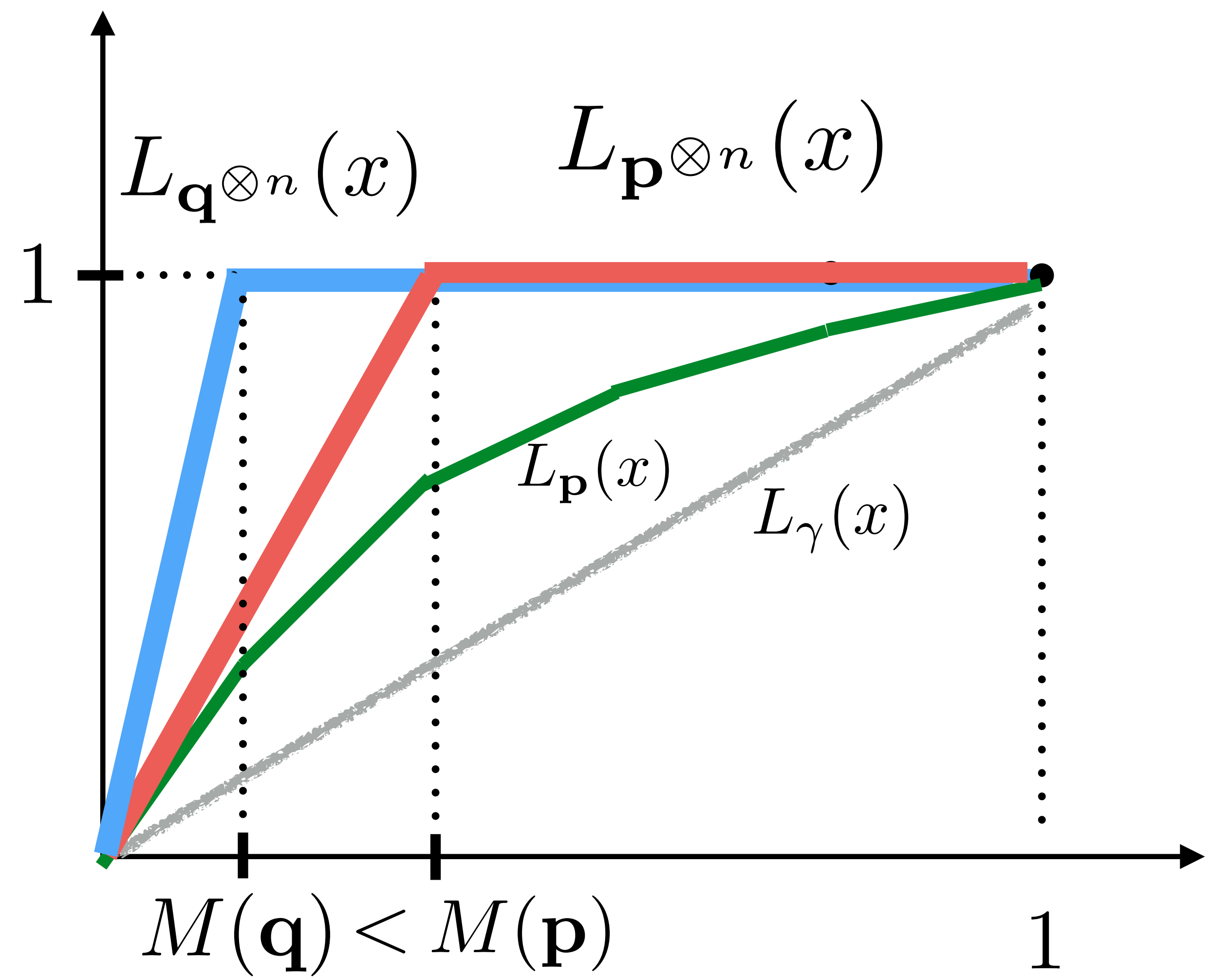
In macroscopic limit all Lorenz curves become **simple elbow functions**!



Macroscopic limit

$\rho^{\otimes n} \quad n \gg 1$

In macroscopic limit all states are **comparable**.



$\mathbf{q}^{\otimes n}$ "more ordered than" $\mathbf{p}^{\otimes n}$

Macroscopic limit— entropy emerges

$\rho = \mathrm{diag}(\mathrm{p}, 1-\mathrm{p})$

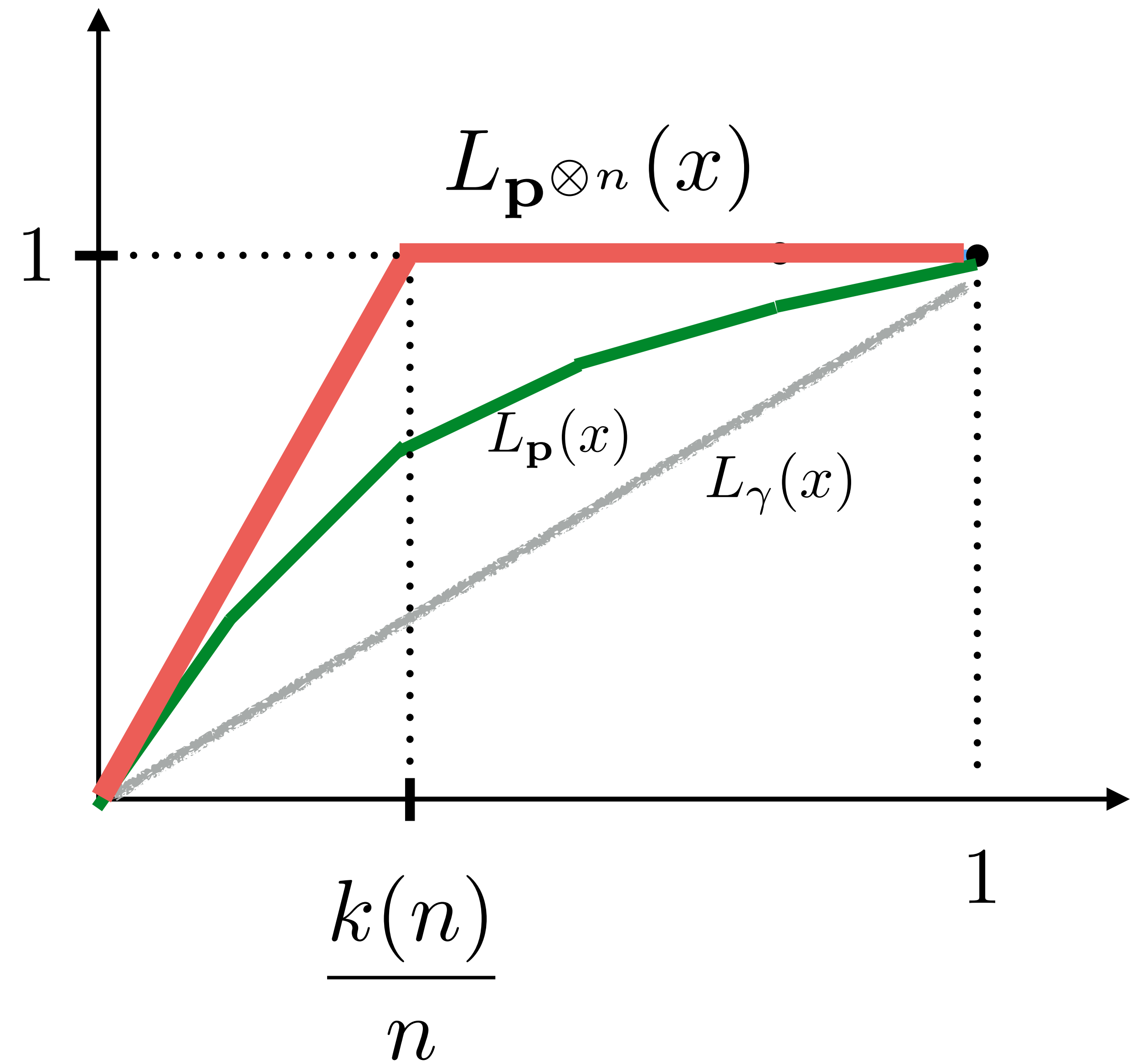
$\rho^{\otimes n} = (1-\epsilon)\frac{1}{k}\mathbf{I}_{typical} + \epsilon\sigma_{other}$

$n \gg 1$

$k(n) \sim \log \binom{n}{pn}$

$= \log[\frac{n!}{(pn)!((1-p)n)!}]$

$\sim H(p)n$



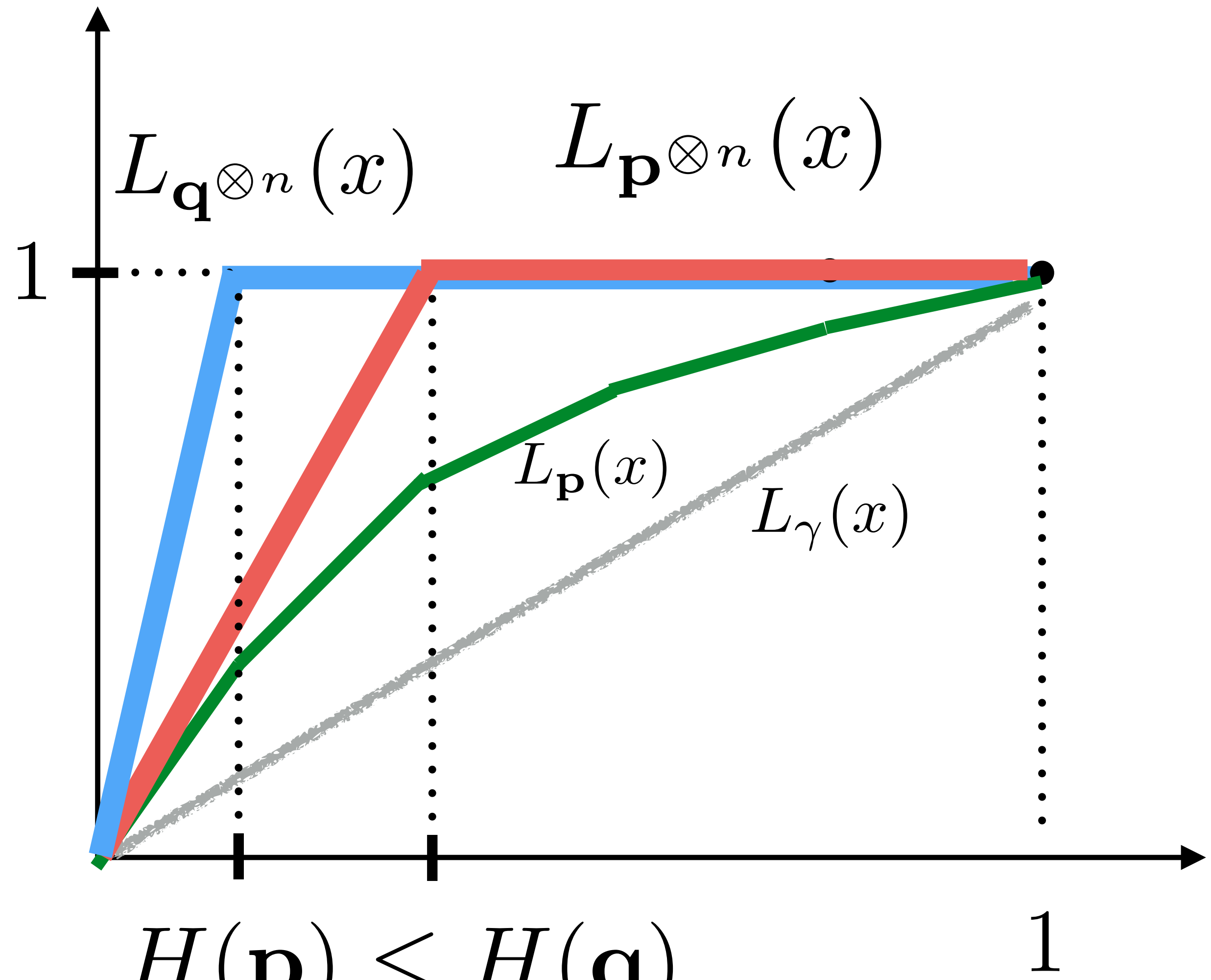
$H(p) = -p\log p - (1-p)\log(1-p) = \text{Shannon entropy!}$

Macroscopic limit

$\rho^{\otimes n} \quad n \gg 1$

In macroscopic limit
all states are
comparable.

single measure for ordering



$H(\mathbf{p}) \leq H(\mathbf{q})$

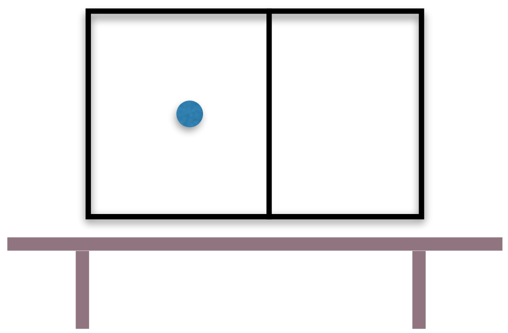
$\mathbf{q}^{\otimes n}$ "more ordered than" $\mathbf{p}^{\otimes n}$

Overview

1. **Motivations and criteria.**
2. **Resource theories and properties.**
3. **Resource formulation of thermodynamics.**
4. **Majorization — order & disorder.**
5. Information & energy.
6. Gibbs-rescaling & work bits.

Information & Energy

Szilard engine

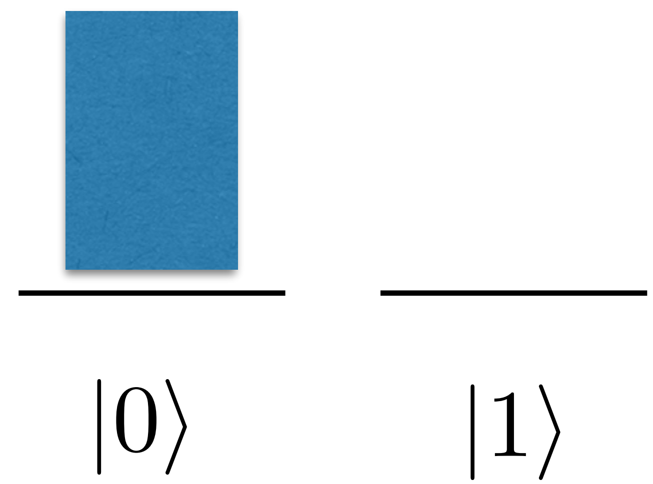


Single unit of information

$$E_1 = E_2$$

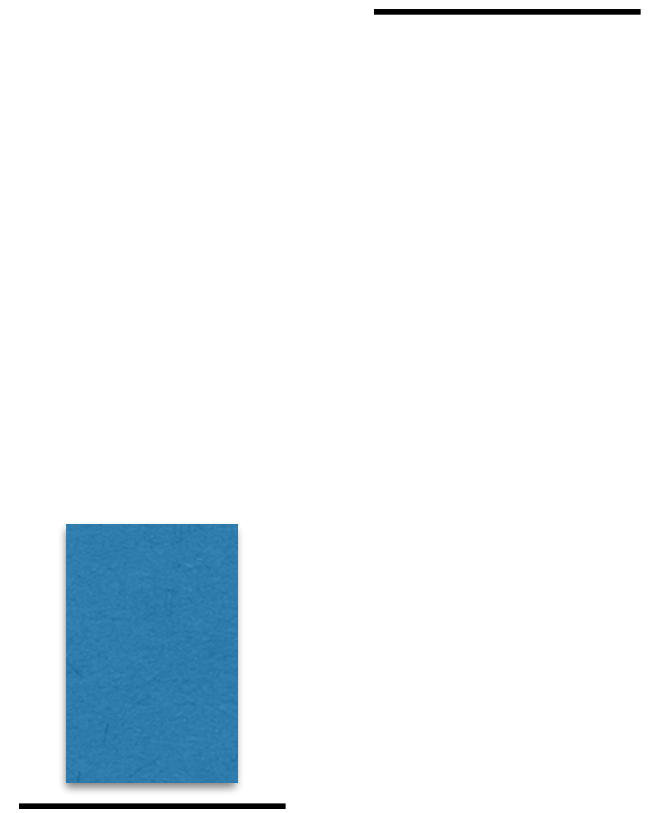
One bit of information can give $W = kT \ln 2$

[1] L. Szilard, Z. Phys. 53, 840 (1929).



One bit of information can give $W = kT\ln 2$

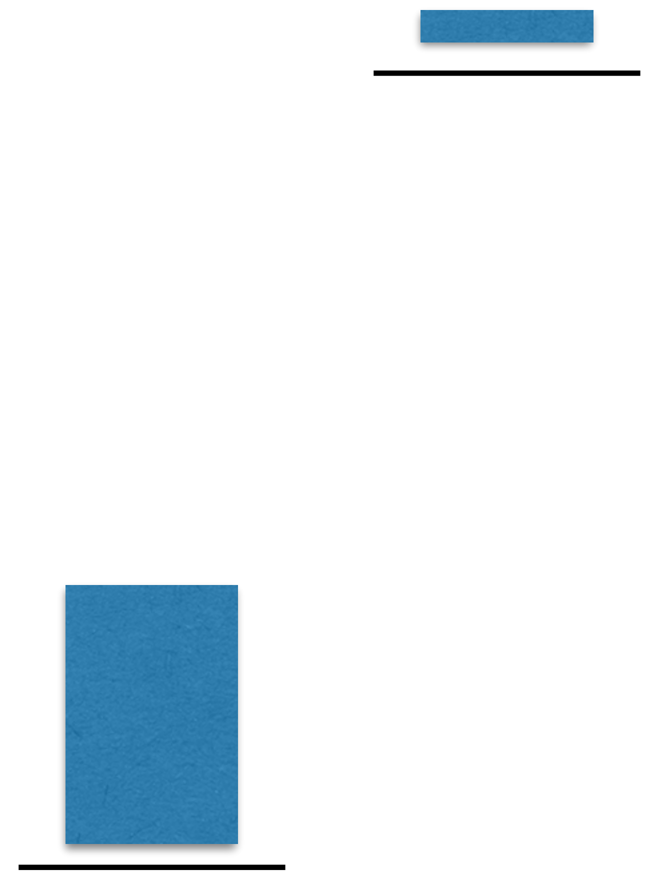
Raise empty level



$$\delta W = 0$$

$$p = \frac{e^{-\beta E}}{Z}$$

Thermalise



$1 - p$

$$\delta W = 0$$

Lower by dE

$$p = \frac{e^{-\beta E}}{Z}$$

dE

$1 - p$

$$\delta W = p dE$$

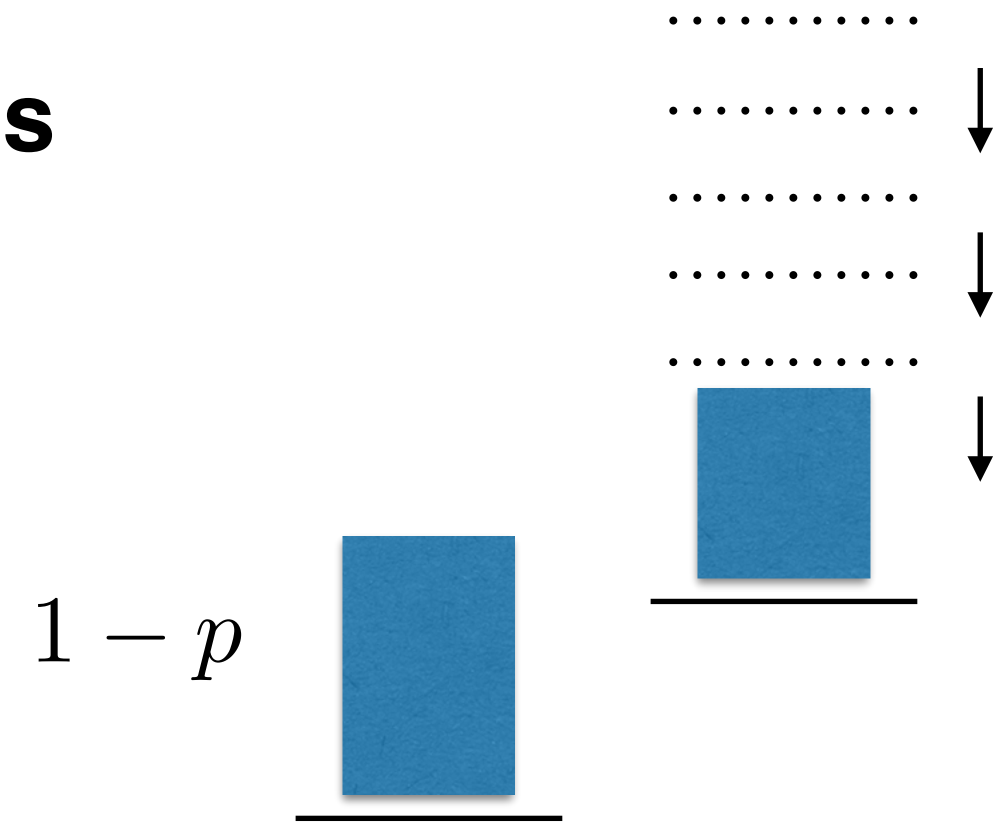
Thermalise

$$p = \frac{1}{Z} e^{-\beta(E-dE)}$$

$$1-p$$

$$\delta W = 0$$

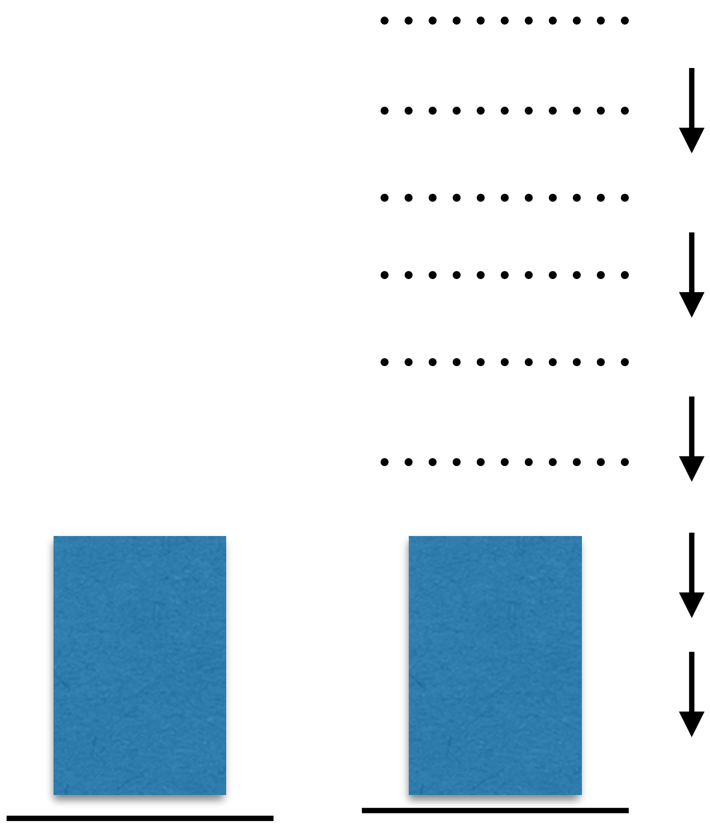
Repeat steps



$$\delta W = \sum_i p^{(i)} dE^{(i)}$$

Repeat steps

$$p = \frac{1}{2}$$



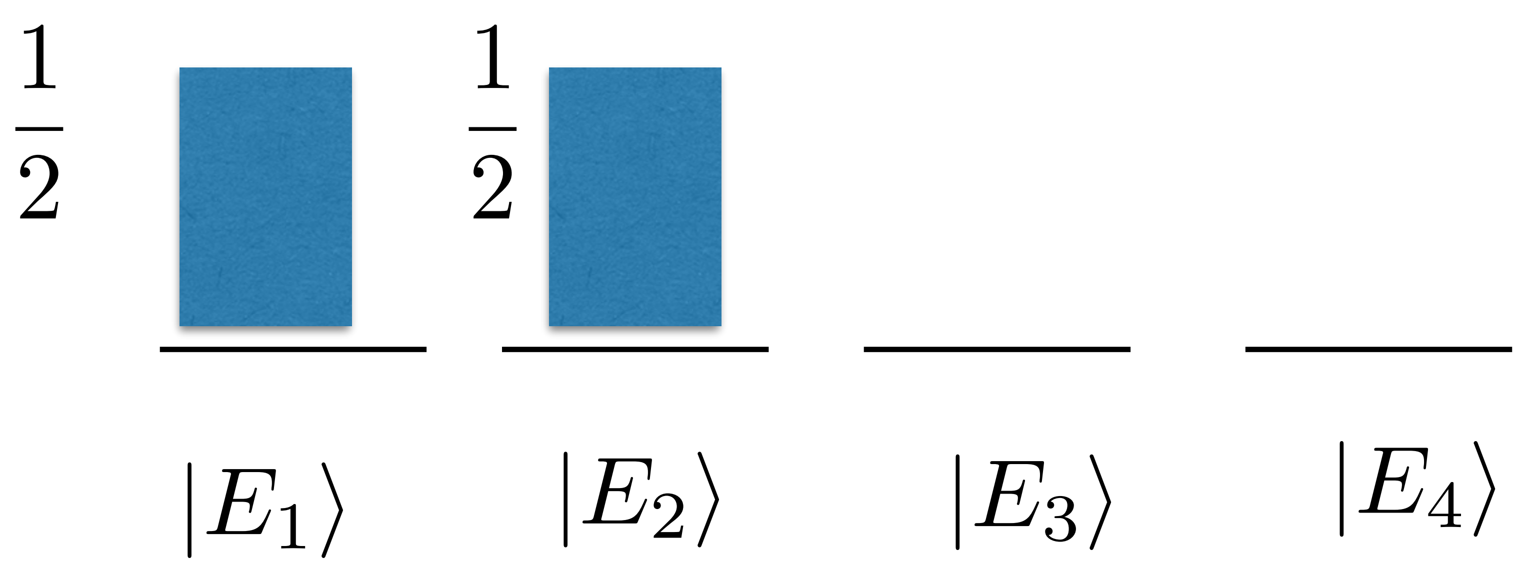
$$W_{tot} = kT \ln 2$$

$$|0\rangle\langle 0| \longrightarrow \frac{1}{2}\mathbb{I}$$

One bit of information can give $W = kT \ln 2$

Data-compression and work

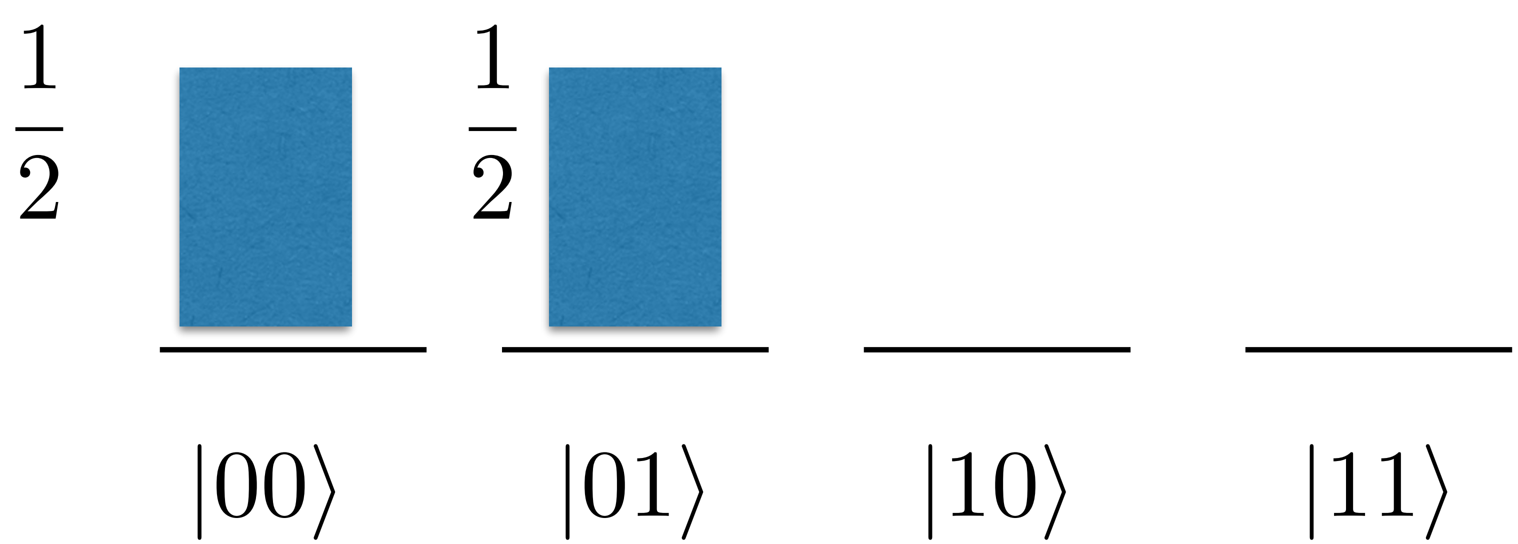
n 'pure' bits: $|0\rangle\langle 0|^{\otimes n}$



One ordered bit in state

Data-compression and work

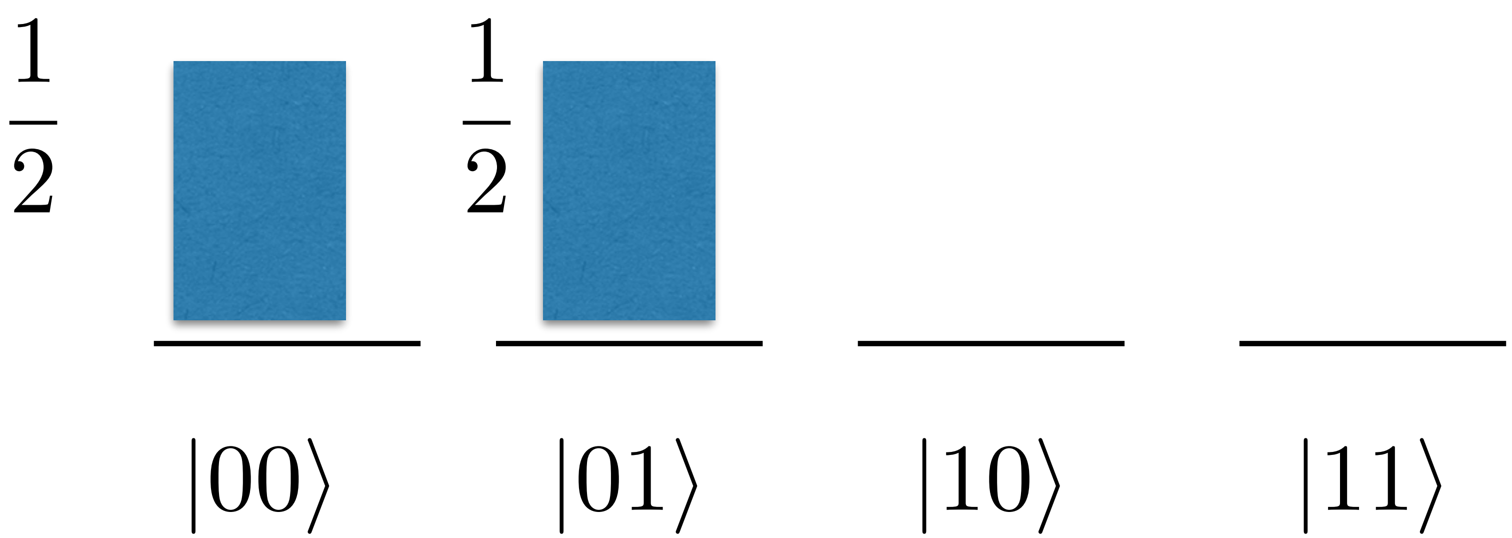
n ‘pure’ bits: $|0\rangle\langle 0|^{\otimes n}$



One ordered bit in state

Data-compression and work

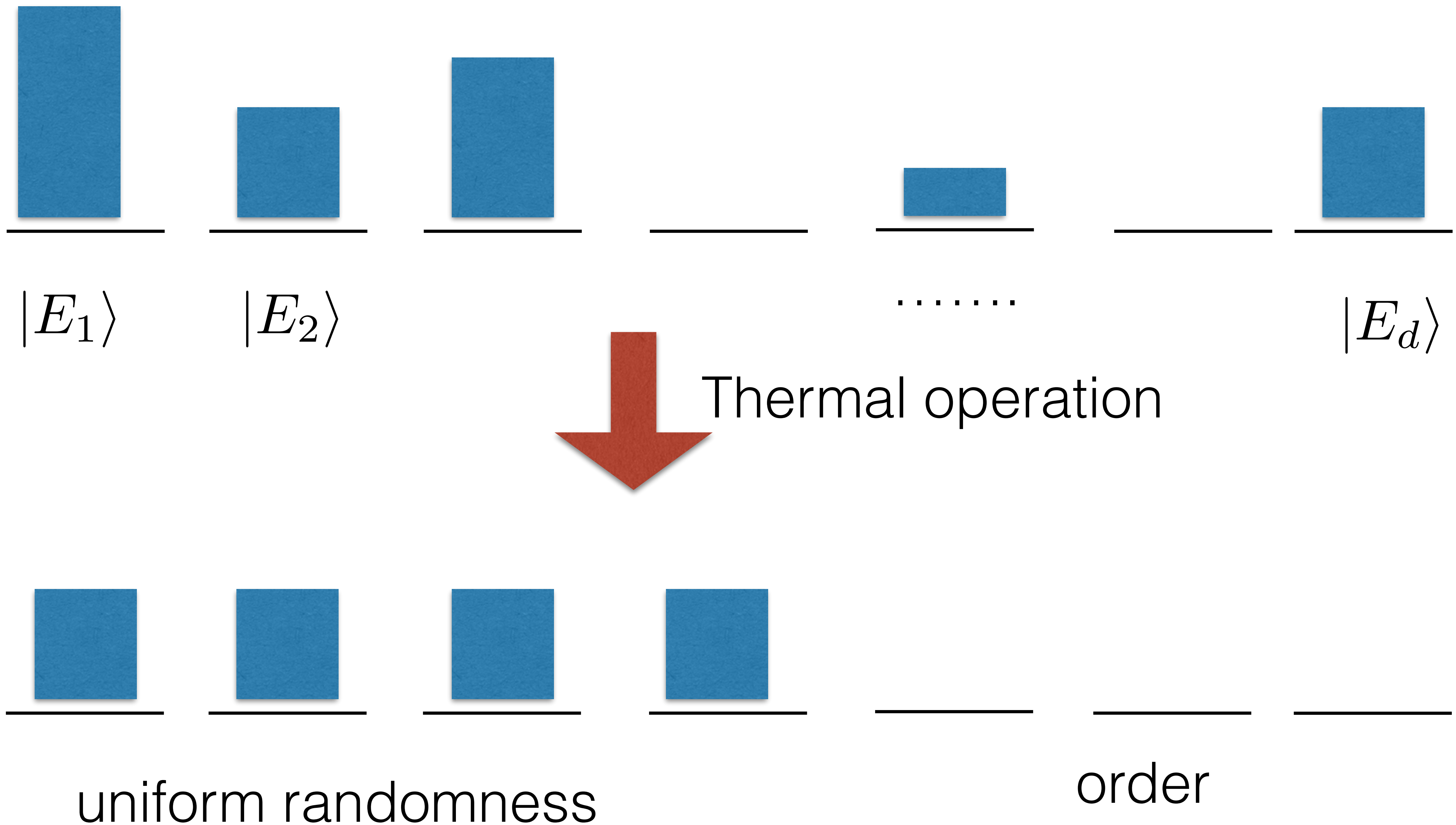
n 'pure' bits: $|0\rangle\langle 0|^{\otimes n}$



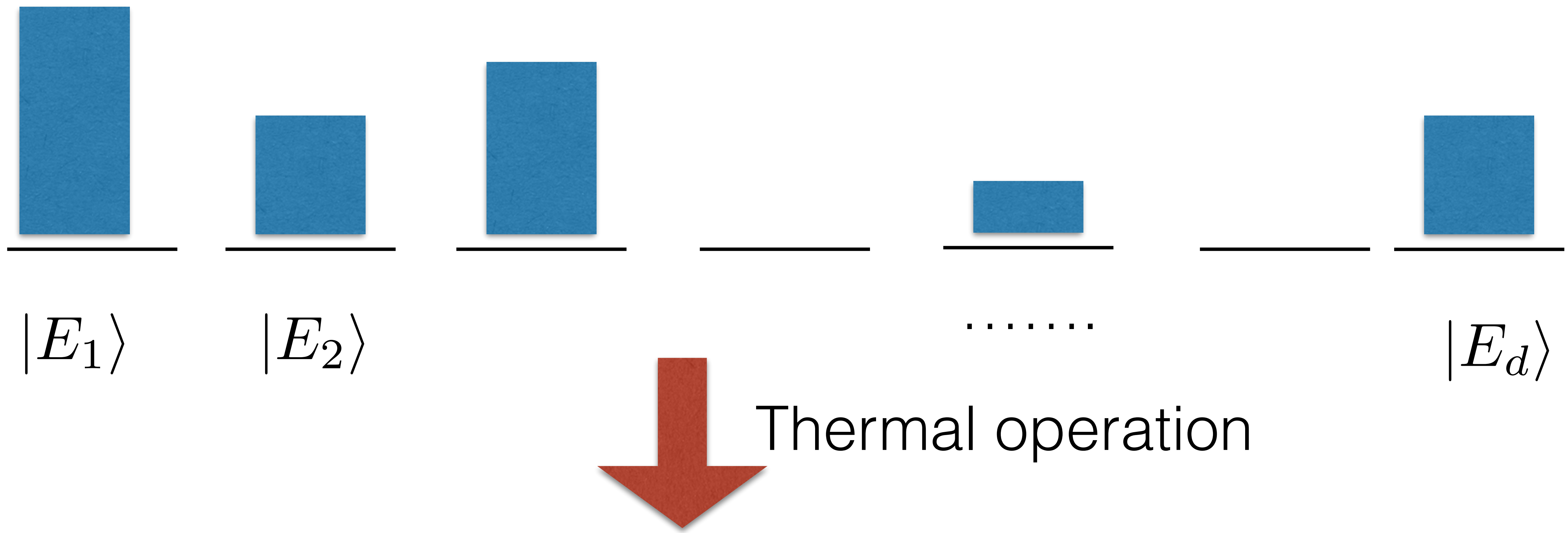
One ordered bit in state $$|0\rangle\langle 0| \otimes \frac{1}{2}\mathbb{I}$$

data "compressed"

Data-compression and work



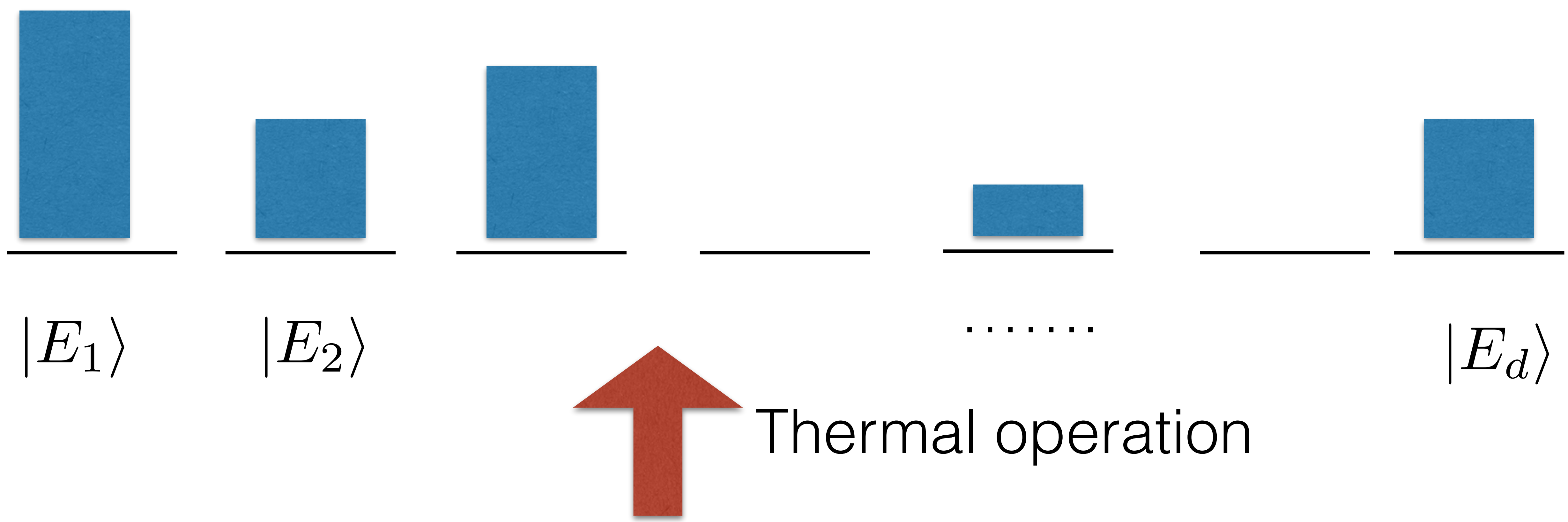
Data-compression and work



$$|0\rangle\langle 0|^{\otimes n} \otimes \frac{1}{2^k}\mathbb{I}^{\otimes k}$$

Maximize n

Work of formation



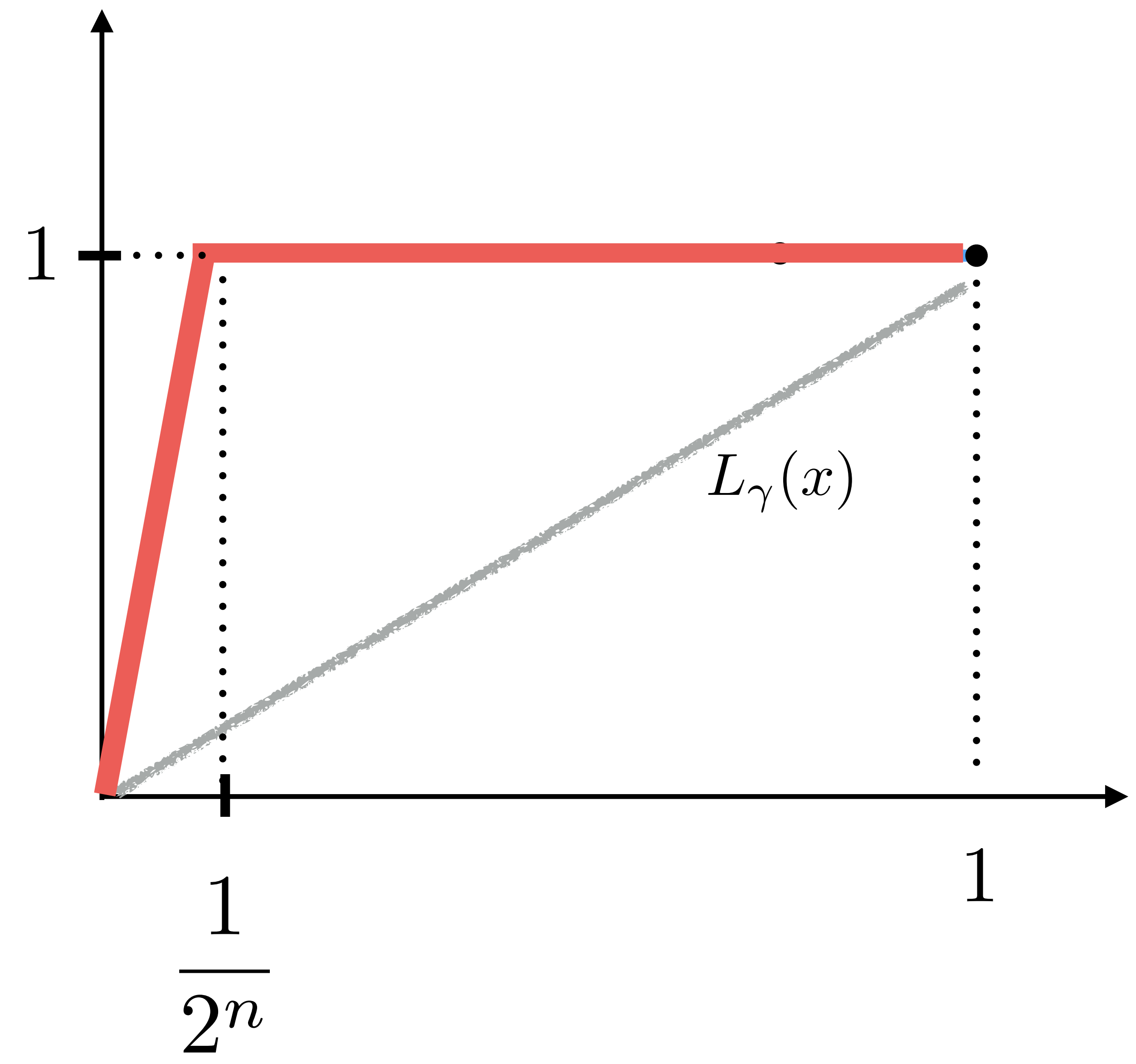
$$|0\rangle\langle 0|^{\otimes n} \otimes \frac{1}{2^k}\mathbb{I}^{\otimes k}$$

Minimize n

k-sharp states: order/ disorder rulers

$$|0\rangle\langle 0|^{\otimes n} \otimes \frac{1}{2^k}\mathbb{I}^{\otimes k}$$

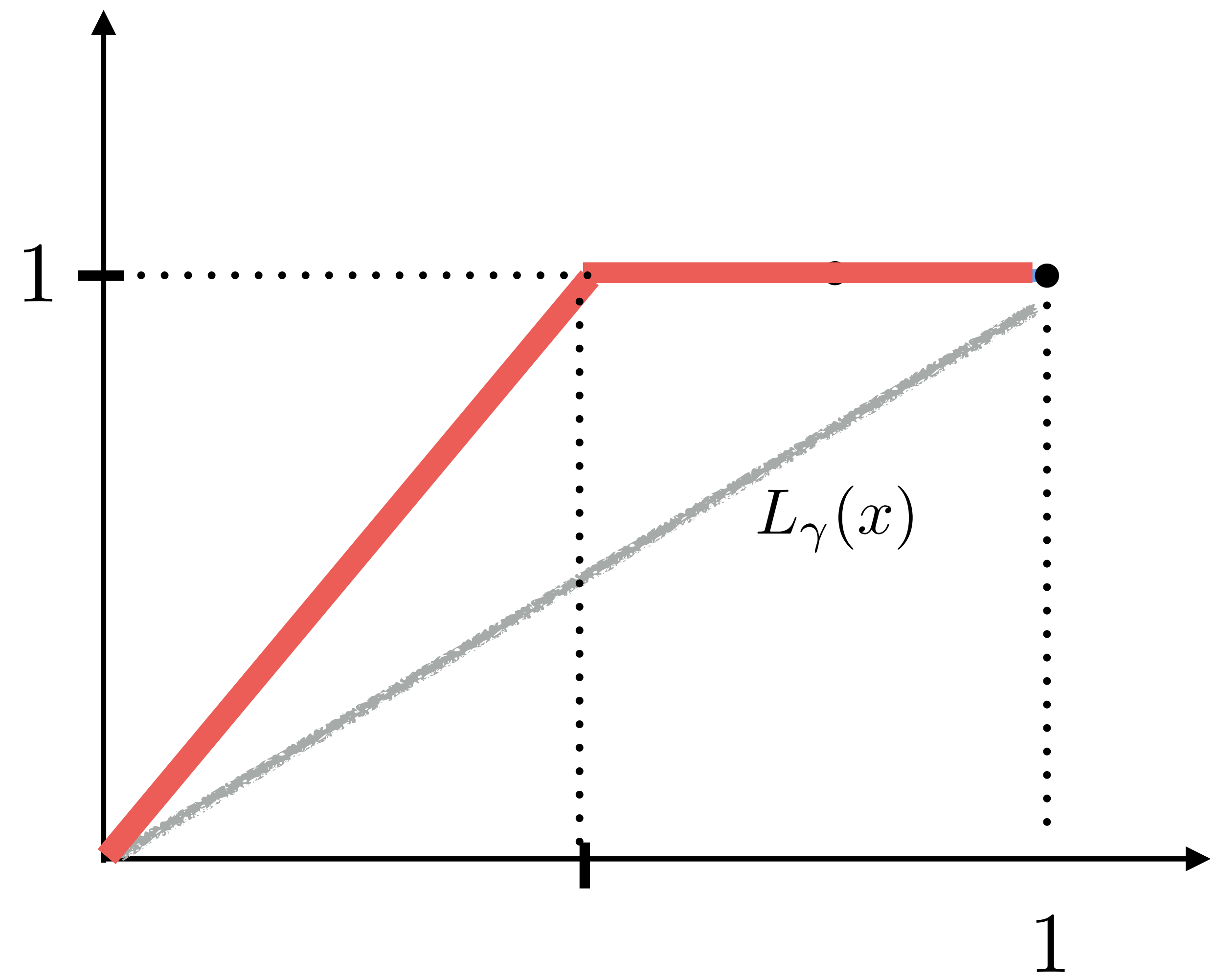
n very large



k-sharp states: order/ disorder rulers

$$|0\rangle\langle 0|^{\otimes n} \otimes \frac{1}{2^k}\mathbb{I}^{\otimes k}$$

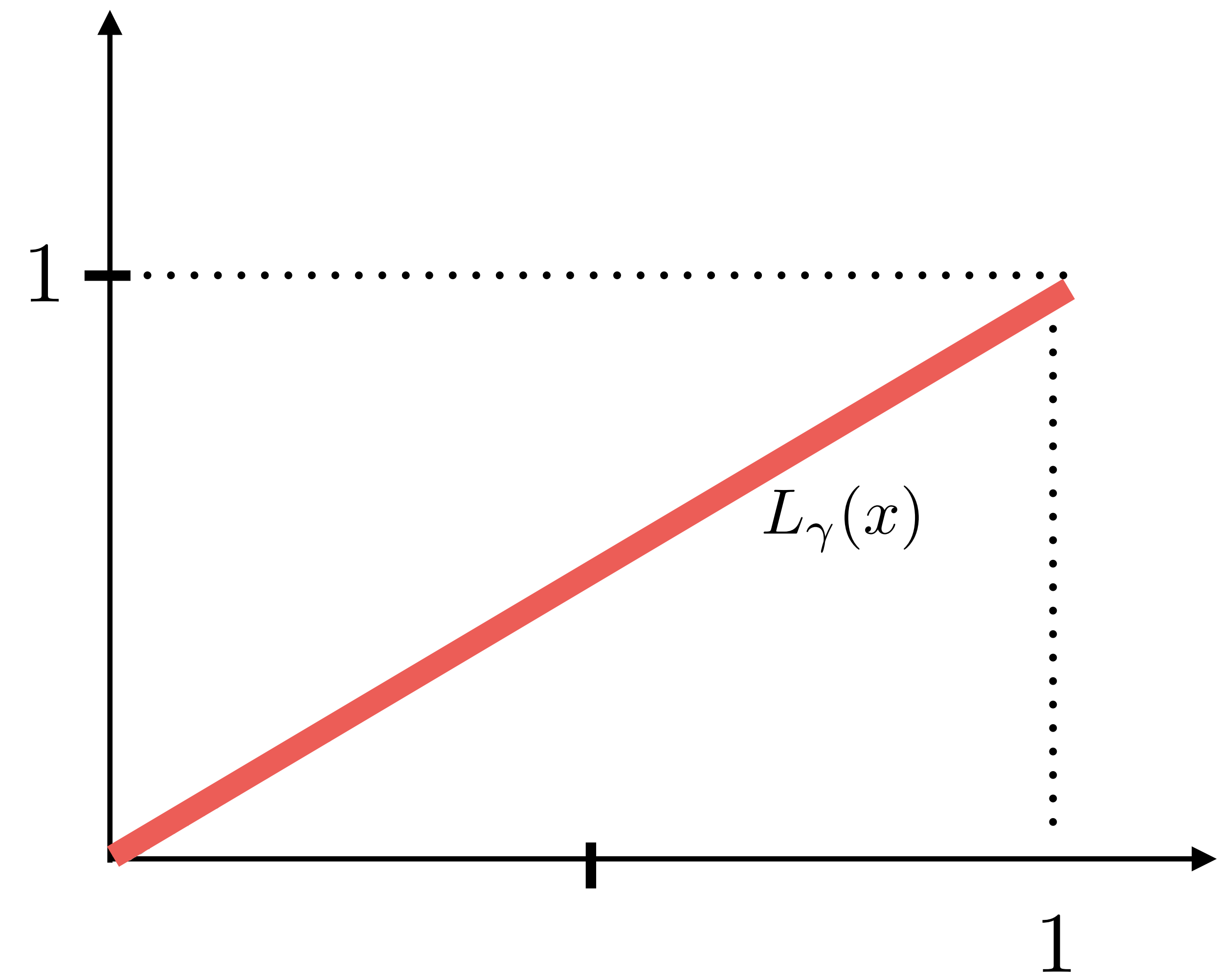
n =1



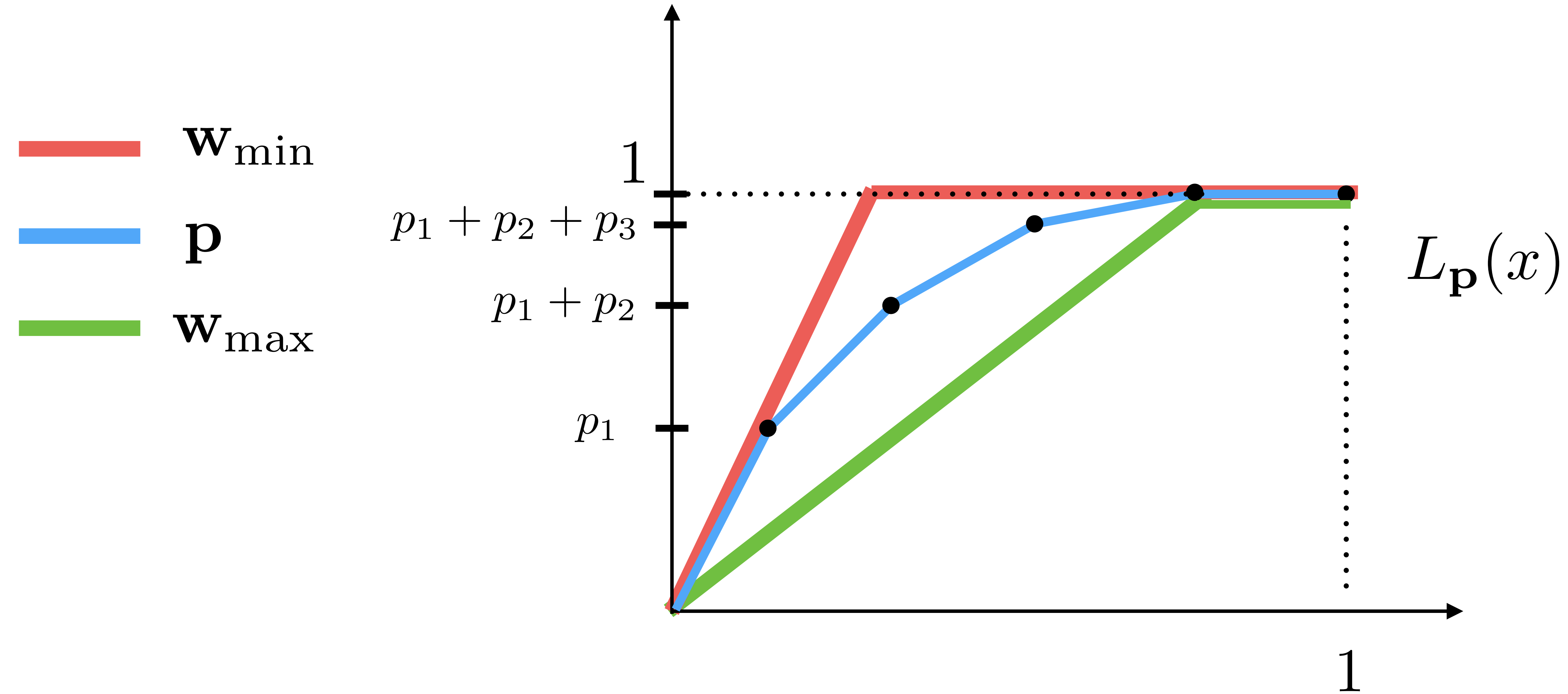
k-sharp states: order/ disorder rulers

$$|0\rangle\langle 0|^{\otimes n} \otimes \frac{1}{2^k}\mathbb{I}^{\otimes k}$$

n =0



"Work" of formation & distillation — nanoscale

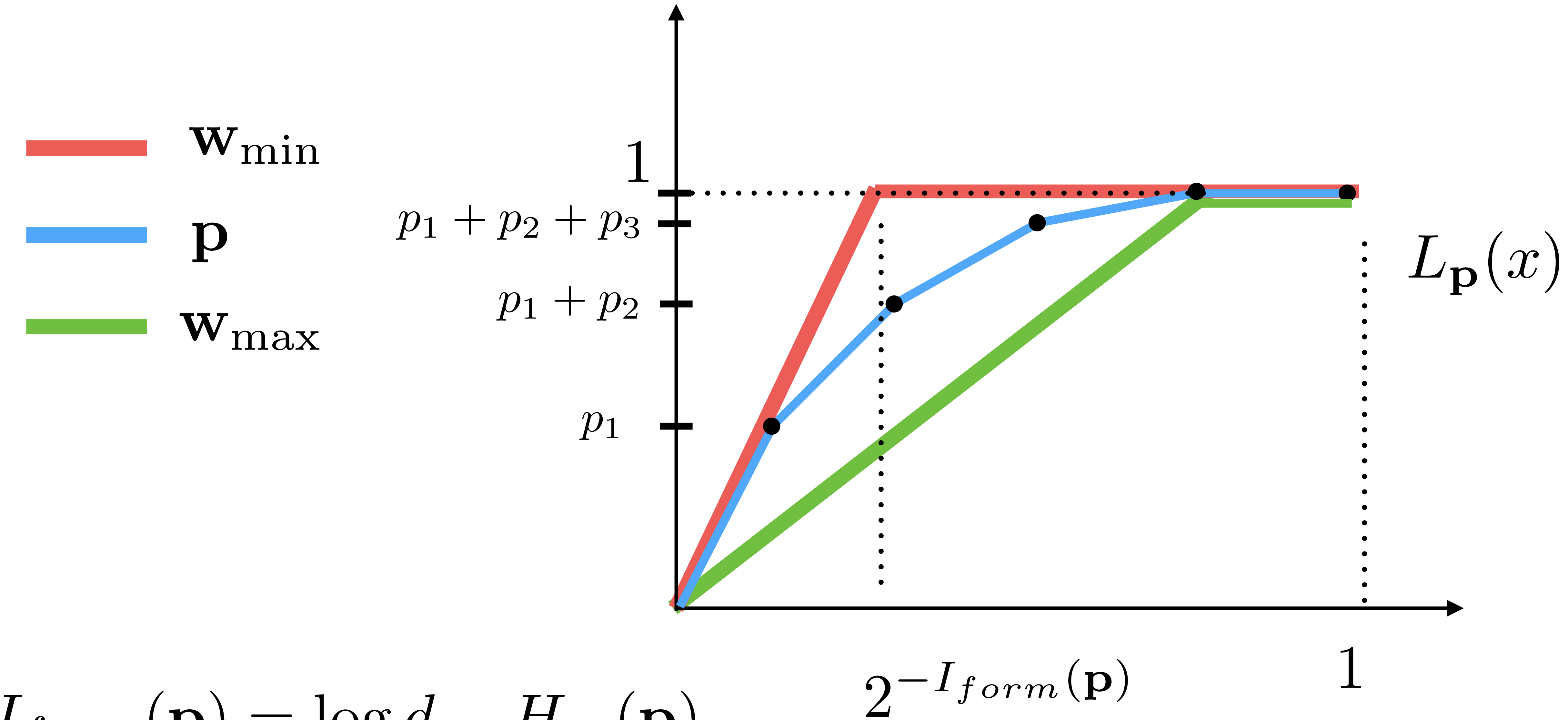


$\mathbf{w}_{\min}$ Minimal order needed to get **to**

$\mathbf{w}_{\max}$ Maximal order obtainable **from**

$\Big\} \; \rho = \mathrm{diag}(\mathrm{p}_1, \mathrm{p}_2, \ldots \mathrm{p}_\mathrm{d})$

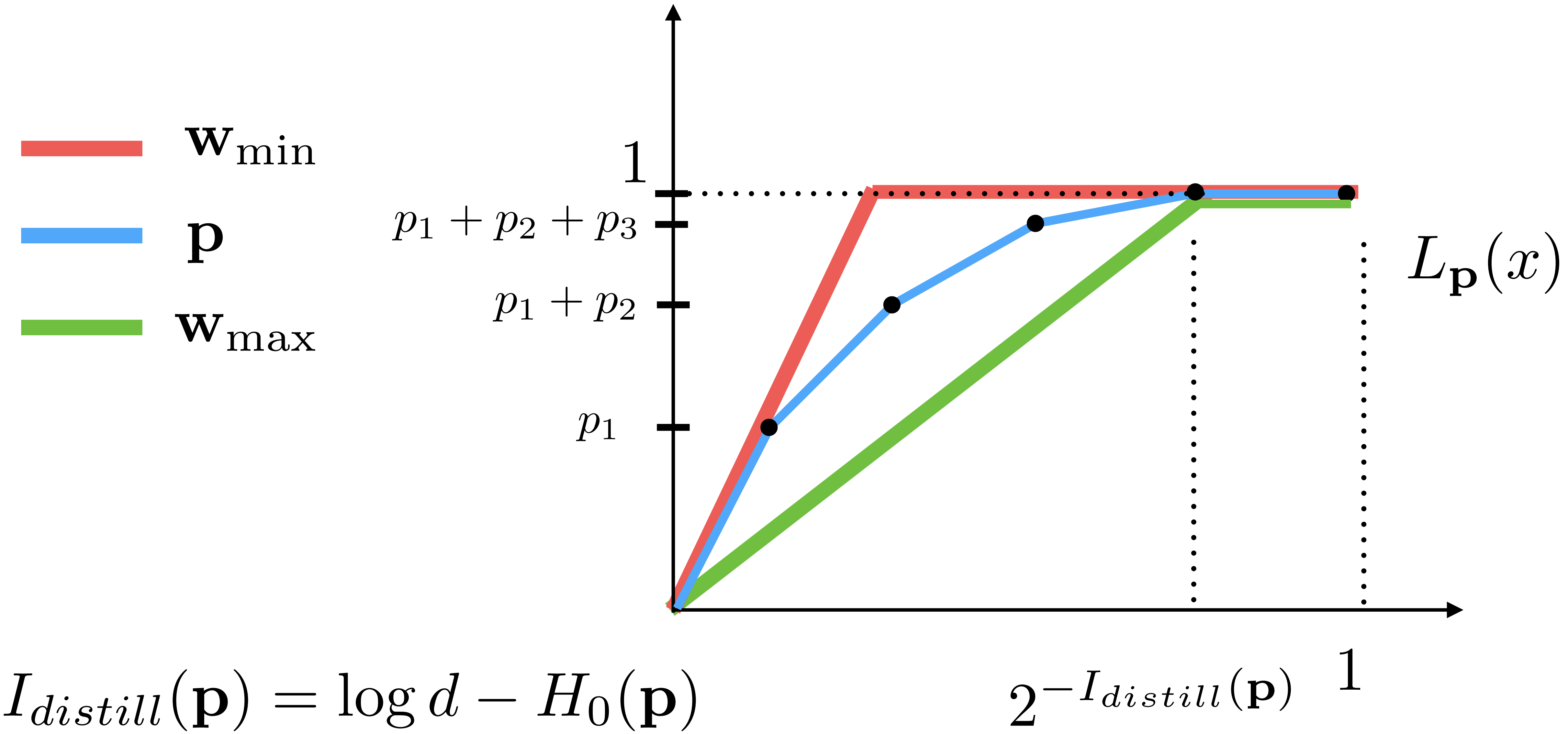
"Work" of formation & distillation — nanoscale



$$I_{form}(\mathbf{p}) = \log d - H_\infty(\mathbf{p})$$

$$H_\infty(\mathbf{p}) = -\log \max(\mathrm{p_k})$$

“Work” of formation & distillation — nanoscale

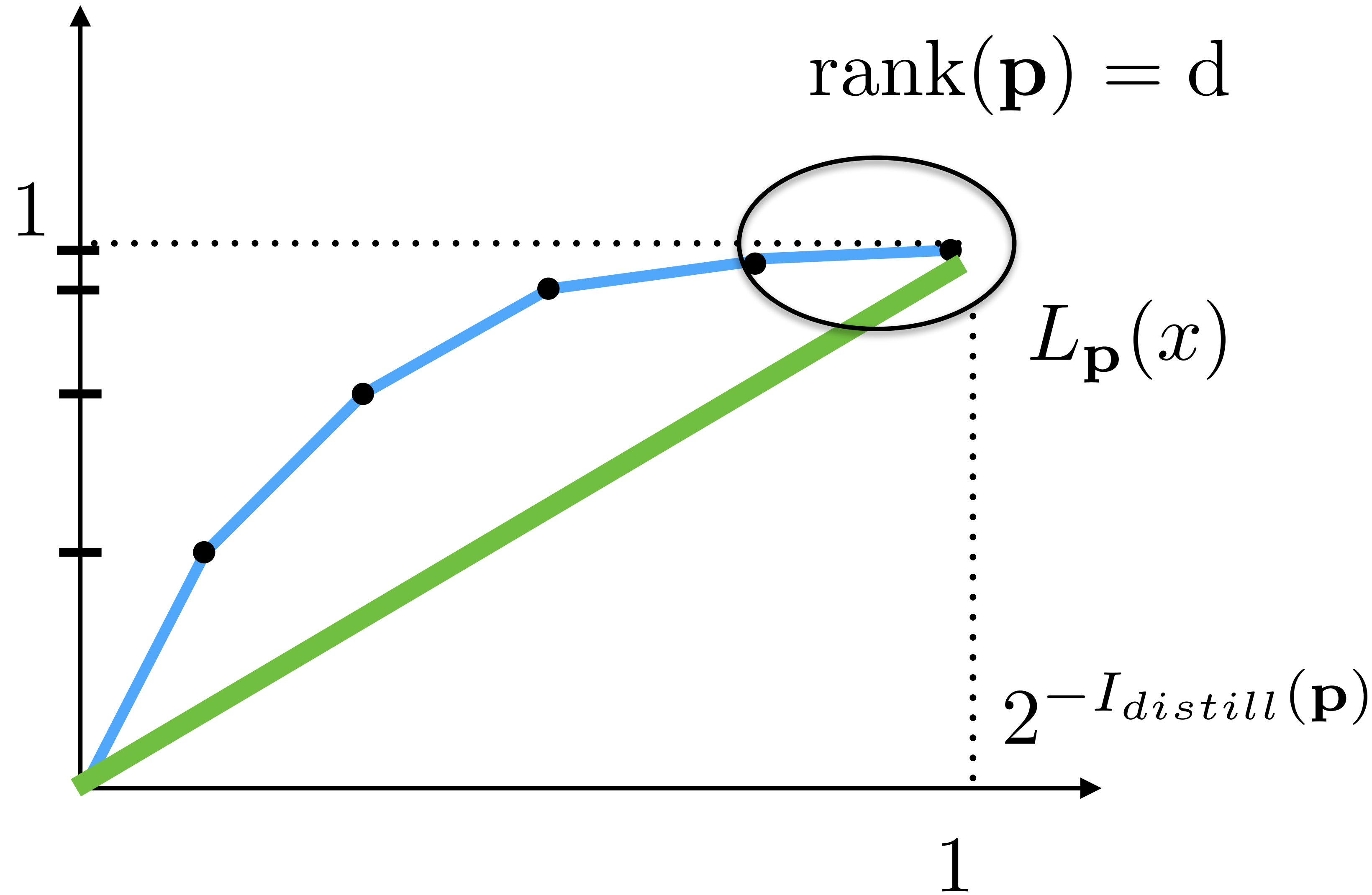


$$I_{distill}(\mathbf{p}) = \log d - H_0(\mathbf{p})$$

$$H_0(\mathbf{p}) = \log \mathrm{rank}(\mathbf{p})$$

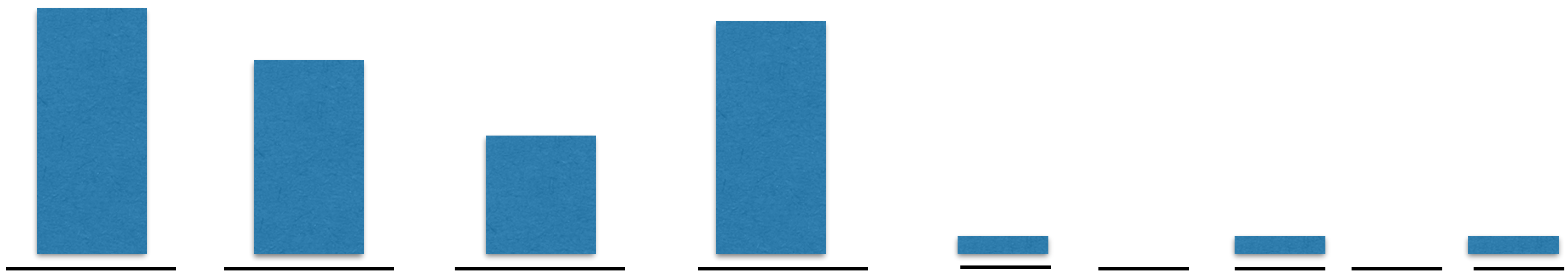
Smoothing

L(x) only reaches 1 at x=1



$$I_{distill}(\mathbf{p}) = \log d - H_0(\mathbf{p})$$
$$= 0$$

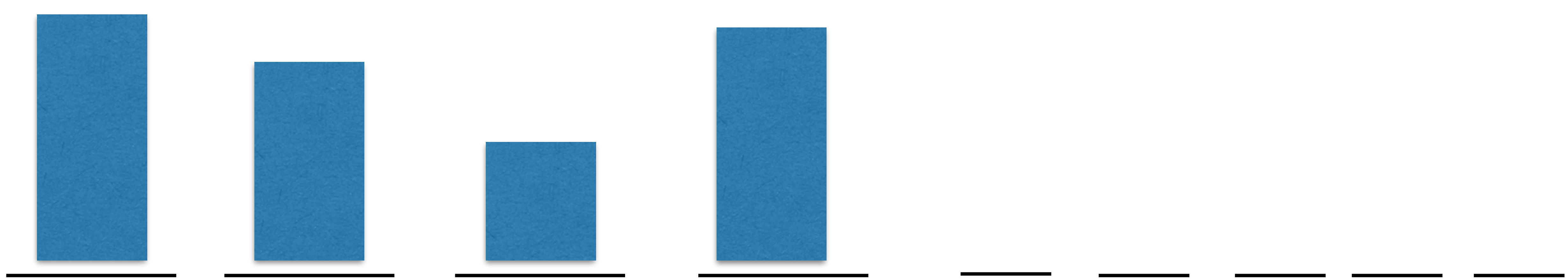
But state is still "ordered"!!



robust randomness

imperfect order

Practically the same as:



robust randomness

perfect order

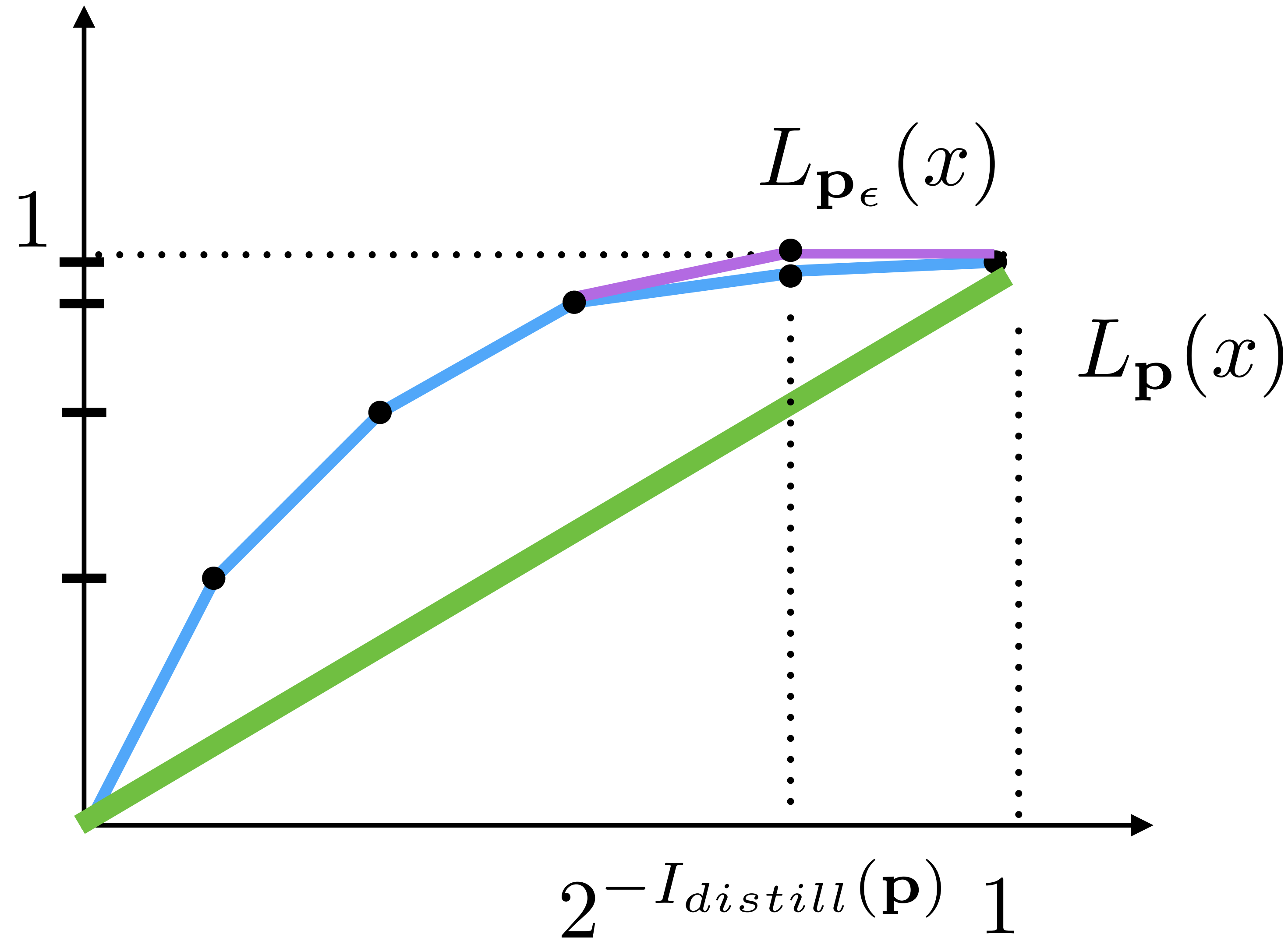
Smoothing

$\mathbf{p}_\epsilon$ has distance ϵ from $\mathbf{p}$

$\mathrm{rank}(\mathbf{p}_\epsilon) \neq \mathrm{d}$

so

$I_{distill}(\mathbf{p}_\epsilon) \neq 0$



with failure probability ϵ **we can distill this amount.**

“Work” of formation & distillation

Exercise: verify that:

$I_{distill} = I_{form}$ only coincide in macroscopic limit!

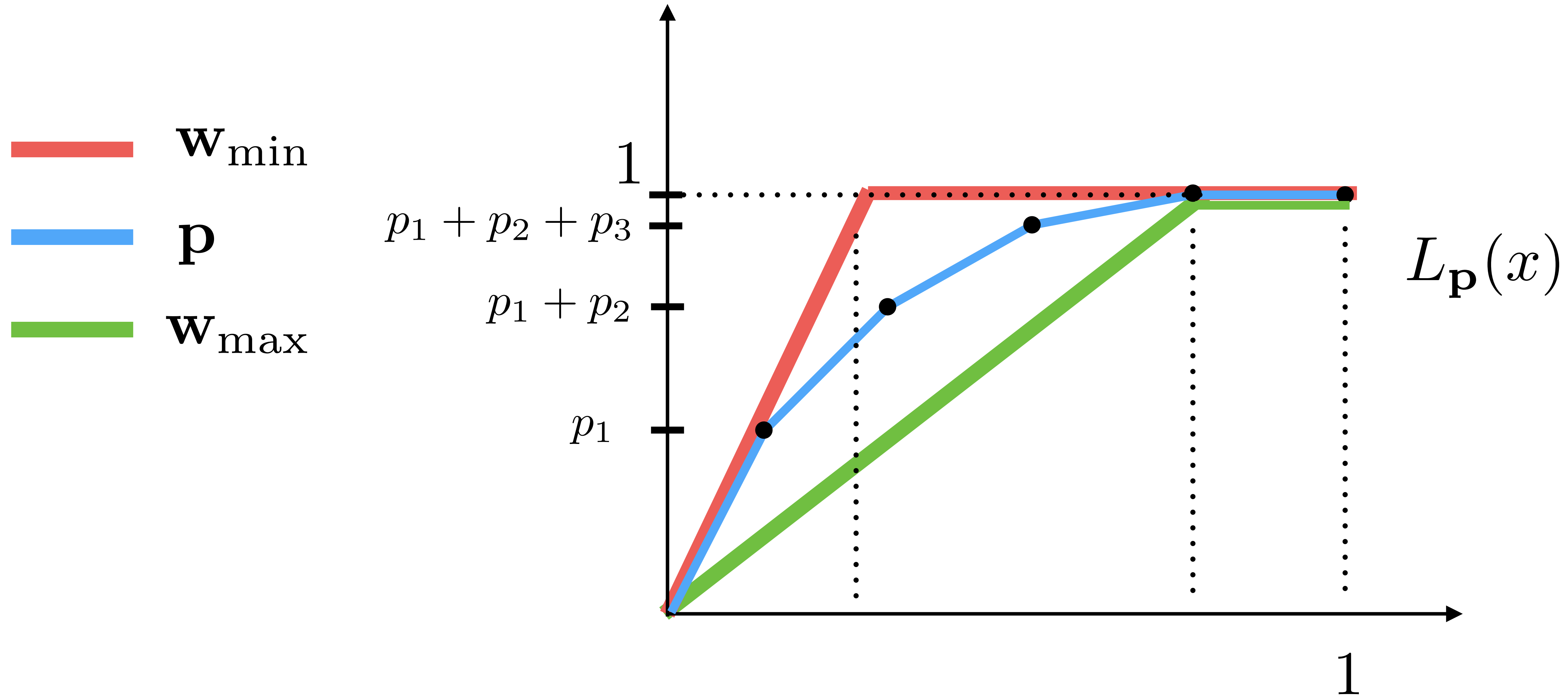
since everything becomes an elbow function

“Fundamental irreversibility at nanoscale”

$$I_{distill} \neq I_{form}$$

[1] M. Horodecki, Oppenheim, Nature Comm 4 (2013)

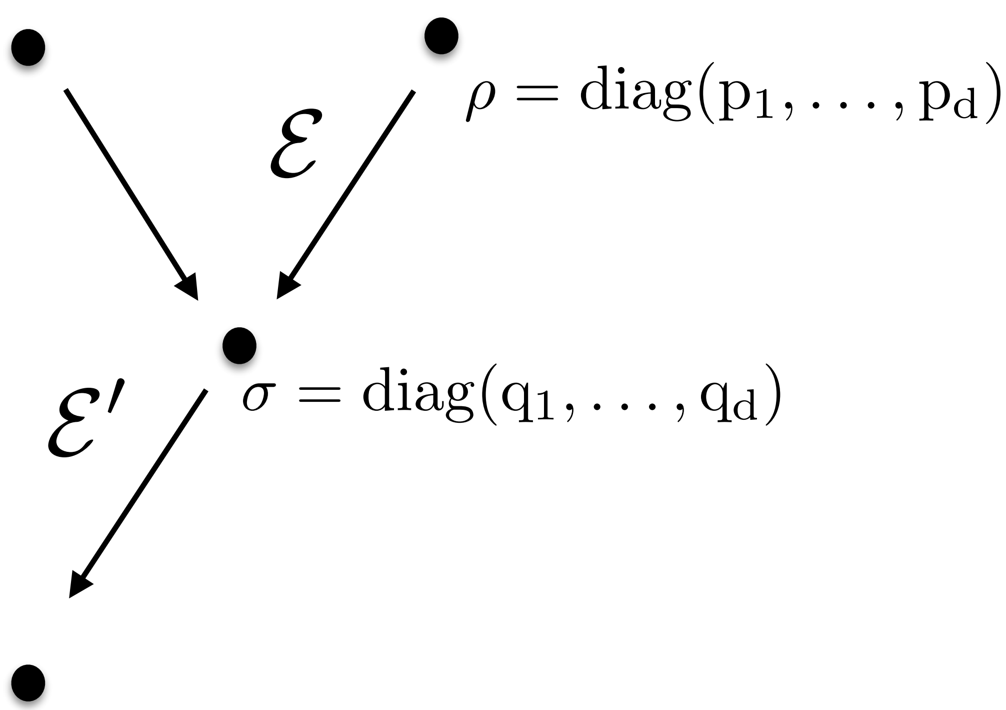
"Work" of formation & distillation — nanoscale



$$I_{form}(\mathbf{p}) = \log d - H_\infty(\mathbf{p})$$
$$I_{distill}(\mathbf{p}) = \log d - H_0(\mathbf{p})$$

Are ***entropic*** measures of order

A complete set of entropies



$$H_0(\mathbf{p}) = \log \mathrm{rank}(\mathbf{p})$$

$$H_\infty(\mathbf{p}) = -\log \max(\mathrm{p_k})$$

$$H_1(\mathbf{p}) = -\sum_k p_k \log p_k$$

$$H_\alpha(\mathbf{p}) := \frac{1}{1-\alpha} \sum_k p_k^\alpha$$

Give a **complete set** for classical, catalytic thermal operations

$$\frac{1}{n} H_\alpha(\mathbf{p}^{\otimes n}) \to H(\mathbf{p})$$

[1] Brandao et al, PNAS (2015)

Macroscopic thermodynamics

$$n \gg 1 \quad \frac{1}{n} H_\alpha(\mathbf{p}^{\otimes n}) \to H(\mathbf{p}) =: S(\rho)$$

for all α

Note it can be proved:

$$\rho \to \sigma = \mathcal{E}(\rho) \qquad S(\rho) = S(\sigma)$$

then can realised as

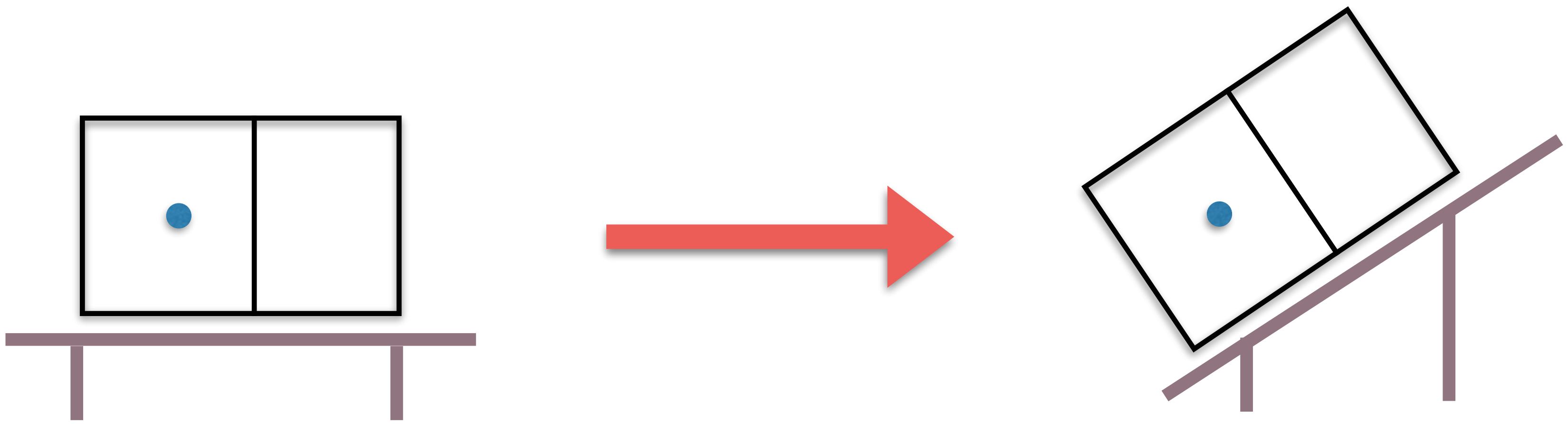
$$\mathcal{E}(\rho) \approx \mathrm{tr_C U}(\rho^{\otimes \mathrm{n}} \otimes \gamma)\mathrm{U}^\dagger$$

von Neumann entropy is “always macroscopic”.

Overview

1. **Motivations and criteria.**
2. **Resource theories and properties.**
3. **Resource formulation of thermodynamics.**
4. **Majorization — order & disorder.**
5. **Information & energy.**
6. Gibbs-rescaling & work bits.

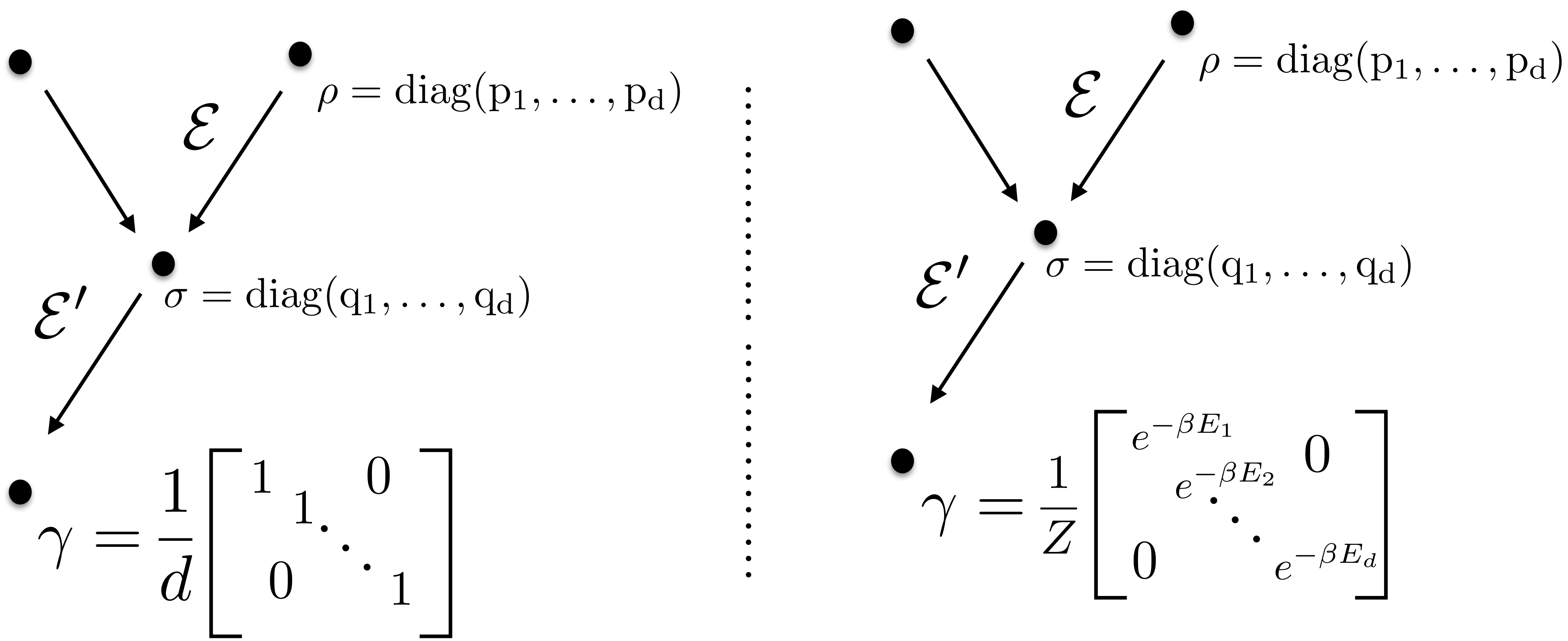
Non-trivial Hamiltonians



$E_1 = E_2$ $E_1 \neq E_2$

Gibbs-rescaling: switching on Hamiltonians

$$H = 0 \longrightarrow H \neq 0$$



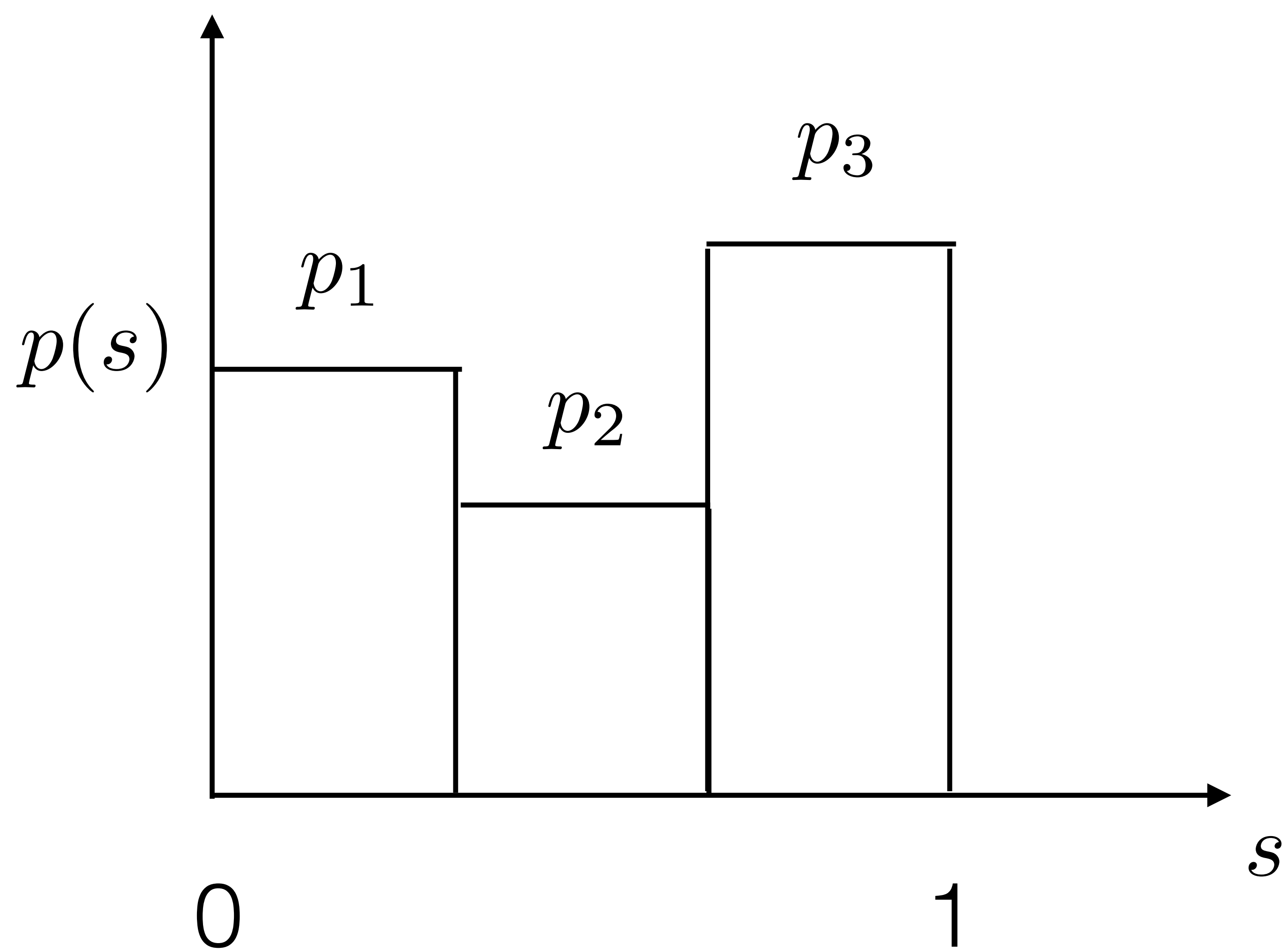
Thermo-majorization

$H \neq 0$

$\mathbf{p} \succ_T \mathbf{q}$??

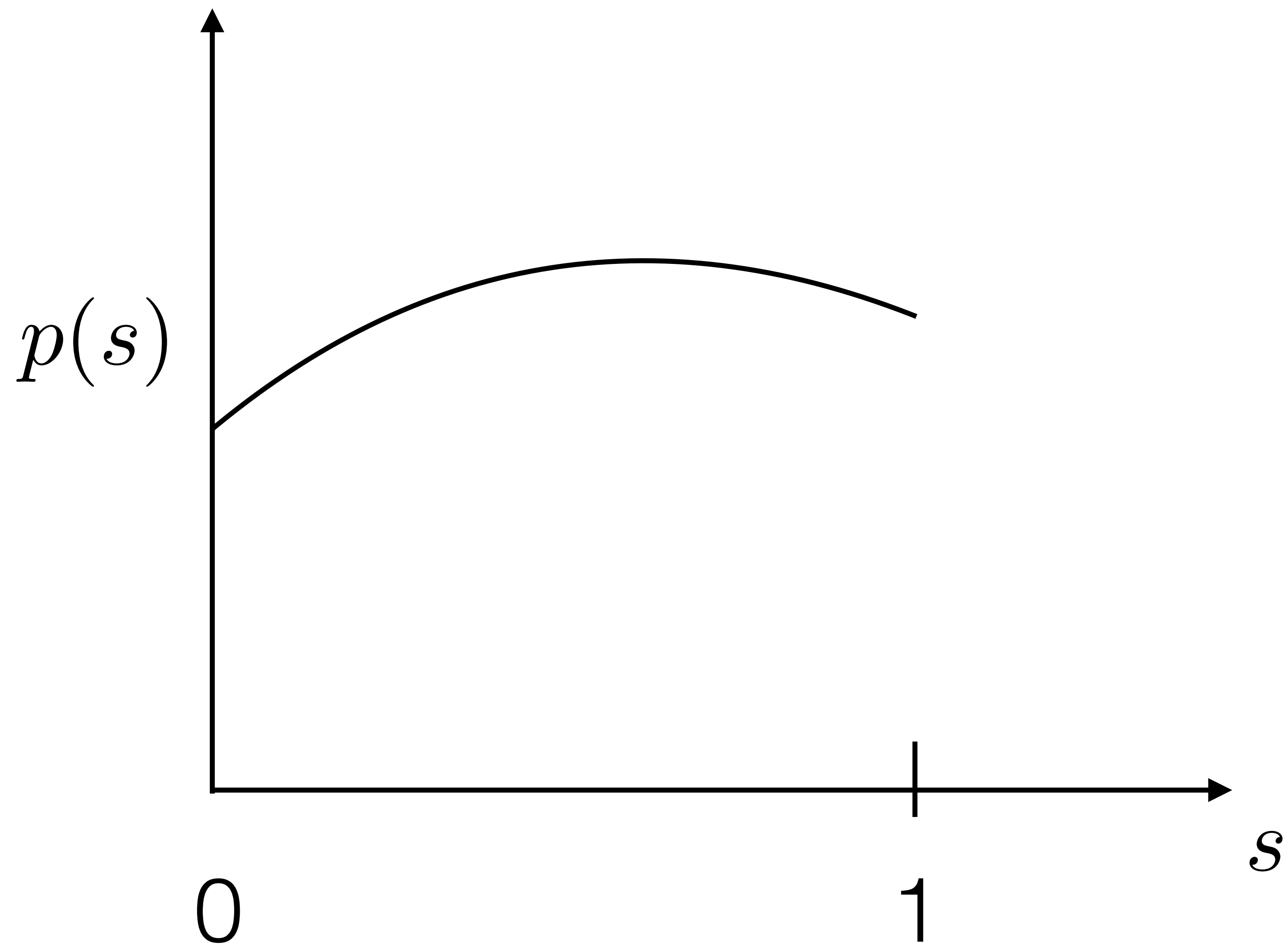
Describe as distribution function on [0,1]

$\mathbf{p} \to p(s)$



$\mathbf{p} \succ_T \mathbf{q} \longrightarrow p(s) \succ_T q(s)$

Thermo-majorization



Thermo-majorization

$H \neq 0$

Define:

$p(s) \succ_T q(s)$

if and only if

$p_\gamma(s) \succ_m q_\gamma(s)$

$$\gamma(s) = \frac{1}{Z} e^{-\beta E(s)}$$

$$L_\gamma(s) = \int_0^s ds\ \gamma(s)$$

$$p_\gamma(s) := \frac{p(L_\gamma^{-1}(s))}{\gamma(L_\gamma^{-1}(s))}$$

"Gibbs re-scaling of p"

[1] E. Ruch, R. Schranner, and T. H. Seligman, J. Chem. Phys. 69 (1978)

Gibbs-preserving maps

$p(s) \succ_T q(s)$ if and only if

There exists a stochastic map S such that

$$q(s) = \int dx \; S(s,x) p(x)$$

$$\gamma(s) = \int dx \; S(s,x) \gamma(x)$$

Gibbs-preserving maps

$p(s) \succ_T q(s)$ if and only if $p_\gamma(s) \succ_m q_\gamma(s)$

if and only if $\int ds \; |p_\gamma(s) - t| \geq \int ds \; |q_\gamma(s) - t|$ for all $t \geq 0$

if and only if there is bistochastic B

$$q_\gamma(s) = \int dx \; B(s,x) p_\gamma(x)$$

$$\mathbf{I}(s) = \int dx \; B(s,x) \mathbf{I}(x)$$

uniform distribution is invariant

Gibbs-preserving maps

$p(s) \succ_T q(s)$ if and only if

$$q_\gamma(s) = \int dx \; B(s,x) p_\gamma(x)$$

$$\mathbf{I}(s) = \int dx \; B(s,x) \mathbf{I}(x)$$

sub in:

$$p_\gamma(s) := \frac{p(L_\gamma^{-1}(s))}{\gamma(L_{\boldsymbol{\gamma}}^{-1}(s))}$$

$$q_\gamma(s) := \frac{q(L_\gamma^{-1}(s))}{\gamma(L_{\boldsymbol{\gamma}}^{-1}(s))}$$

if and only if

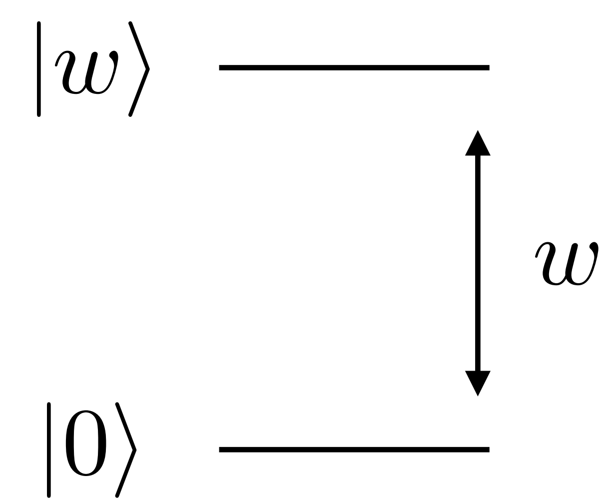
$$q(s) = \int dx \; S(s,x) p(x)$$

$$\gamma(s) = \int dx \; S(s,x) \gamma(x)$$

$\blacksquare$

"Work bits"

- *Conceptual* unit.
- "When bits have non-zero H".

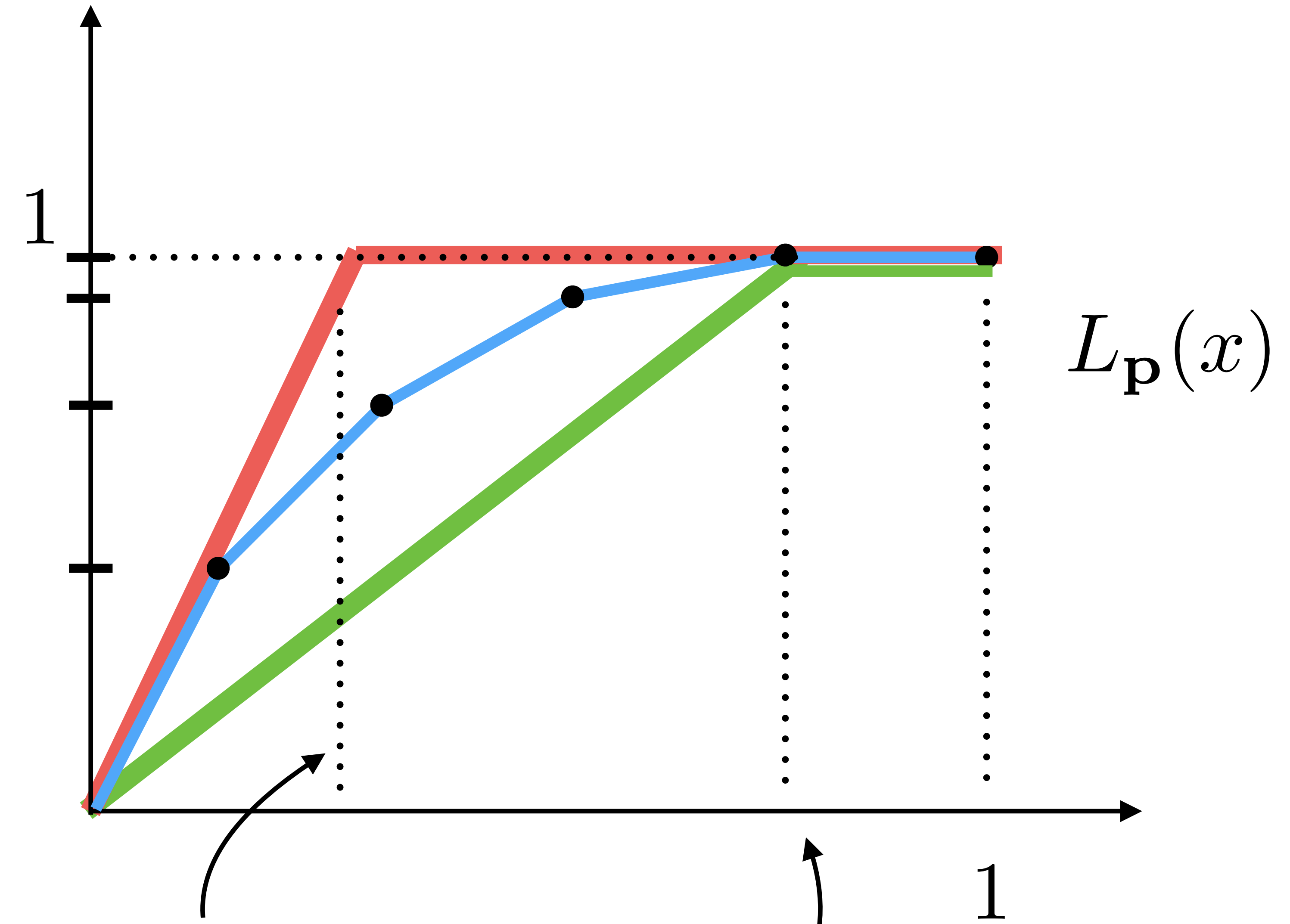


$H = 0$ $\quad\quad \rho \longrightarrow \frac{1}{2^k}\mathbb{I}^{\otimes k} \otimes |0\rangle\langle 0|^{\otimes n}$ maximize n

$H \neq 0$ $\quad\quad \rho \longrightarrow \gamma \otimes |w\rangle\langle w|$ maximize w

"Work" of formation & distillation — nanoscale

$$H_\alpha(\mathbf{p}||\boldsymbol{\gamma}) = \frac{1}{\alpha - 1} \ln \sum p_k^\alpha \gamma_k^{1-\alpha}$$



$$W_{form} \geq kTH_\infty(\rho||\gamma) - kT\ln Z$$

$$W_{distill} \leq kTH_0(\rho||\gamma) - kT\ln Z$$

The "Second Laws of Thermodynamics"

Theorem: For zero coherence states, the transformation $\rho \rightarrow \sigma$ is possible

if and only if $F_\alpha(\rho) \geq F_\alpha(\sigma) \qquad \forall \alpha \in [0, \infty)$

Renyi-divergences:

$$F_\alpha(\rho) := D_\alpha(\rho||\gamma)$$

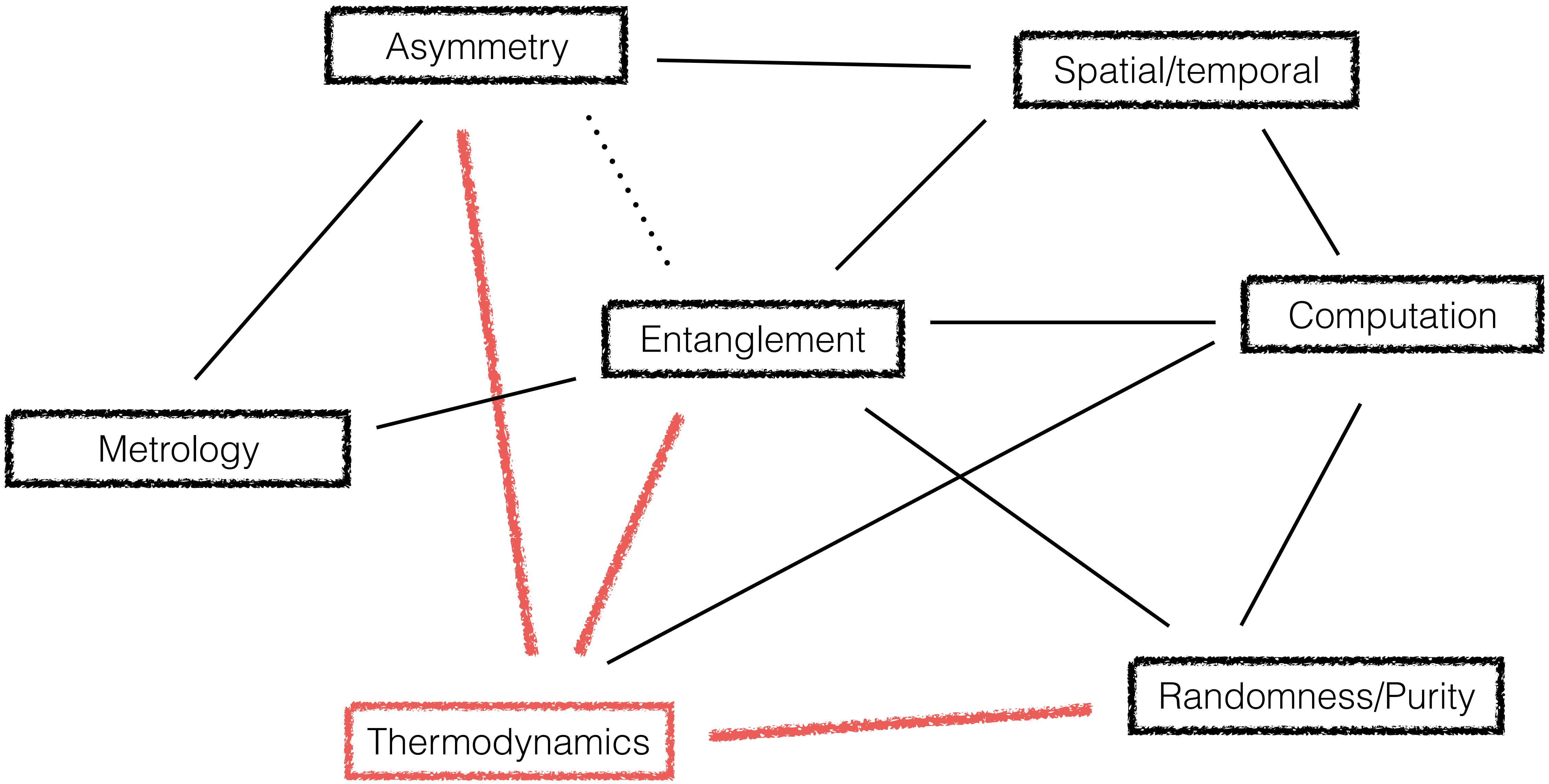
$$D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log\left[\mathrm{tr}(\sigma^\kappa \rho \sigma^\kappa)^\alpha)\right] \qquad \kappa = \frac{1-\alpha}{2\alpha}$$

Overview

1. **Motivations and criteria.**
2. **Resource theories and properties.**
3. **Resource formulation of thermodynamics.**
4. **Majorization — order & disorder.**
5. **Information & energy.**
6. **Gibbs-rescaling & work bits.**

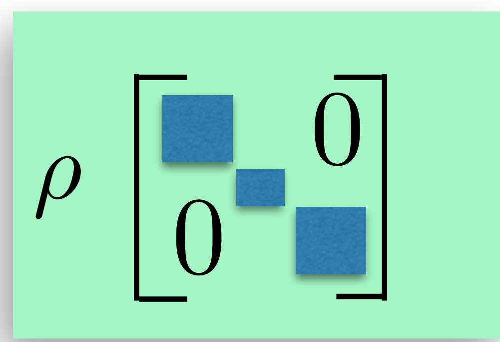
Quantum Features

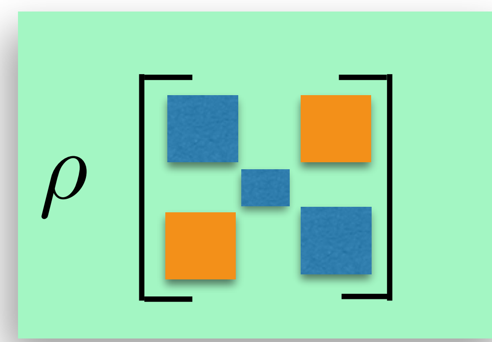
Different quantum resources



Quantum Coherence

1. *Energy conserved microscopically.*
2. *An equilibrium state exists.*
3. ***Quantum coherence has thermodynamic cost.***





$\cong$ Classical stochastic component

Coherent Oscillations ???

Symmetry & Coherence

$$|\Psi\rangle = |E_k\rangle \qquad e^{-iHt}|\Psi\rangle\langle\Psi|e^{iHt} = |\Psi\rangle\langle\Psi|$$

$$|\Phi\rangle = \sqrt{\frac{2}{3}}|E_1\rangle - \sqrt{\frac{1}{3}}|E_9\rangle \qquad e^{-iHt}|\Phi\rangle\langle\Phi|e^{iHt} \neq |\Phi\rangle\langle\Phi|$$

$$|\chi\rangle = \sqrt{0.1}|E_2\rangle - \sqrt{0.2}|E_3\rangle + i\sqrt{0.5}|E_7\rangle + \sqrt{0.2}|E_{11}\rangle \qquad e^{-iHt}|\chi\rangle\langle\chi|e^{iHt} \neq |\chi\rangle\langle\chi|$$

$$\rho = \frac{1}{4}|\Phi\rangle\langle\Phi| + \frac{3}{4}|\chi\rangle\langle\chi|$$

Coherence "Asymmetric"

Clocks - resource theory of time-asymmetry

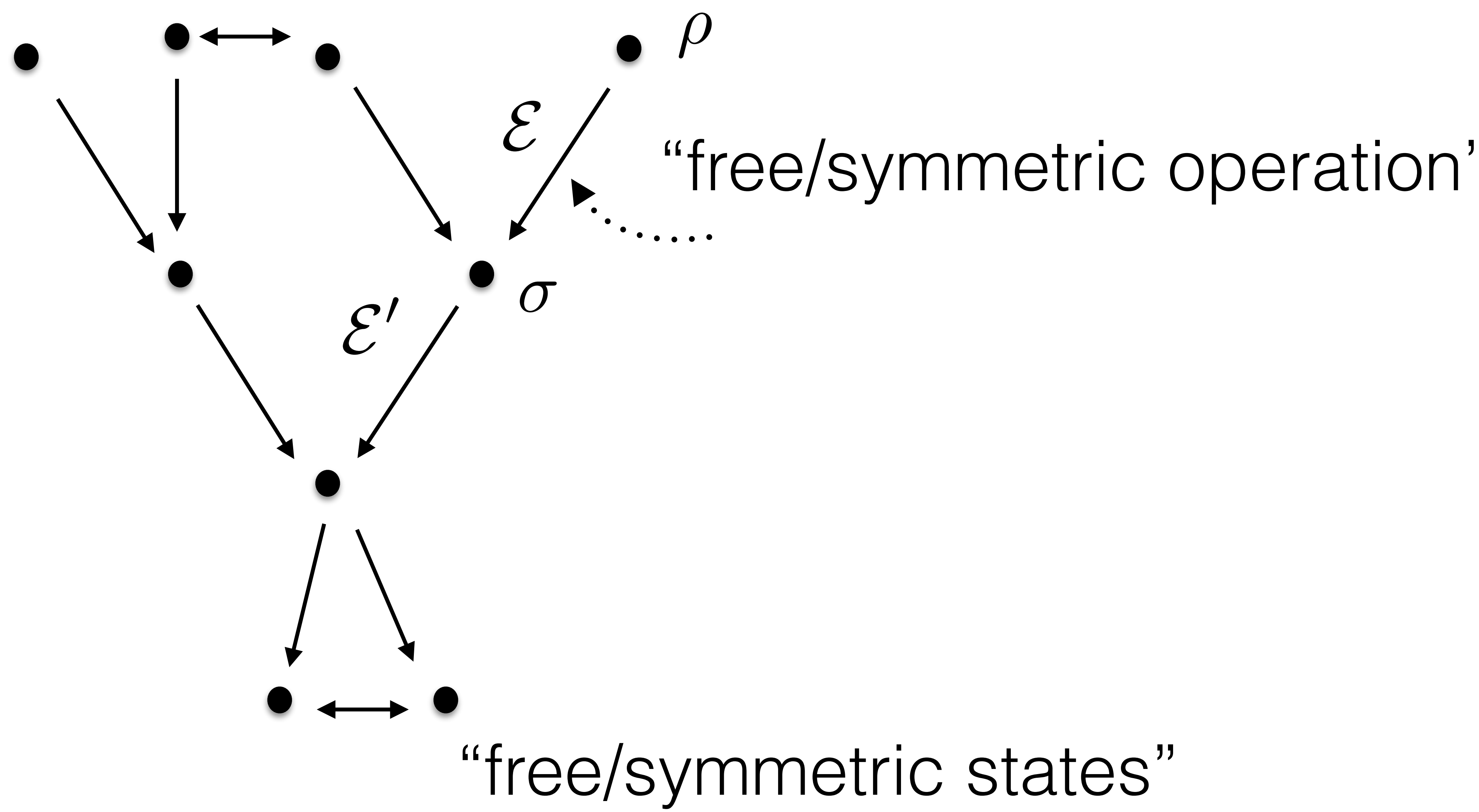
Free states ("symmetric states"):

$$\sigma \text{ such that } e^{-iHt}\sigma e^{iHt} = \sigma \text{ for all } t$$

Free operations ("symmetric operations"):

$$\mathcal{E} \text{ such that } \mathcal{E}(U(t)\rho U(t)^\dagger) = U(t)\mathcal{E}(\rho)U(t)^\dagger \text{ for all } t$$

Partial order — “time-asymmetry”



Overview

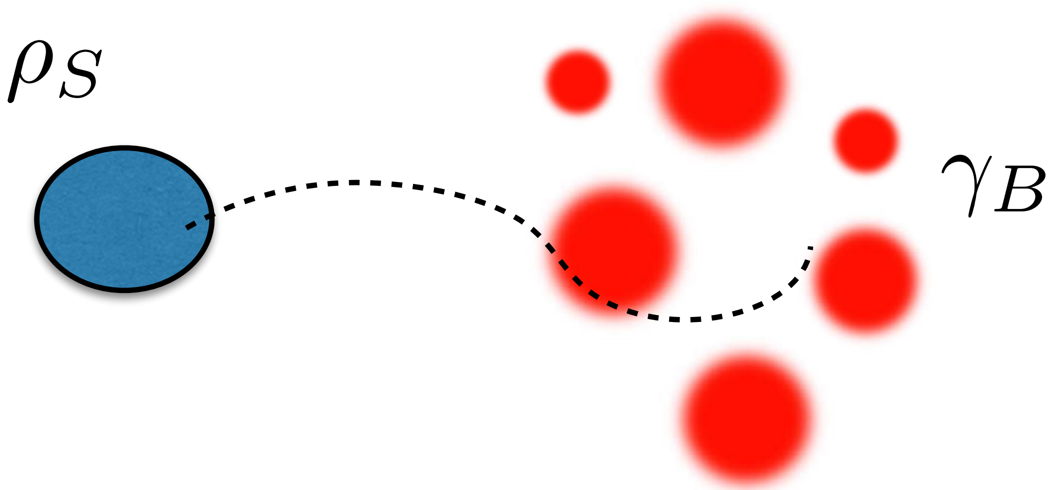
- **Coherent energy.**
- **Quantum asymmetry.**
- Symmetry and quantum thermodynamics.
- Mode analysis and Measures.
- Complete conditions & the role of Time.

Asymmetry & thermodynamic coherence

Underlying symmetry

$$[U, H_{\rm tot}] = 0$$

$$\mathcal{E}(\rho) = \mathrm{tr_C}\left[\mathrm{U}(\rho_\mathrm{S} \otimes \gamma_\mathrm{B})\mathrm{U}^\dagger\right]$$



$$\mathcal{E}(e^{-iHt}\rho e^{iHt}) = e^{-i\tilde{H}t}\mathcal{E}(\rho)e^{i\tilde{H}t}$$

thermal maps 'commute' with time-translation

[1] Lostaglio, DJ, Rudolph, Nat Comm (2015)
[2] Lostaglio, Korzekwa, DJ, Rudolph, Phys. Rev. X (2015)

Insufficiency of free energies

Consider all functions $\{F_\alpha(\cdot)\}_\alpha$ that "behave like free energies": $F_\alpha(\rho) \geq c||\rho - \gamma||$

Theorem:

"There cannot exist a set of free energies $\{F_\alpha(\rho) \geq F_\alpha(\sigma)\}_\alpha$ that captures the full thermodynamic quantum ordering."

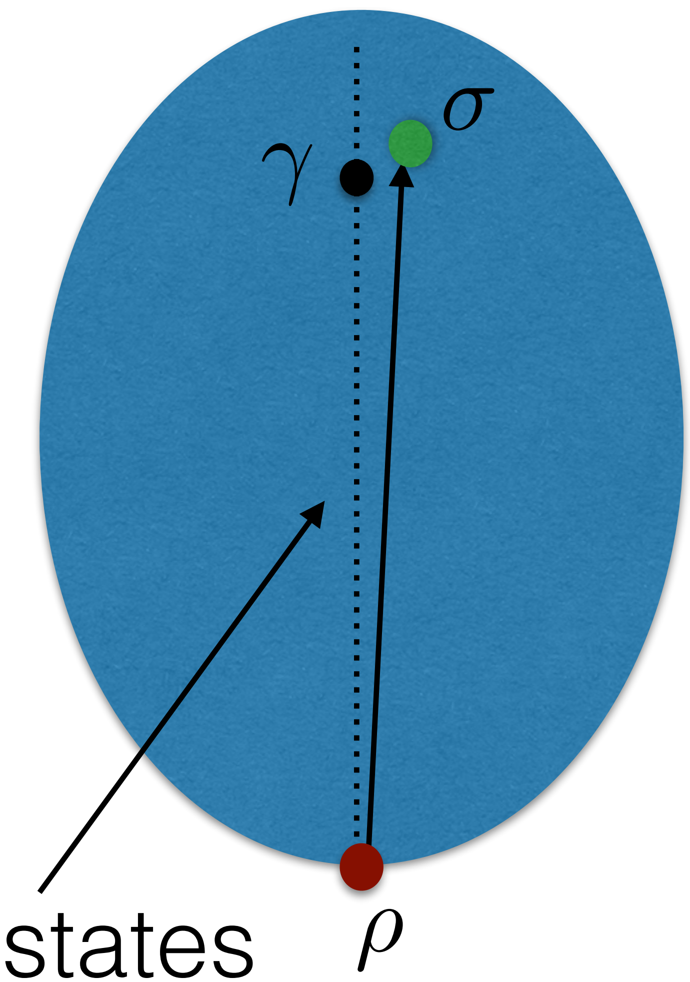
$$\rho \to \sigma \quad \Leftrightarrow \Delta F_\alpha \leq 0$$

Insufficiency of free energies

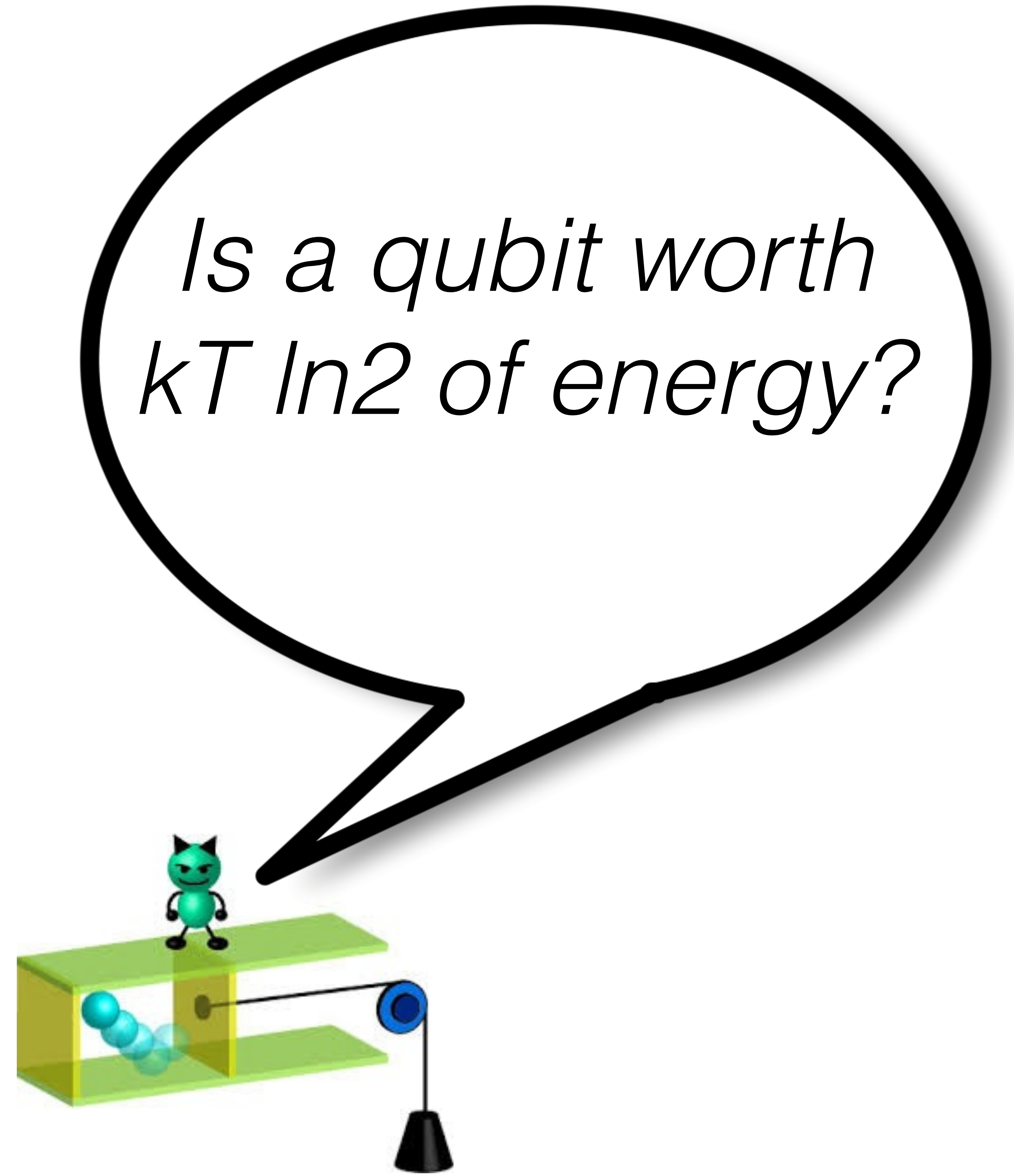
Proof:

F_α say "get closer to γ."

Symmetry says:
"coherence non-increasing." ∎

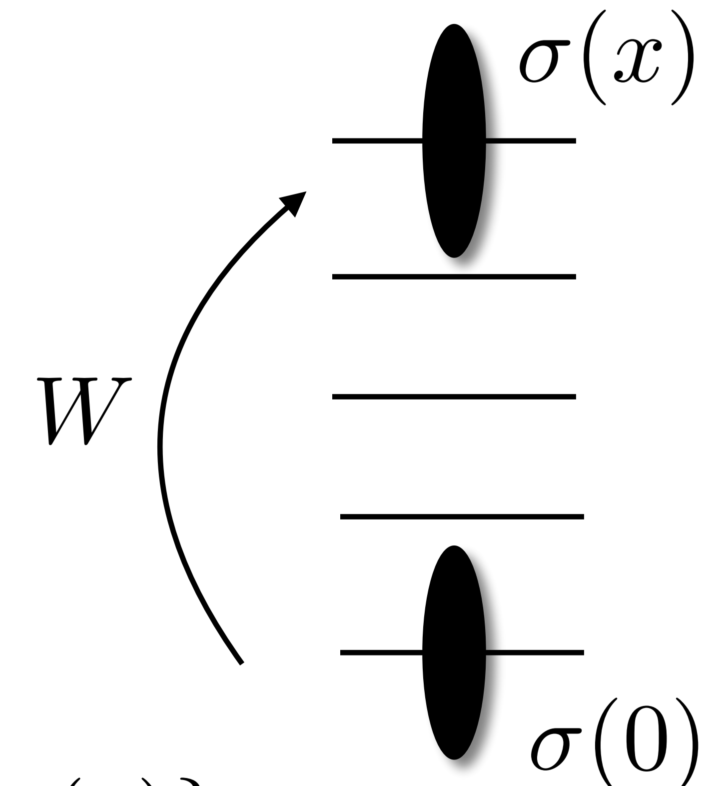


$$\sigma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$


Is a qubit worth
kT ln2 of energy?

An "inclusive" approach

Broad notion of "work":
"raising a weight up a ladder by height W"



$$W := \sup\{\mathrm{x} : \mathcal{E} \text{ thermal \& sends } \rho \otimes \sigma(0) \rightarrow \sigma(\mathrm{x})\}$$

[1] Deffner, Jarzynski, Phys Rev. X 3 (2013)
[2] Aberg Phys. Rev. Lett 113 (2014)

An "inclusive" approach

$\sigma(x)$ → "classical parameter for family of states"

$$e^{-iHt}\sigma(x)e^{iHt} = \sigma(x)$$

Includes sharpest notion: x=energy eigenvalue

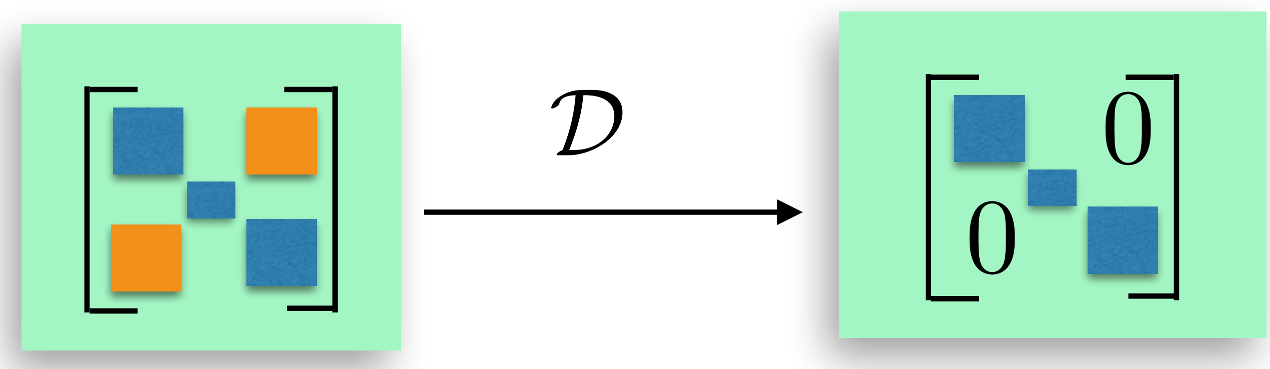
Work-locking superselection

Theorem:

If an amount of work W can be extracted from ρ under thermal processes then the same amount W can be extracted from $\mathcal{D}(\rho)$.

$$\mathcal{D}(\rho) = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\; e^{-iHt}\rho e^{iHt}$$

(complete dephasing map)



ρ $\mathcal{D}(\rho)$

Proof

Let $\mathcal{E}$ be any thermal map that does the following:

$$\mathcal{E}(\rho \otimes \sigma(0)) = \gamma \otimes \sigma(x)$$

Now we have

$$\mathcal{D}(\mathcal{E}(\rho \otimes \sigma(0))) = \mathcal{D}(\gamma \otimes \sigma(x))$$

$$= \gamma \otimes \sigma(x)$$

since RHS has no coherences.

Proof

But we have that $\mathcal{D}$ *commutes* past $\mathcal{E}$

$$\mathcal{D}(\mathcal{E}(\rho \otimes \sigma(0)) = \int dt U(t)\mathcal{E}(\rho \otimes \sigma(0))U(t)^\dagger$$

$$= \int dt \mathcal{E}(U(t)(\rho \otimes \sigma(0))U(t)^\dagger)$$

$$= \mathcal{E}(\int dt U(t)(\rho \otimes \sigma(0))U(t)^\dagger)$$

$$= \mathcal{E}(\mathcal{D}(\rho \otimes \sigma(0))) = \mathcal{E}(\mathcal{D}(\rho) \otimes \sigma(0))$$

Proof

Thus if

$$\mathcal{E}(\rho \otimes \sigma(0)) = \gamma \otimes \sigma(x)$$

then we also have

$$\mathcal{E}(\mathcal{D}(\rho) \otimes \sigma(0)) = \gamma \otimes \sigma(x)$$

Therefore: off-diagonal coherences can never be contribute to the extracted work under thermal processes.

$\blacksquare$

Extreme example

$$|\gamma\rangle = \sum_k \sqrt{\frac{e^{-\beta E_k}}{Z}} |E_k\rangle$$

$$\mathcal{D}(|\gamma\rangle\langle\gamma|) = \gamma = \frac{1}{Z}e^{-\beta H}$$

(zero work extractable)

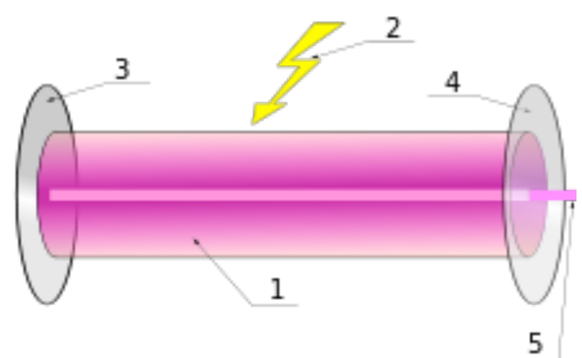
$|\gamma\rangle\langle\gamma|$ = a pure state from which **zero** work can be extracted under thermal processes!

Equivalent to a “superselection" constraint.

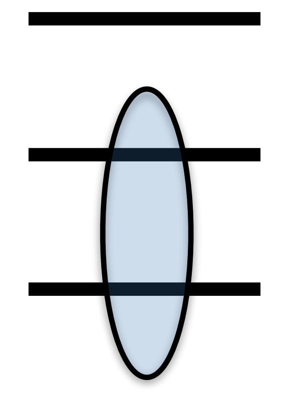
Resolution — Quantum Reference Frames

1. Quantum coherences are relational.
2. Coherence is not "free".
3. Coherences must be explicitly modelled.

Clock



R



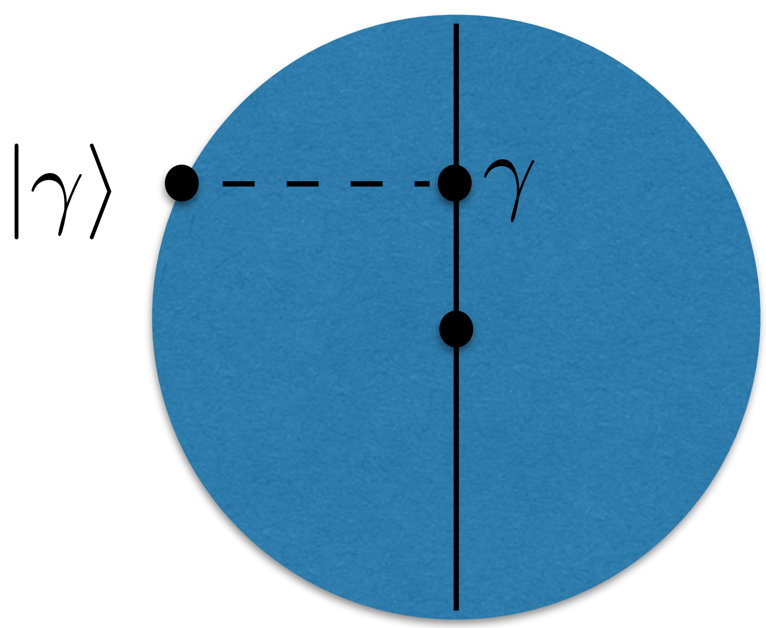
ρ

$$\frac{1}{\sqrt{2}}(|E_1\rangle + e^{i\omega t}|E_2\rangle)$$

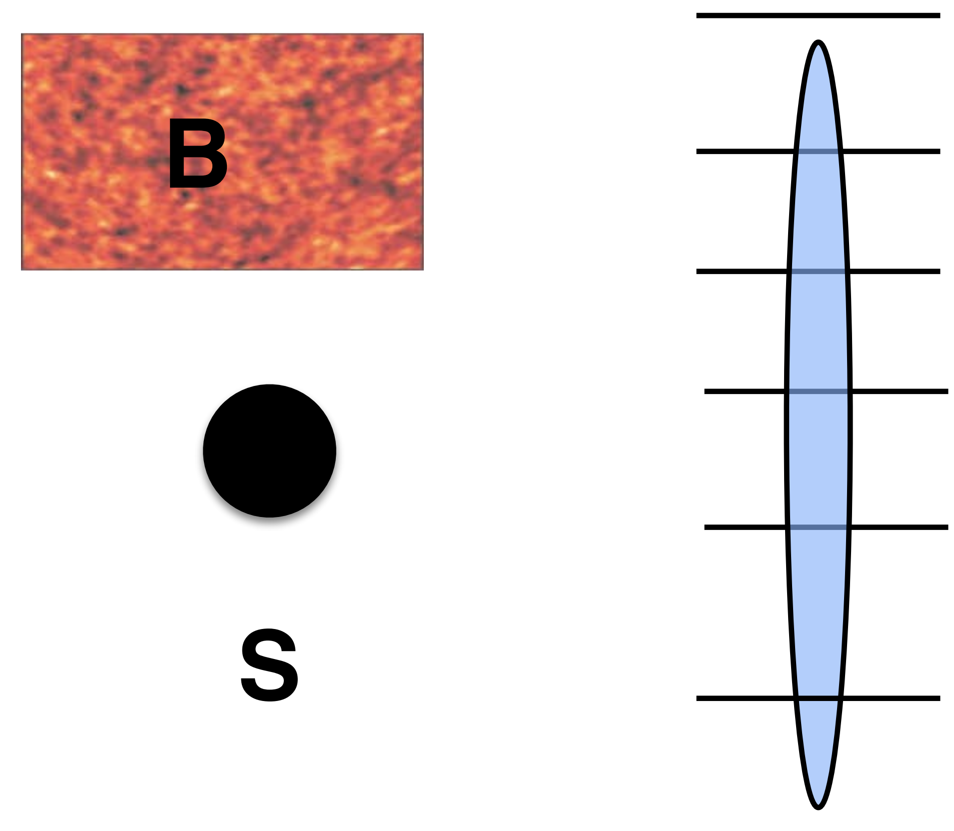
Always relative to a <u>phase</u> <u>reference</u> **R**

Extreme example revisited

$$|\gamma\rangle = \sum_k \sqrt{\frac{e^{-\beta E_k}}{Z}}\,|E_k\rangle$$



qubit



large phase reference **R**

$$|\Psi\rangle \longrightarrow W = -\Delta F$$

M. Lostaglio, K. Korzekwa, J. Oppenheim, DJ, "Extracting work from quantum coherence" NJP (2015)

H. Callen - *"Thermodynamics & an introduction to thermostatistics"* (1985)

"The statistical feature veils the incoherent complexity of the atomic dynamics, thereby revealing the coherent effects of the underlying symmetries."

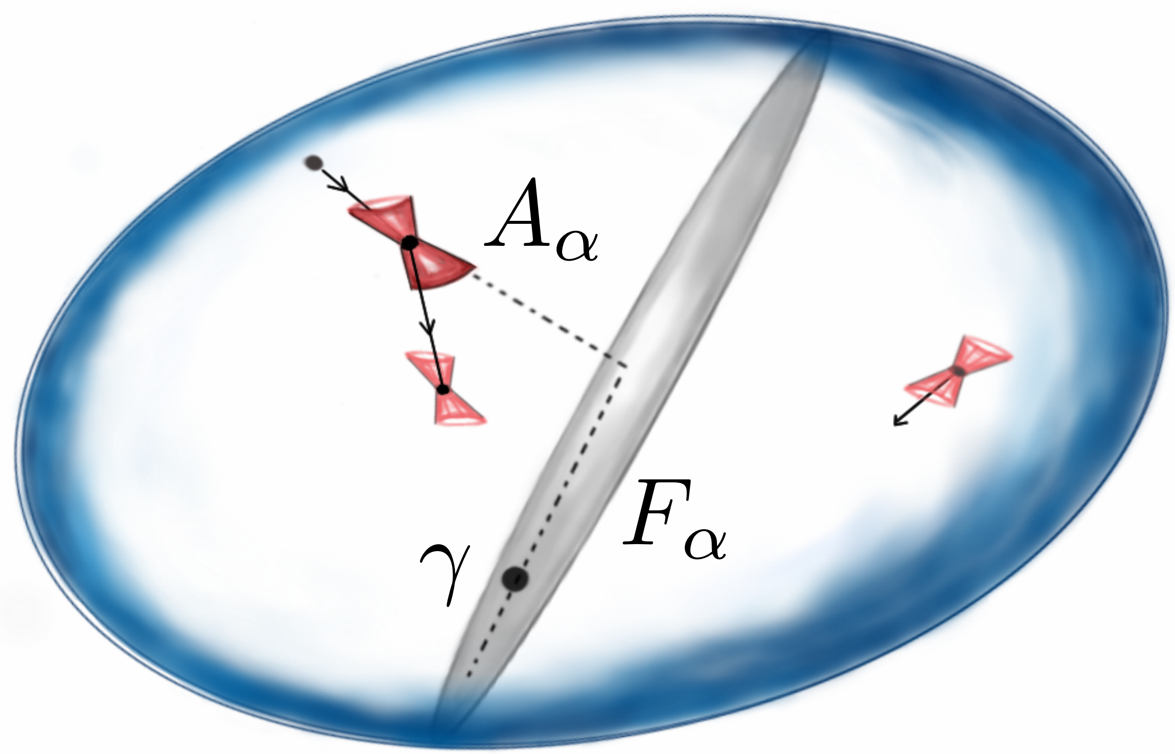
Time-translation on microstates. $\longrightarrow$ Time-translation on quantum states.

$\text{Symmetry} \Leftrightarrow \text{Conservation}$

$\text{Symmetry} \not\Rightarrow \text{Conservation}$

Thermodynamic structure?

Quantum thermodynamics
~ **classical athermality** X **time-asymmetry**



Overview

- **Coherent energy.**
- **Quantum asymmetry.**
- **Symmetry and quantum thermodynamics.**
- Mode analysis and Measures.
- Complete conditions & the role of Time.

Harmonic Analysis

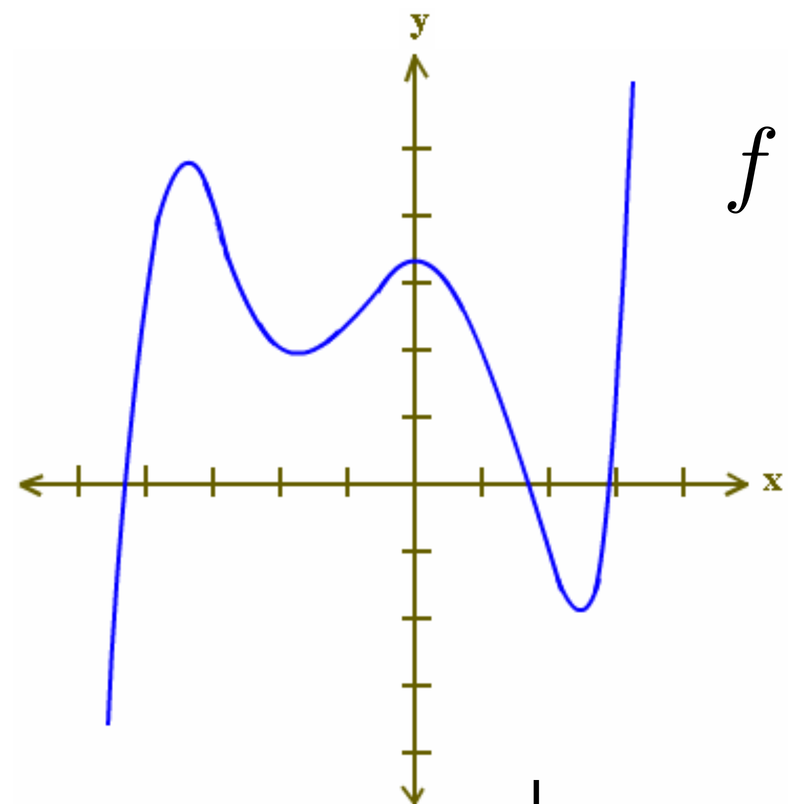
$f(x)$

↓

$$f(x) = \sum_k a_k e^{-ikx}$$

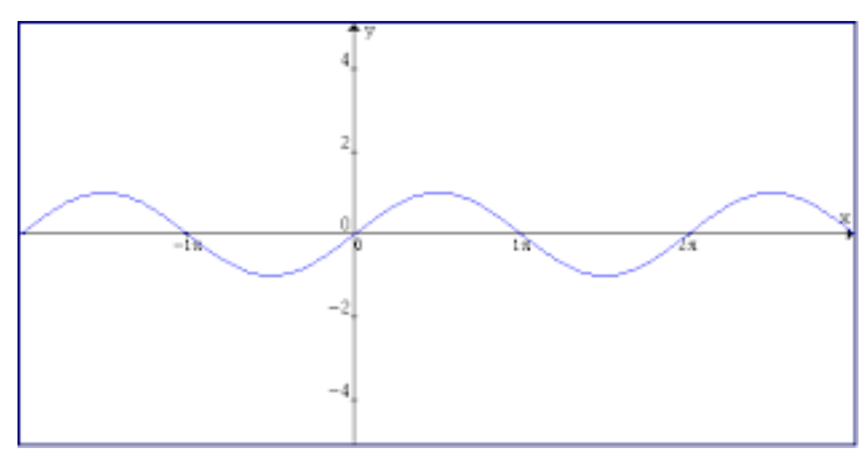
↓

$$\{a_0, a_1, a_2, \ldots\}$$



$f(x)$

↓



Operator harmonic analysis

Decompose quantum states

$$\rho = \sum_{\nu=-d}^{d} \rho^{(\nu)}$$

$$U(t)\rho^{(\nu)}U(t)^\dagger = e^{-i\nu t}\rho^{(\nu)}$$

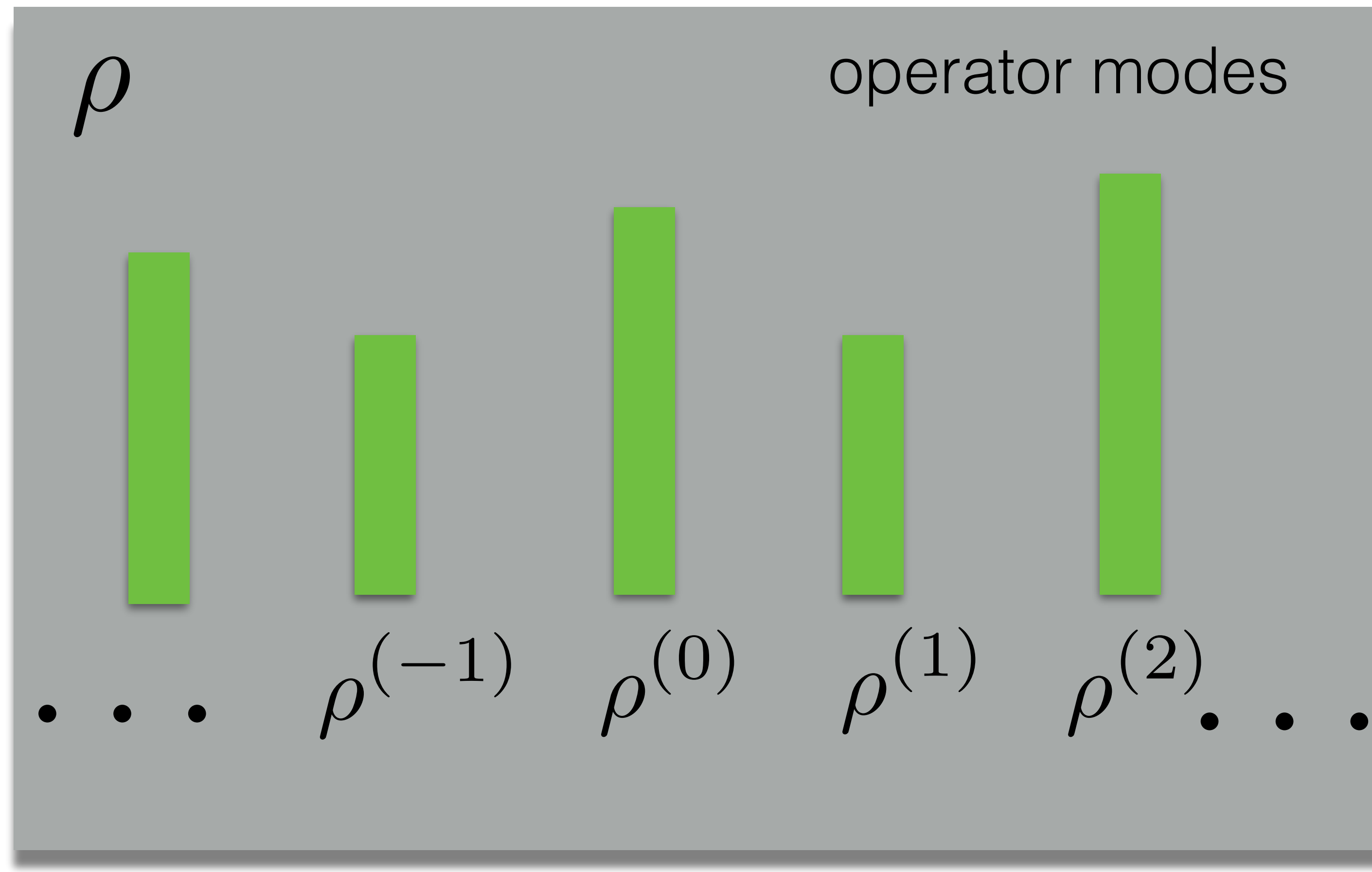
[1] Lostaglio, Korzekwa, DJ, Rudolph, Phys. Rev. X (2015)

Operator harmonic analysis

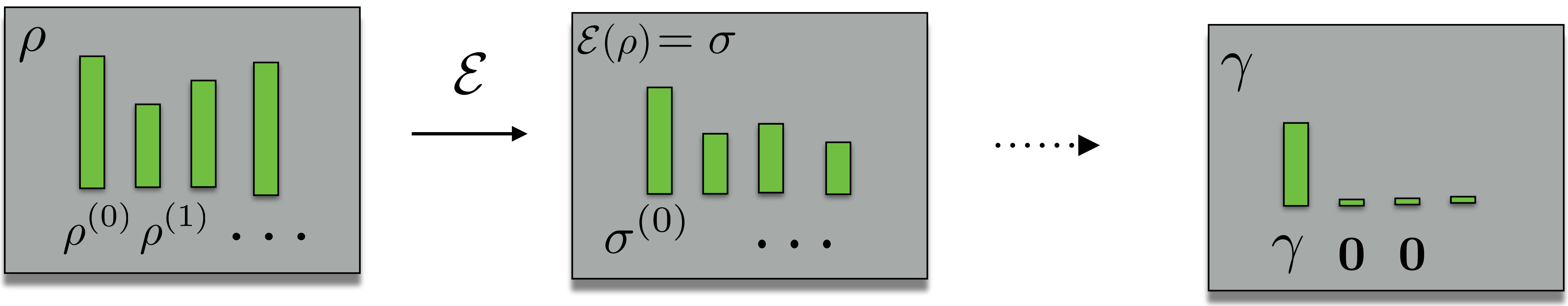
Decompose quantum states

$$\rho = \sum_{\nu=-d}^{d} \rho^{(\nu)}$$

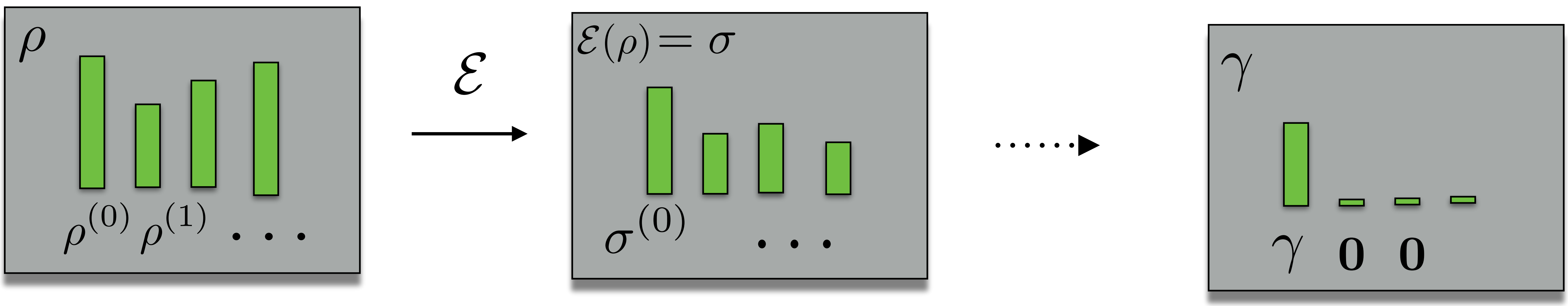
$$U(t)\rho^{(\nu)}U(t)^\dagger = e^{-i\nu t}\rho^{(\nu)}$$



Thermal operations



Thermal operations



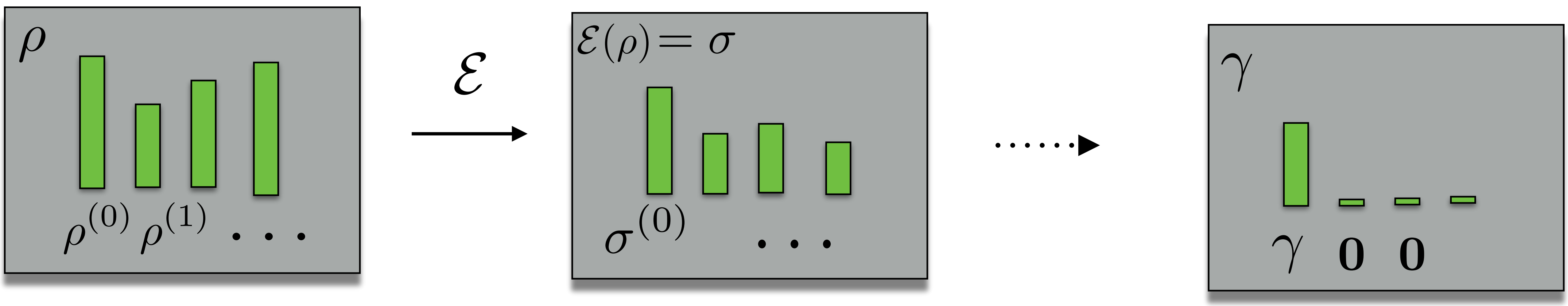
Independence $[\mathcal{E}(\rho)]^{(\nu)} = \mathcal{E}(\rho^{(\nu)})$

Monotonicity $||\mathcal{E}(\rho)^{(\nu)}||_1 \leq ||\rho^{(\nu)}||_1$ ("Data processing" inequality)

$||\rho - \sigma||_1$: optimal discrimination of ρ from σ

$$||X||_1 := \mathrm{tr}\left[\sqrt{\mathrm{X}^\dagger \mathrm{X}}\right]$$

Thermal operations



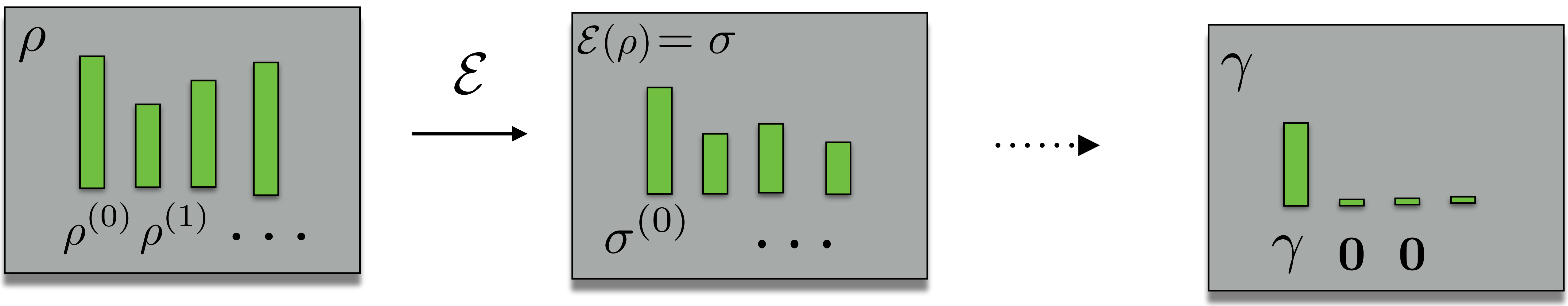
$$\gamma = \gamma + \mathbf{0} + \mathbf{0} \ldots$$

$$\rho^{(\nu)} \to \sigma^{(\nu)}$$

$$\mathcal{E}(\rho^{(\nu)}) = \sum_n A_n \rho^{(\nu)} A_n^\dagger$$ Kraus decomposition

$$\mathcal{E}(\gamma) = \gamma$$ Invariance of Gibbs

Thermal operations



$$\rho^{(\nu)} \to \sigma^{(\nu)}$$

Upper bound: $$|\sigma_k^{(\nu)}| \leq \sum_{c:\omega_c \leq \omega_k} |\rho_c^{(\nu)}| e^{-\beta\hbar(\omega_k - \omega_c)} + \sum_{c:\omega_c > \omega_k} |\rho_c^{(\nu)}|$$

Lower bound: $$\sigma^{(\nu)} = \lambda_\star \rho^{(\nu)}$$

Necessary entropic constraints

Theorem: For arbitrary quantum states, and thermodynamic transformation $\rho \to \sigma$ then

$$F_\alpha(\rho) \geq F_\alpha(\sigma) \quad \forall \alpha \geq 0$$
$$A_\alpha(\rho) \geq A_\alpha(\sigma)$$

Asymmetry Monotones:

$$A_\alpha(\rho) = S_\alpha(\rho || \mathcal{D}(\rho))$$

$$\mathcal{D}(\rho) = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\ e^{-iHt} \rho e^{iHt}$$

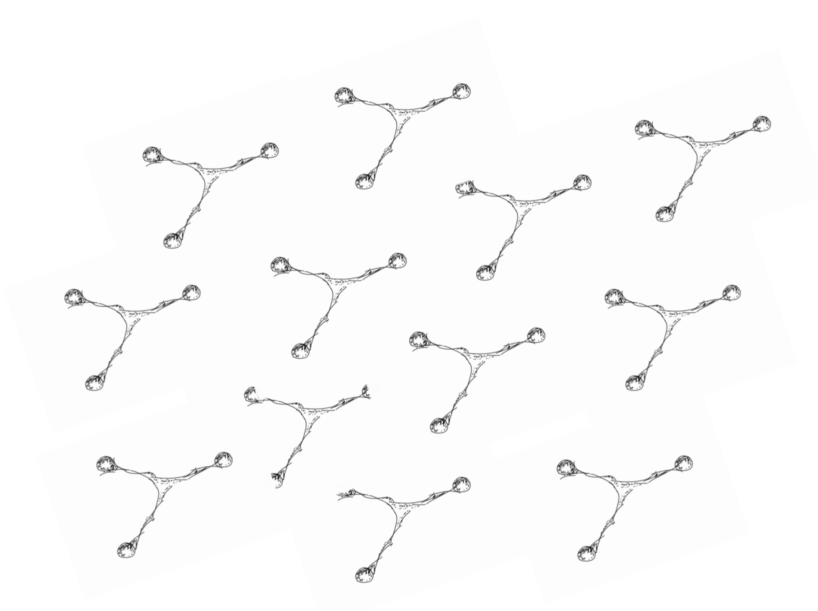
[1] M. Lostaglio, DJ, T. Rudolph, Nature Comm. (2015).

Macroscopic regime

Theorem: for any $\rho \in \mathcal{B}(\mathcal{H})$ we have

$$\lim_{n\to\infty} \frac{1}{n} \begin{bmatrix} F_\alpha(\rho^{\otimes n}) \\ A_\alpha(\rho^{\otimes n}) \end{bmatrix} = \begin{bmatrix} F(\rho) - F(\gamma) \\ 0 \end{bmatrix}$$

$$F = \langle H \rangle - TS$$



Overview

- **Coherent energy.**
- **Quantum asymmetry.**
- **Symmetry and quantum thermodynamics.**
- **Mode analysis and Measures.**
- Complete conditions & the role of Time.

Relational quantum thermodynamics

Relational Quantum Thermodynamics

Minimal assumptions:

1. *Energy conserved microscopically.*
2. *An equilibrium state exists.*
3. *Quantum coherence has thermodynamic cost.*

→

$$\rho \longrightarrow \sigma$$
if and only if
$$S_{\boldsymbol{\eta}}(\rho) \leq S_{\boldsymbol{\eta}}(\sigma)$$

(classical)

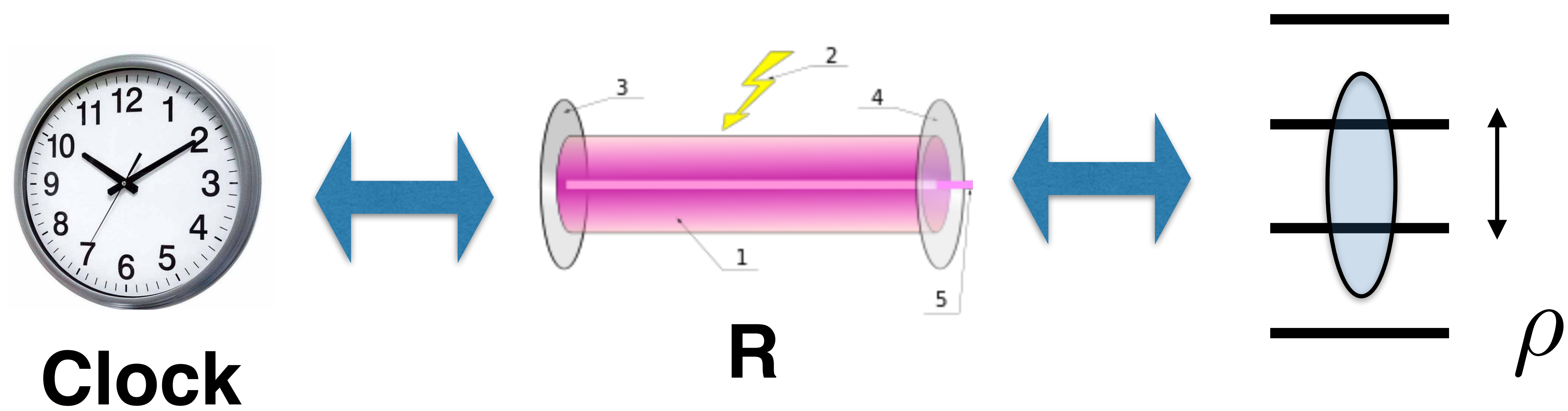
- thermo-majorization.
- no unique ordered energy or "heat".
- Many *derived* entropies.

(quantum)

- Free energies insufficient.
- Superselection rules.
- Quantum clocks are central.

[1] Gour, DJ, Buscemi, Duan, Marvian arXiv:1708.04302 (2017)

(Relational) Quantum Thermodynamics



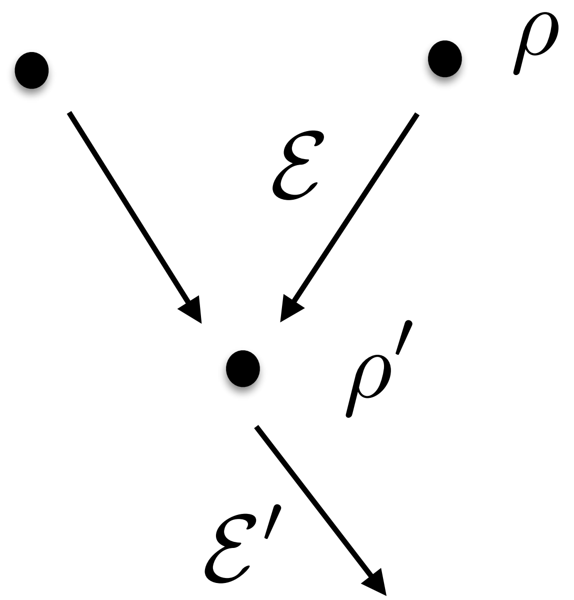
$$\frac{1}{\sqrt{2}}(|E_1\rangle + e^{i\omega t}|E_2\rangle)$$

Always relative to a phase reference **R**

(Relational) Quantum Thermodynamics

Theorem: $\rho \to \rho'$ if and only if for all reference systems η_R

$$S_{\min}(R|A') \geq S_{\min}(R|A)$$



$$\Omega_{RA} = \frac{1}{2}\int dt U(t)(\eta_R \otimes \rho_A + \eta'_R \otimes \gamma_A)U^{\dagger}(t)$$

$$2^{-S_{\min}(R|A)} = \min_{\tau \geq 0}\{\mathrm{tr}(\tau) : \mathbb{I} \otimes \tau \geq \rho_{\mathrm{RA}}\}$$

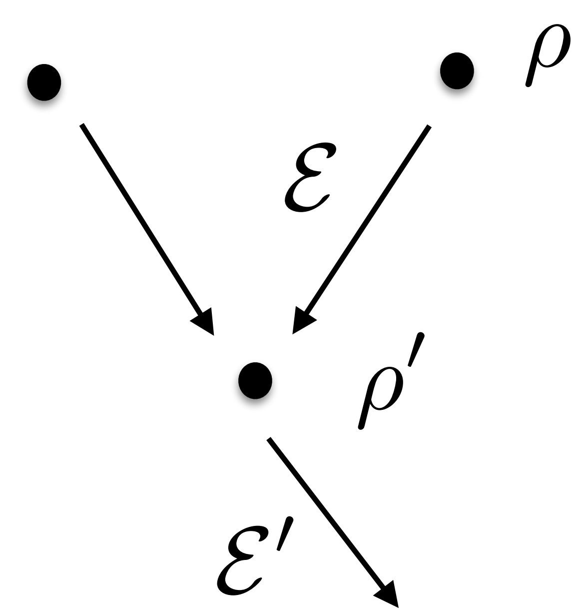
[1] Gour, DJ, Buscemi, Duan, Marvian arXiv:1708.04302 (2017)

(Relational) Quantum Thermodynamics

Theorem: $\rho \to \rho'$ if and only if

for all reference systems η_R

$$S_{\min}(R|A') \geq S_{\min}(R|A)$$



"Quantum Majorization" ~ Majorization + Quantum Reference Frames

(Relational) Quantum Thermodynamics

(I) Energy conserved microscopically.
(II) An equilibrium state exists.
(III) No free quantum coherence.

↓

$S_{\min}(R|A') \geq S_{\min}(R|A)$ $\rightarrow$ thermo-majorization $\rightarrow$ $S(\rho') \geq S(\rho)$

$(\text{coherence} \rightarrow 0)$ $(\text{scale} \rightarrow \infty)$

[1] F. Buscemi, R. Duan, G. Gour, D. Jennings, I. Marvian (in preparation)

[2] Brandao et al, PNAS (2015)

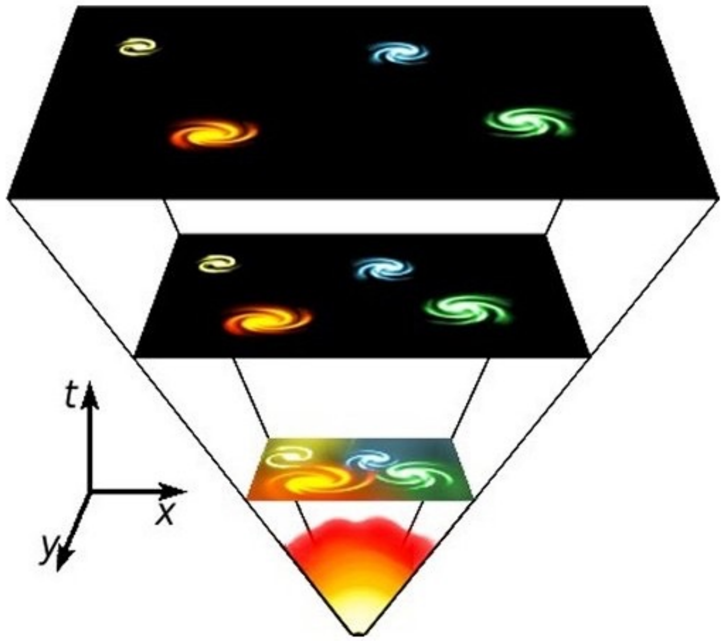
[3] Oppenheim, Horodecki Nat. Comm (2013)

[4] Lieb, Yngvason, Phys. Rept. (1999)

[5] Giles, Pergamon (1964)

Quantum clocks & Thermodynamics

Page-Wootters Time

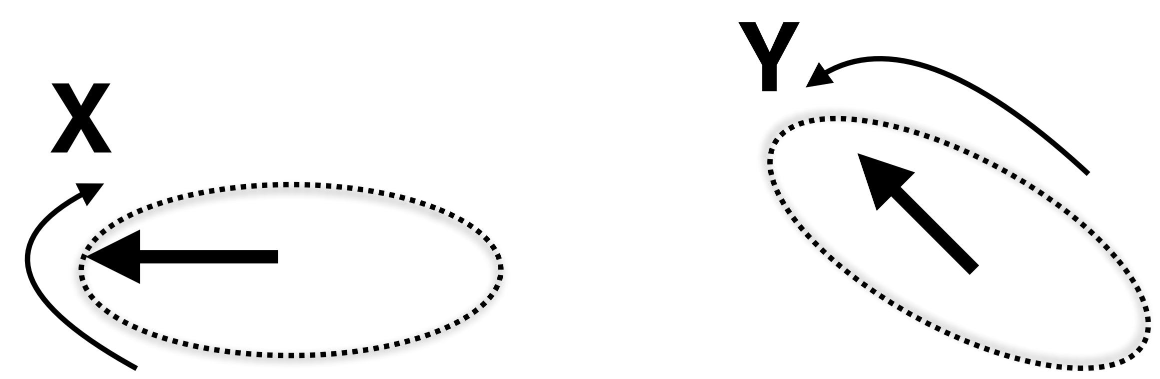


Foundational theory for time in Wheeler deWitt scenario

$$H|\Psi_{universe}\rangle = 0$$

"clock-system" division.
Conditional probabilities describe internal time.

Page-Wootters Time



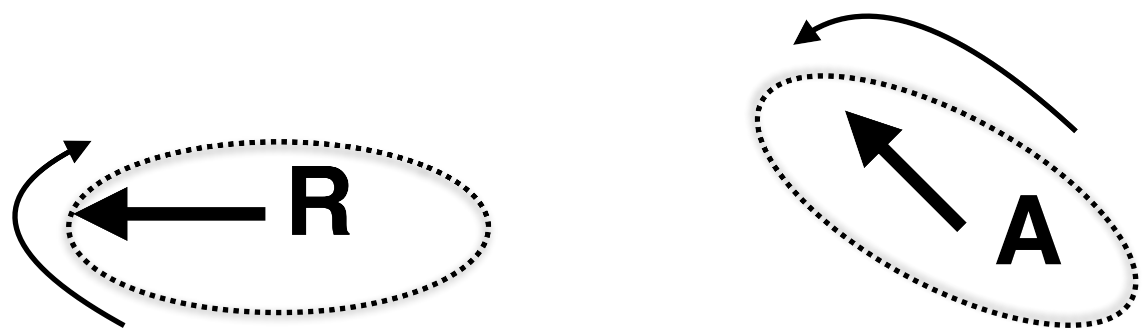
Conditional probability interpretation:

$$P(Y = y|X = x)$$

"time ~ relational correlations"

Thermodynamics and time

$$\Omega_{RA} = \frac{1}{2}\int dt U(t)(\eta_R \otimes \rho_A + \eta'_R \otimes \gamma_A)U^\dagger(t)$$



$$\rho_A(t) = U(t)\rho_A U(t)^\dagger$$

"High coherence regime"

$$\Omega_{RA} \longrightarrow \sum_t p_t |t\rangle_R\langle t| \otimes \rho_A(t)$$

Thermodynamics and time

$$\Omega_{RA} = \frac{1}{2}\int dt U(t)(\eta_R \otimes \rho_A + \eta'_R \otimes \gamma_A) U^\dagger(t)$$

$$\rho_A(t) = U(t)\rho_A U(t)^\dagger$$

"High coherence regime"

$$\Omega_{RA} \longrightarrow \sum_t p_t |t\rangle_R\langle t| \otimes \rho_A(t)$$

$H_{\min}(R|A)$ can never decrease

Thermodynamics and time

$$\Omega_{RA} = \frac{1}{2}\int dt U(t)(\eta_R \otimes \rho_A + \eta'_R \otimes \gamma_A)U^\dagger(t)$$

"High coherence regime"

$$\Omega_{RA} \longrightarrow \sum_t p_t |t\rangle_R\langle t| \otimes \rho_A(t)$$

$$H_{\min}(R|A) = -\log p_{guess}(t|\rho(t))$$

"conditional ***entropy*** condition"!

Thermodynamic Page-Wootters

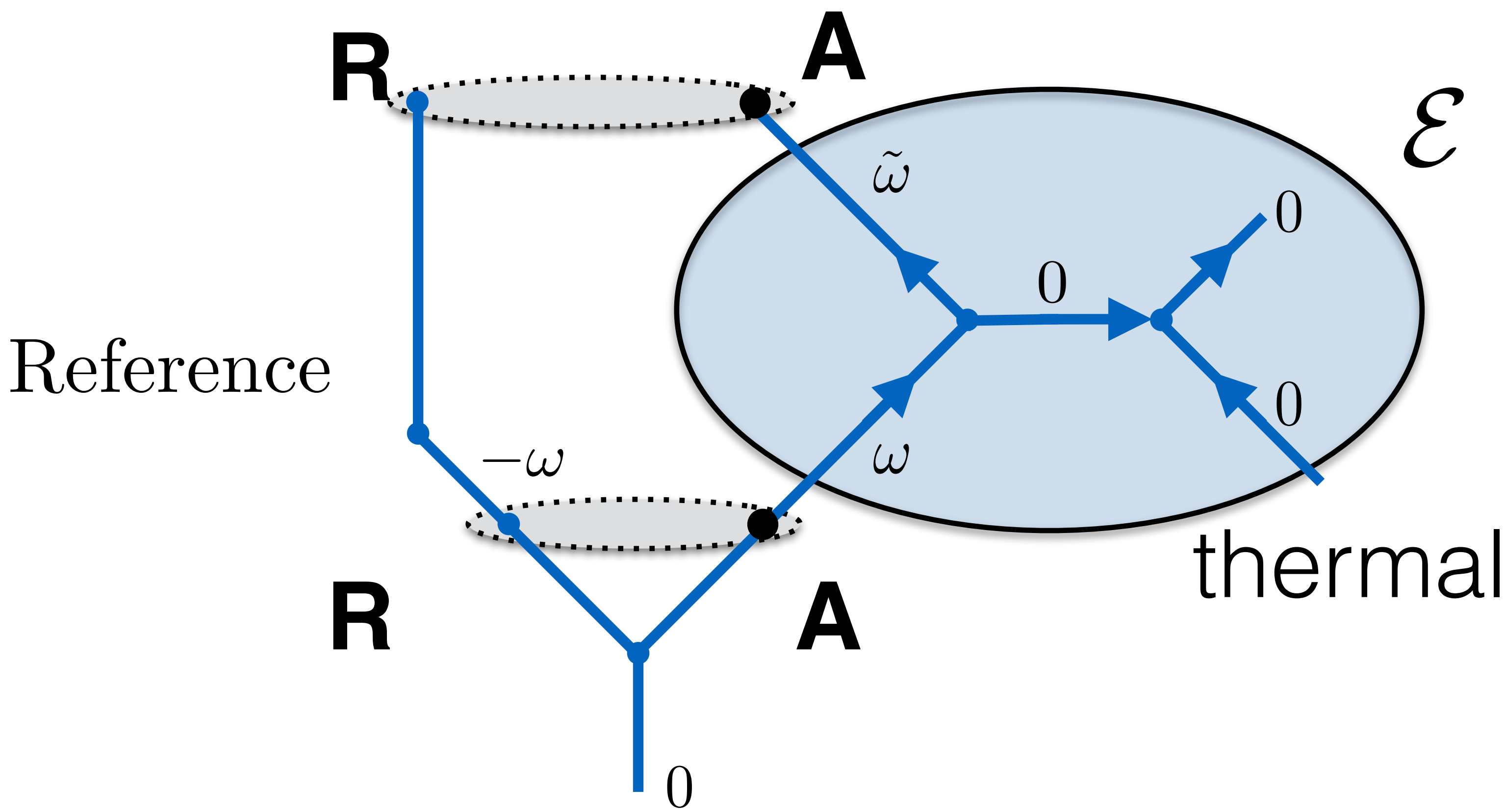
Under thermodynamic processes the ability of a quantum system to act as a Page-Wootters quantum clock can never increase.

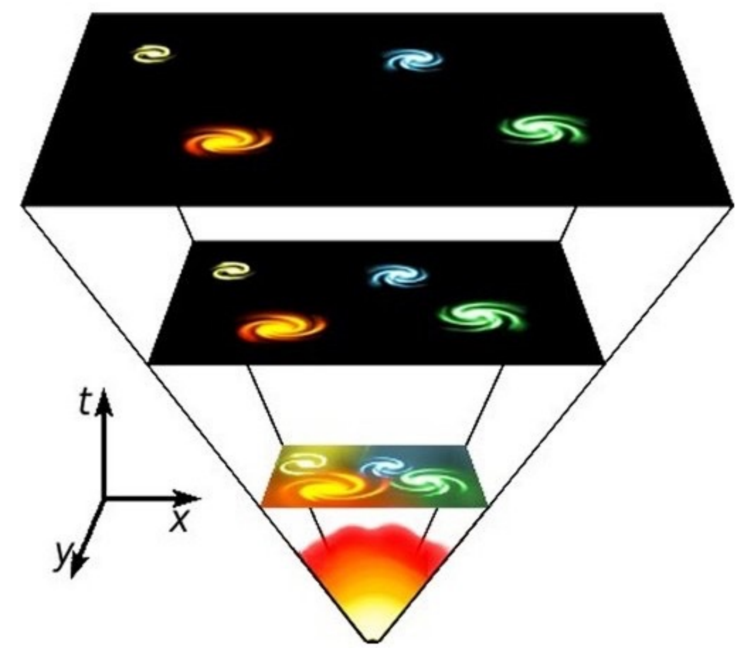
$$H_{\min}(R|A) \leq H_{\min}(R|A')$$

thermal process

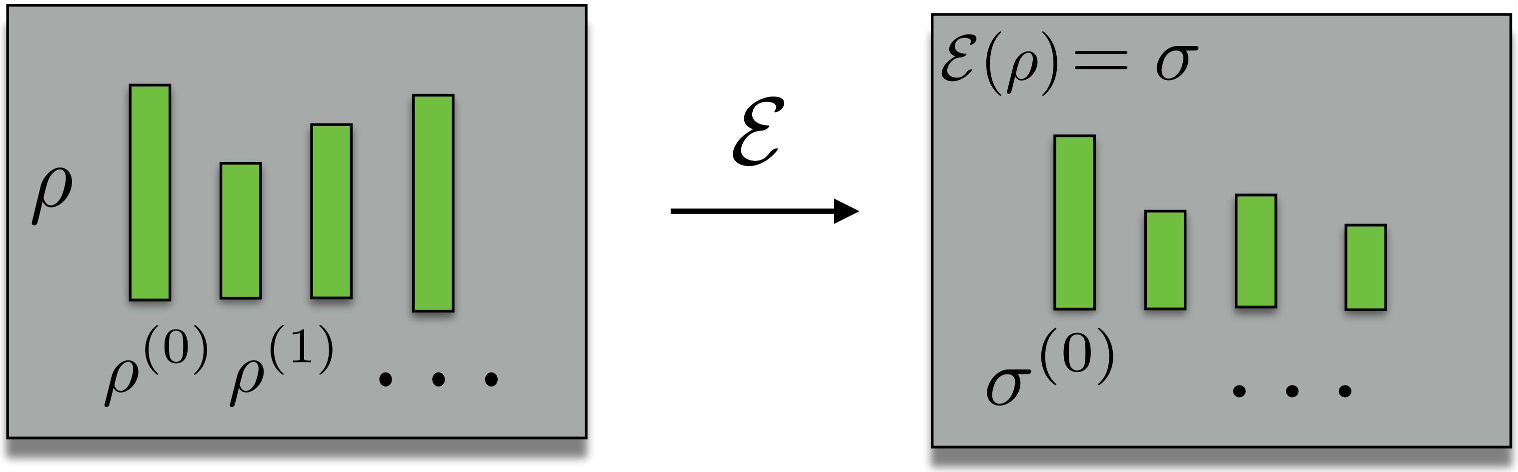
$\rho \to \rho'$

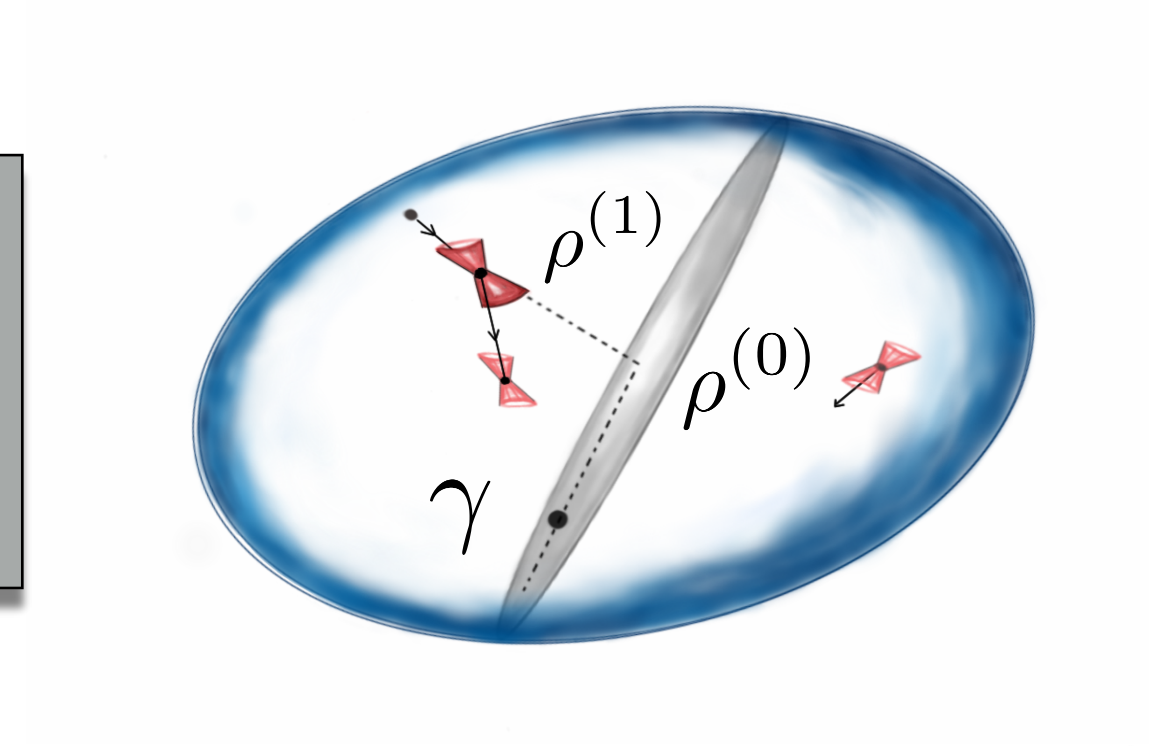
*"The coherent information content of **A** about **R** never increases"*





Theorem: $\rho \to \rho' \Leftrightarrow S_{\min}(R|A') \geq S_{\min}(R|A)$





free energy | **coherence**

- **Work, Heat,**
- **Entropy,**
- **Expectation values.**
- **Markovianity.**
- **Weak/strong coupling.**
- **Coupling rates.**
- **Scale of system…**

- **Minimal energy conditions.**
- **QI techniques.**
- **Consistent with macro regime.**
- **Other resource theories: (Coherence/asymmetry)**

Bonus material

Connections between Entanglement theory & Classical thermodynamics

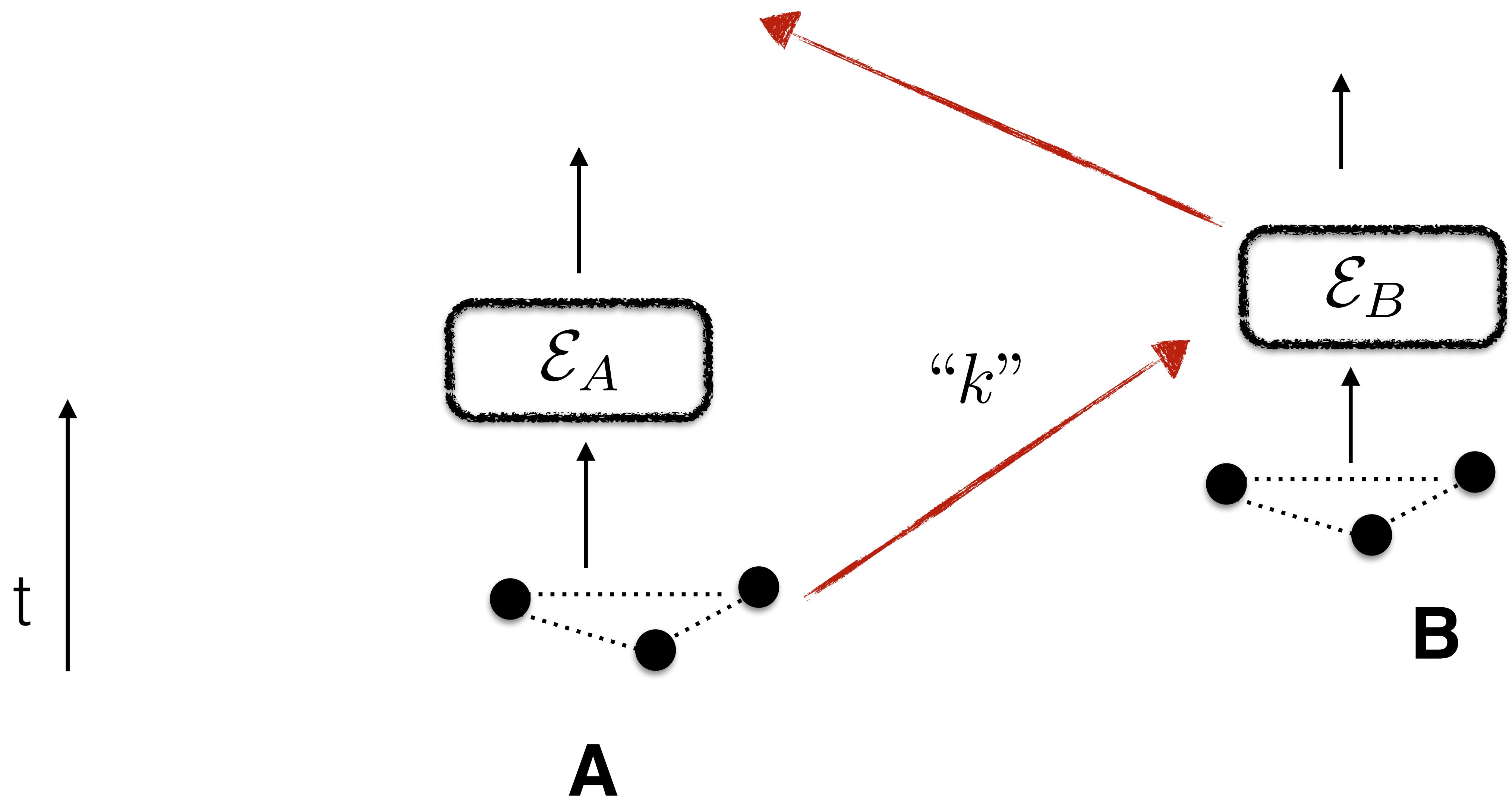
The theory of entanglement

Is the "original" QI resource theory

Free states: $$\rho_{AB} = \sum_k p_k \sigma_k \otimes \chi_k$$ "separable"

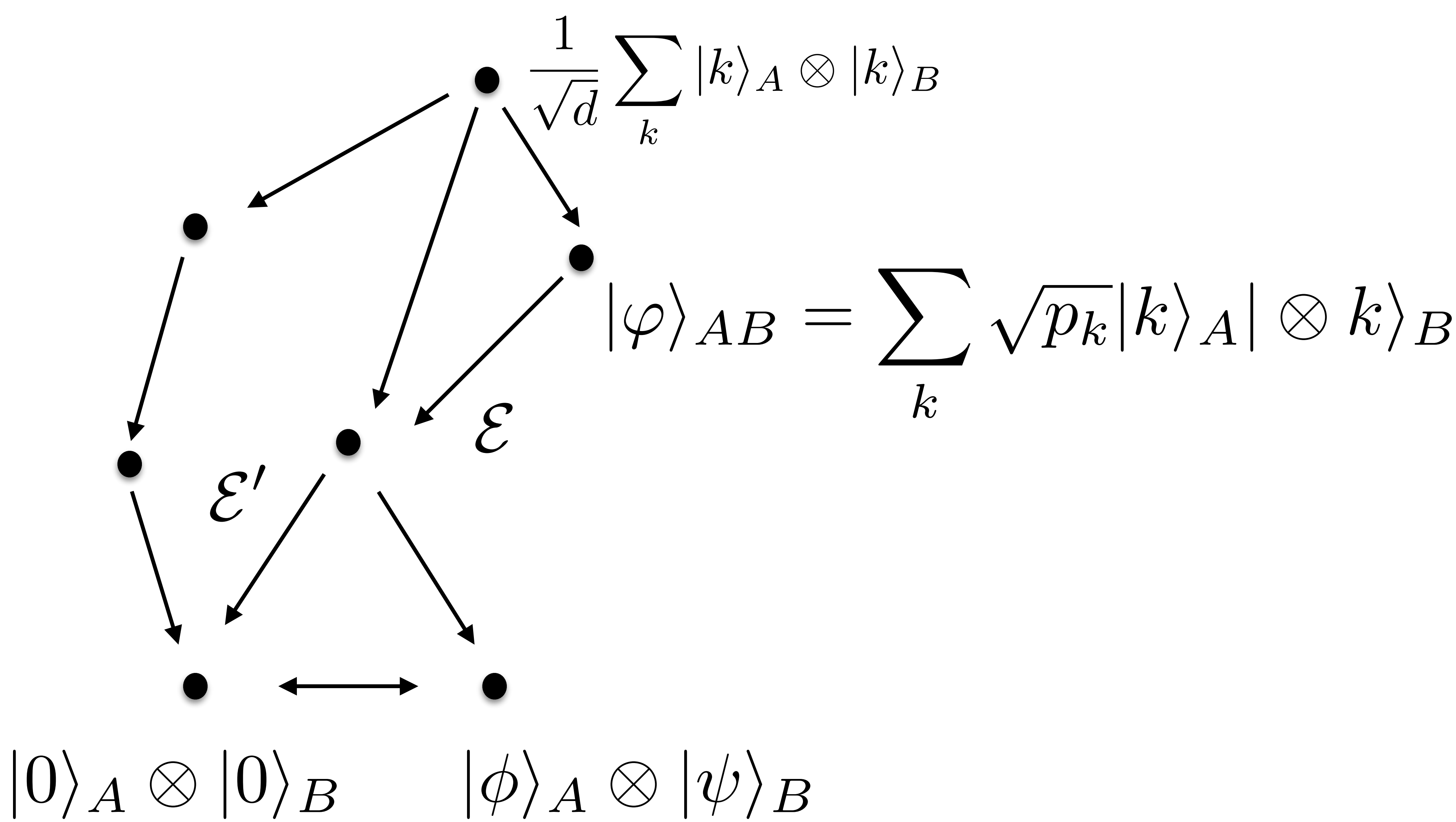
Free operations: Local operations + Classical Communications

Free Operations



LOCC = "Local operations + Classical Communications"

Pure entanglement



Nielsen's Theorem

$$|\psi\rangle_{AB} = \sum_k \sqrt{q_k}|kk\rangle_{AB} \longrightarrow |\phi\rangle_{AB} = \sum_k \sqrt{p_k}|kk\rangle_{AB}$$

under LOCC if and only if $\mathbf{p} \succ_m \mathbf{q}$

$$\mathbf{q} = \text{eigenvalues}(\rho_{\text{A}})$$

"local randomness can never increase"

$|0\rangle_A \otimes |0\rangle_B$: zero entanglement, no local randomness

Structural parallels

	Pure entanglement	**Classical thermo**
free states	product states	equilibrium states
free ops	LOCC	Thermal operations
resource	local noise	ordered energy
task	teleportation/QC	cooling/work

the ***same*** structure of states
(but "inverted"!)

Entanglement distillation & formation

Two basic measures of entanglement:

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

1 unit of entanglement

Distillation $\rho_{AB}^{\otimes N} \longrightarrow |\psi^-\rangle\langle\psi^-|^{\otimes M}$ maximize M

Formation $|\psi^-\rangle\langle\psi^-|^{\otimes M} \to \rho_{AB}^{\otimes N}$ minimize M

Pure asymptotic regime

Theorem: $|\phi\rangle_{AB}^{\otimes N} \longleftrightarrow |\psi^-\rangle_{AB}^{\otimes S(\phi)N} \qquad N \to \infty$

$$S(\phi) = -\mathrm{tr}[\rho_{\mathrm{A}} \log \rho_{\mathrm{A}}] \text{ where } \rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}[\phi_{\mathrm{AB}}]$$

S="The entropy of entanglement"

Uniqueness

For pure, bipartite asymptotic entanglement the entropy of entanglement is the **unique** measure of entanglement that obeys:

1. Monotone under LOCC
2. S(Bell state) = 1 $\qquad E(\psi^-) = 1$
3. Additivity. $\qquad E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$

Uniqueness

For pure, bipartite asymptotic entanglement the entropy of entanglement is the **unique** measure of entanglement that obeys:

Let E be any measure obeying 1-3.

$$|\phi\rangle_{AB}^{\otimes N} \longleftrightarrow |\psi^-\rangle_{AB}^{\otimes S(\phi)N} \qquad \text{(reversibility/Carnot)}$$

$$E[\phi^{\otimes N}] \leq E[(\psi^-)^{S(\rho)N}] \leq E[\phi^{\otimes N}] \qquad \text{(monotone)}$$

$$NE[\phi] \leq S(\phi)NE[(\psi^-)] \leq NE[\phi] \qquad \text{(additive)}$$

$$E[\phi] = S(\phi) \qquad \text{only.} \qquad \blacksquare$$