Scrambling of Quantum Information in Quantum Many-Body Systems

E. Iyoda, T. Sagawa, arXiv:1704.04850

E. Iyoda, H. Katsura, T. Sagawa, in preparation

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Outline

- General background
- Part 1: Scrambling of quantum information
  - Tripartite mutual information and OTOC
  - Our setup and results
- Part 2: “Wishart” Sachdev-Ye-Kitaev model

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Dynamics of isolated quantum systems

Microscopically reversible unitary dynamics of a pure state $\left|\Psi(t)\right\rangle$

→ **Relaxes towards a macroscopically steady state**

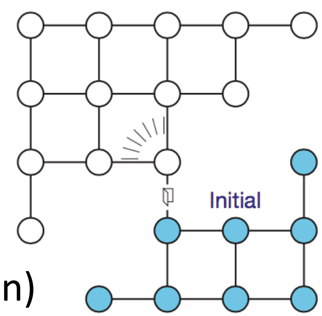
(Recurrence time is very long: almost irreversible!)

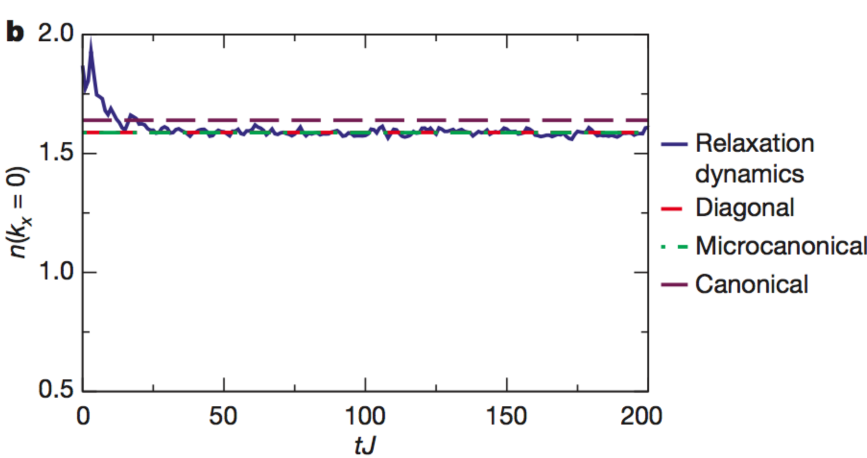
A seminal work by von Neumann in 1929 (arXiv:1003.2133)

Theory:

ex. Hard-core bosons

(numerical exact diagonalization)

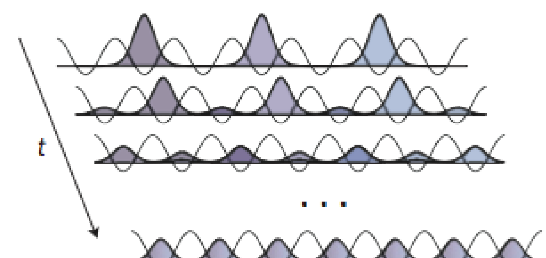


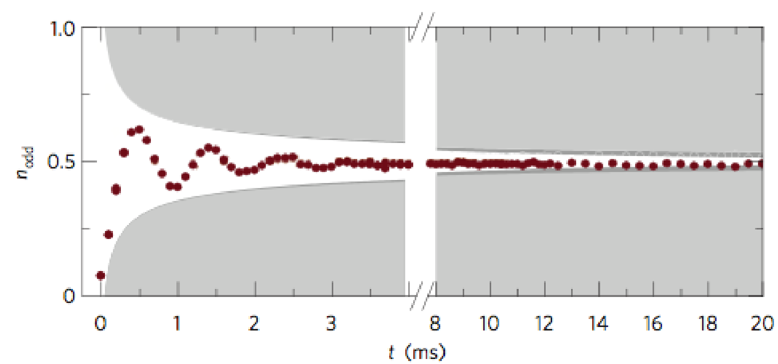


M. Rigol et al., Nature **452**, 854 (2008)

Experiment: Ultracold atoms

ex. 1d Bose-Hubbard, ^{87}Rb





S. Trotzky et al., Nature physics **8**, 325 (2012)

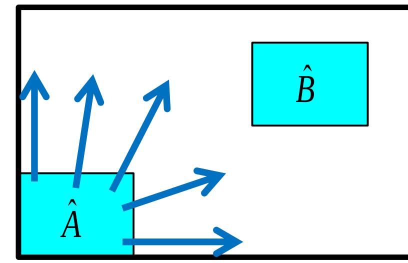
Scrambling of quantum information

How does locally-encoded quantum information spread out?

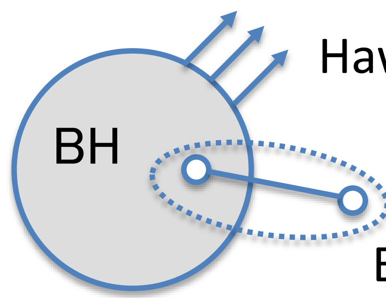
Characterized by out-of-time-ordered correlator (OTOC)

$$\left\langle \hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0)\right\rangle := \mathrm{tr}[\hat{\rho}\hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0)]$$

or **tripartite mutual information (TMI)**



Black hole = "fastest" scrambler?



Sachdev-Ye-Kitaev (SYK) model

$$\hat{H}_{\mathrm{SYK}} = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1}^{N} J_{ij;kl}\hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l$$

Maldacena and Stanford, PRD 94, 106002 (2016)
Sachdev and Ye, PRL 70, 3339 (1993)

- ✓ Believed to be a dual of AdS_2
- ✓ Saturates the upper bound of "Lyapunov exponent"

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Motivation of Part 1

E. Iyoda, T. Sagawa, arXiv:1704.04850

Scrambling = Delocalization of quantum information: "Chaotic"

Quantified by: Decay of out-of-time ordered correlator (OTOC)
Negativity of tripartite mutual information (TMI)

Quantum chaos

Integrability matters!

Wigner-Dyson level statistics and **Eigenstate Thermalization Hypothesis (ETH)** are true only for non-integrable systems

How about scrambling?

- Does integrability matter?
- Relevant to quantum chaos?

Systematic investigation of TMI by numerical exact diagonalization

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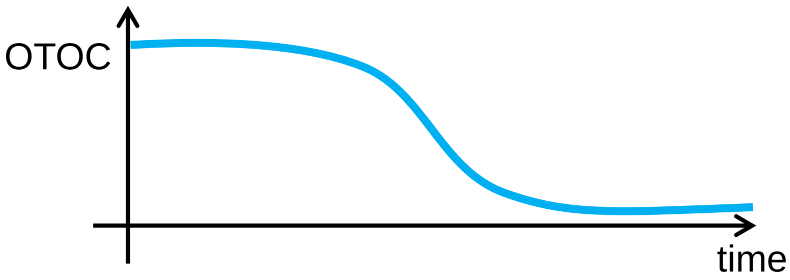
Out-of-time ordered correlator (OTOC)

$$\mathrm{tr}[\hat{\rho}\hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0)]$$



- Early time: "Lyapunov exponent"

  Maldacena, Shenker, Stanford, JHEP 08, 106 (2016)

- Late time: Delocalization of quantum information
  - Some theorems (with *particular* averages of OTOC):

✓ Small OTOC ⇒ Negative tripartite mutual information

Hosur, Qi, Roberts, Yoshida, JHEP02(2016)004

✓ Small OTOC ⇔ Unitary 2-design

(approximation of the Haar random up to the 2nd moment)

Roberts, Yoshida, arXiv:1610.04903

Tripartite mutual information (TMI)

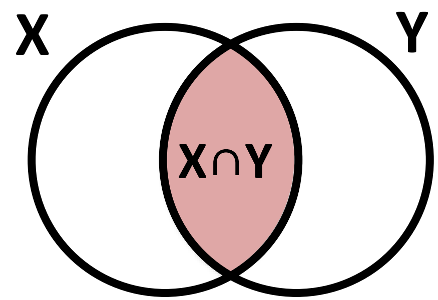
- **von Neumann entropy**

$$S[\hat{\rho}] := \mathrm{tr}[-\hat{\rho}\ln\hat{\rho}] \qquad \hat{\rho}_{\mathrm{X}}$$: Reduced density operator

$$S_{\mathrm{X}} := S[\hat{\rho}_{\mathrm{X}}]$$

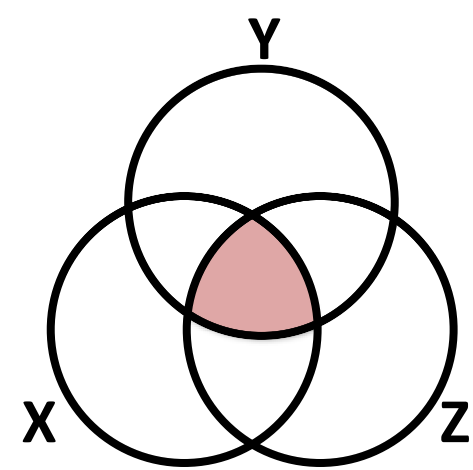
- **Bipartite mutual information (BMI)**

$$I_2(\mathrm{X}:\mathrm{Y}) := S_{\mathrm{X}} + S_{\mathrm{Y}} - S_{\mathrm{XY}}$$



- **Tripartite mutual information (TMI)**

$$\begin{aligned} &I_3(\mathrm{X}:\mathrm{Y}:\mathrm{Z}) \\ :=&S_{\mathrm{X}} + S_{\mathrm{Y}} + S_{\mathrm{Z}} - S_{\mathrm{XY}} - S_{\mathrm{YZ}} - S_{\mathrm{ZX}} + S_{\mathrm{XYZ}} \\ =&I_2(\mathrm{X}:\mathrm{Y}) + I_2(\mathrm{X}:\mathrm{Z}) - I_2(\mathrm{X}:\mathrm{YZ}) \end{aligned}$$



Negativity of TMI

Example: three classical bits

$x = y = z$ x: random ➡ $I_3 = \ln 2$ (three-body correlation)

x, y, z: random, independent ➡ $I_3 = 0$ (no correlation)

$x = y \oplus z$ y, z: random, independent ➡ $I_3 = -\ln 2$

Information about x is delocalized to y and z

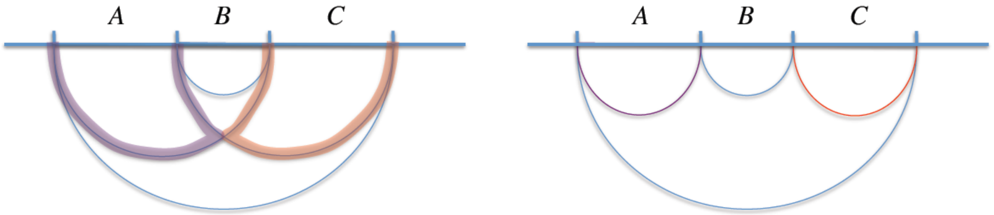
TMI is negative when $I_2(\mathrm{X}:\mathrm{Y}) + I_2(\mathrm{X}:\mathrm{Z}) < I_2(\mathrm{X}:\mathrm{YZ})$

→ Indicates delocalization of quantum information: **Scrambling**

Cf. Ryu-Takayanagi formula

➡ $I_3 \leq 0$

Hayden, Headrick, Maloney, PRD 87, 046003 (2013)



i) $S(AB)+S(BC)+S(AC)$ ii) $S(A)+S(B)+S(C)+S(ABC)$

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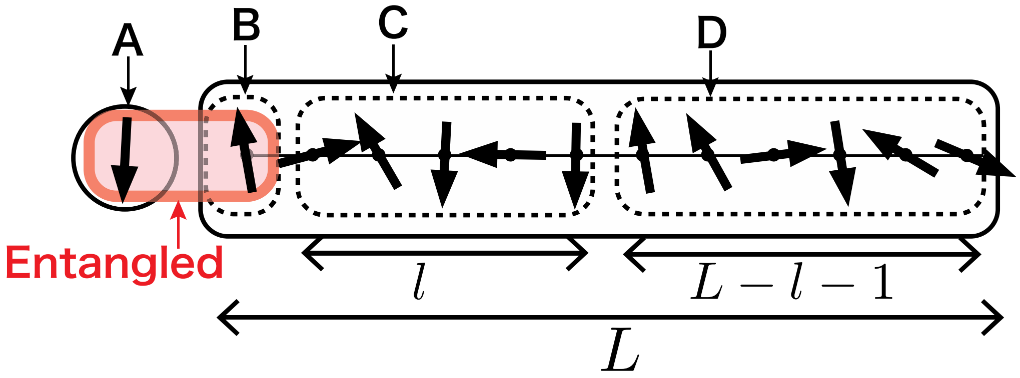
Our setup

Total system :

- **Qubit A**
- **Many-body system BCD**



Prepare $$\frac{1}{\sqrt{2}}(|0\rangle_{\mathrm{A}} + |1\rangle_{\mathrm{A}}) \otimes |\Xi\rangle_{\mathrm{BCD}}$$

$|\Xi\rangle_{\mathrm{BCD}}$: product state (e.g. Néel state, all-up state)

Apply CNOT gate (A: control qubit, B: target qubit)

⇒ Information about A is encoded in B through entanglement

Only BCD evolves with a Hamiltonian

Calculate dynamics of TMI $I_3(\mathrm{A:B:C})$

Preliminary: Bipartite mutual information (BMI)

Non-integrable XXX model

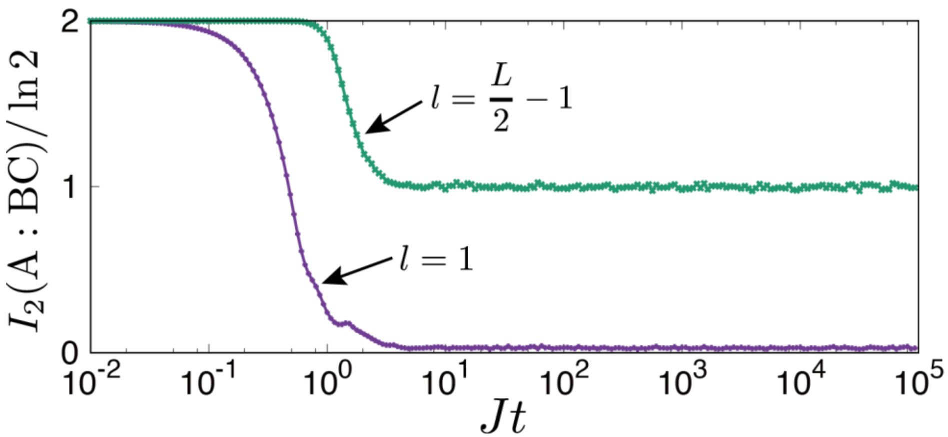
$L = 14,\ J' = 0.8J$

$$H = J \sum_{<ij>} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y + \hat{\sigma}_i^z \hat{\sigma}_j^z) + J' \sum_{<<ij>>} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y + \hat{\sigma}_i^z \hat{\sigma}_j^z)$$

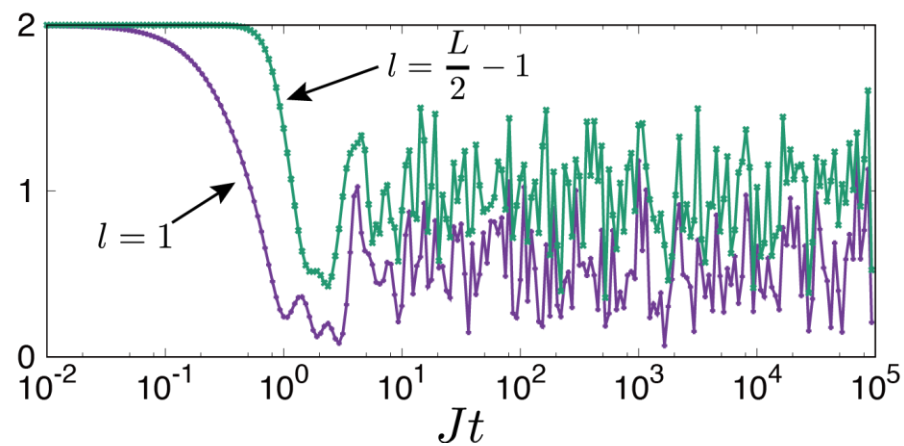
Nearest neighbor

Next nearest neighbor

Néel



All-up



BMI decays, entanglement spreads ballistically

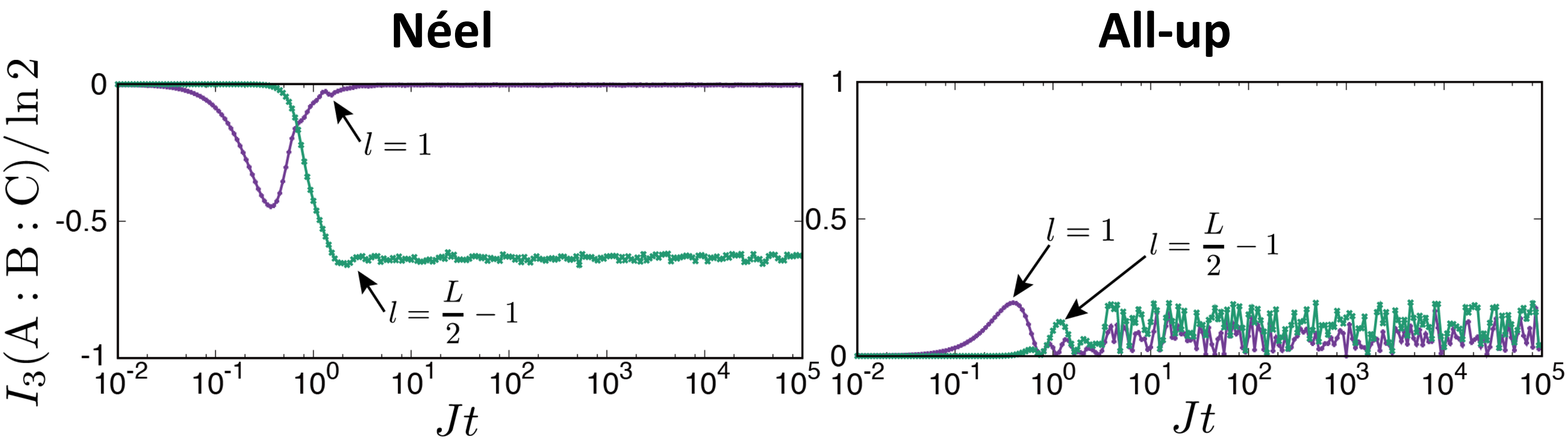
Non-integrable 1d XXX model

$L = 14,\ J' = 0.8J$

$$H = J\sum_{<ij>}(\hat{\sigma}_i^x\hat{\sigma}_j^x + \hat{\sigma}_i^y\hat{\sigma}_j^y + \hat{\sigma}_i^z\hat{\sigma}_j^z) + J'\sum_{<<ij>>}(\hat{\sigma}_i^x\hat{\sigma}_j^x + \hat{\sigma}_i^y\hat{\sigma}_j^y + \hat{\sigma}_i^z\hat{\sigma}_j^z)$$

Nearest neighbor

Next nearest neighbor (n.n.n.)



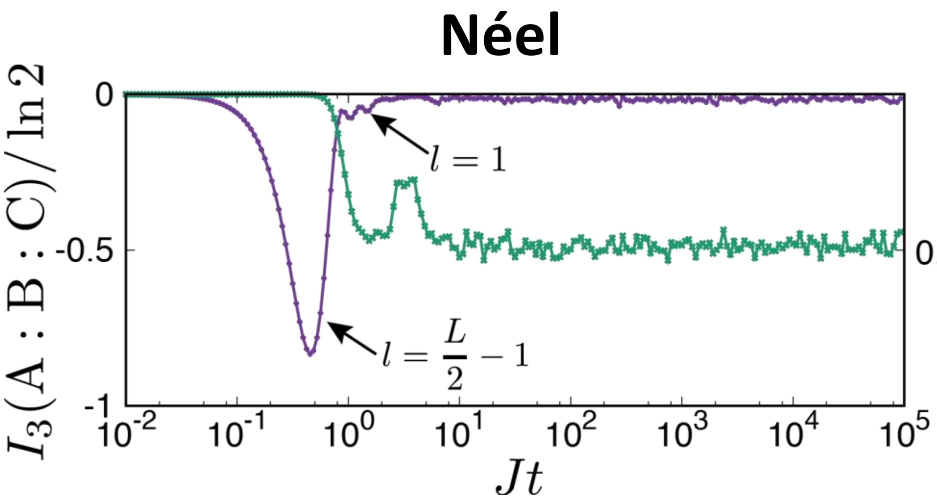
Néel: Negative TMI: **Scrambled**

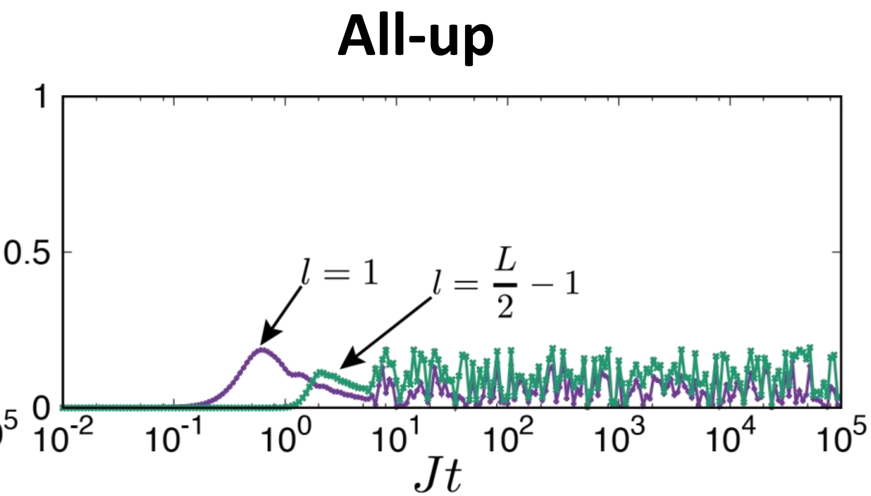
All-up: Positive TMI: **Not scrambled**

Integrable 1d XXX model

$$H = J\sum_{<ij>}(\hat{\sigma}_i^x\hat{\sigma}_j^x + \hat{\sigma}_i^y\hat{\sigma}_j^y + \hat{\sigma}_i^z\hat{\sigma}_j^z)$$

Only nearest neighbor $L = 14$





Néel: Negative TMI: **Scrambled**

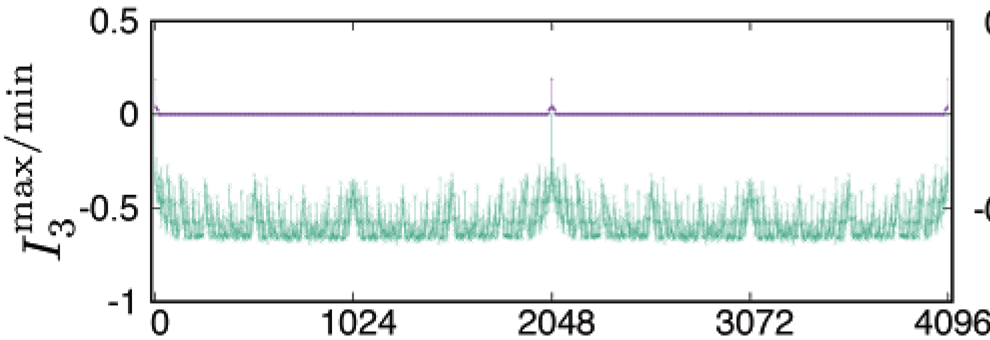
All-up: Positive TMI: **Not scrambled**

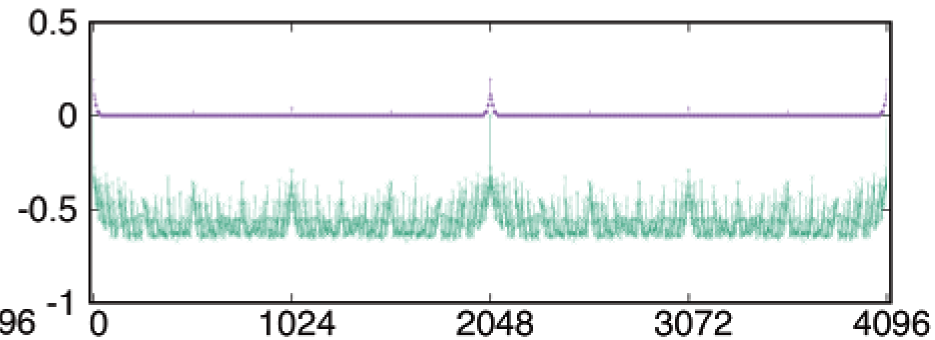
Qualitatively the same as the non-integrable case!

Initial state dependence: XXX model

$L = 12$

Non-integrable; $J' = 0.8J$ **Integrable**





$|\Xi\rangle_{\mathrm{BCD}}$: 2^L product states from $|000\cdots0\rangle$ to $|111\cdots1\rangle$

$I_3^{\mathrm{max/min}}$: max/min of TMI in $0 \le Jt < 10^4$

$I_3^{\mathrm{min}} = 0$ for only four states $|000\cdots0\rangle, |100\cdots0\rangle$ $|011\cdots1\rangle, |111\cdots1\rangle$

Scrambling occurs for most of the initial states

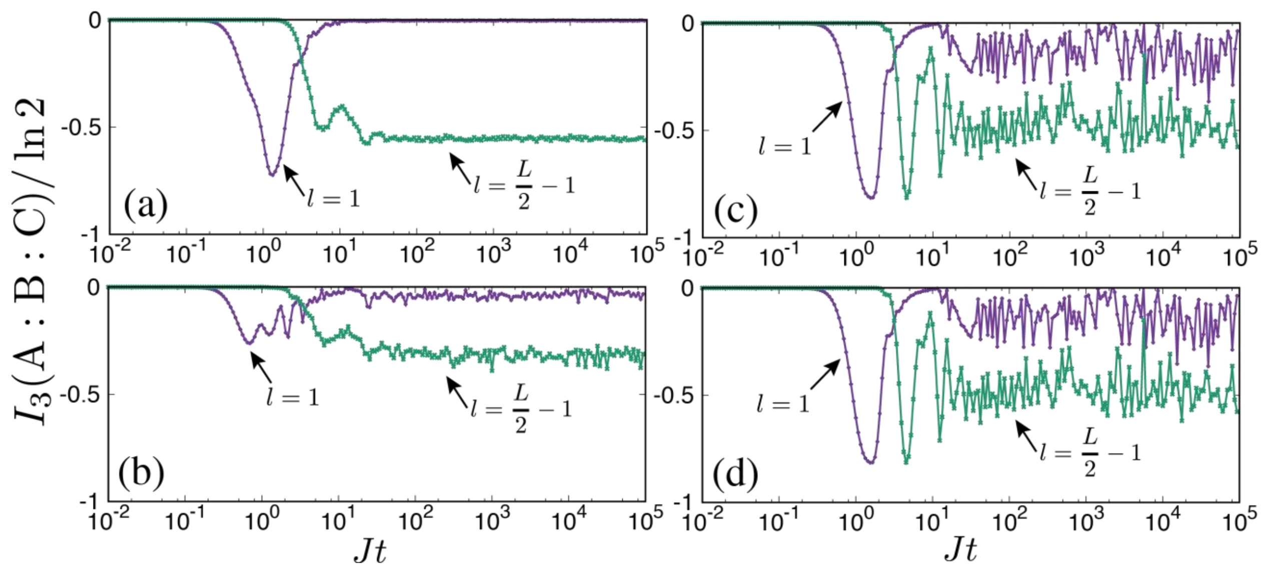
Transverse-field Ising model (TFI) (Integrable/non-integrable)

$$H = J\sum_{<ij>} \hat{\sigma}_i^z \hat{\sigma}_j^z + h_x \sum_i \hat{\sigma}_i^x + h_z \sum_i \hat{\sigma}_i^z$$

Integrable: $h_x = J, h_z = 0$

Non-integrable: $h_x = 2.1J, h_z = 1.1J$

$L = 14$



(a) non-integrable/Néel (b) non-integrable/all-up

(c) integrable/Néel (d) integrable/all-up

Scrambling occurs for all the initial states, both integrable and non-integrable cases

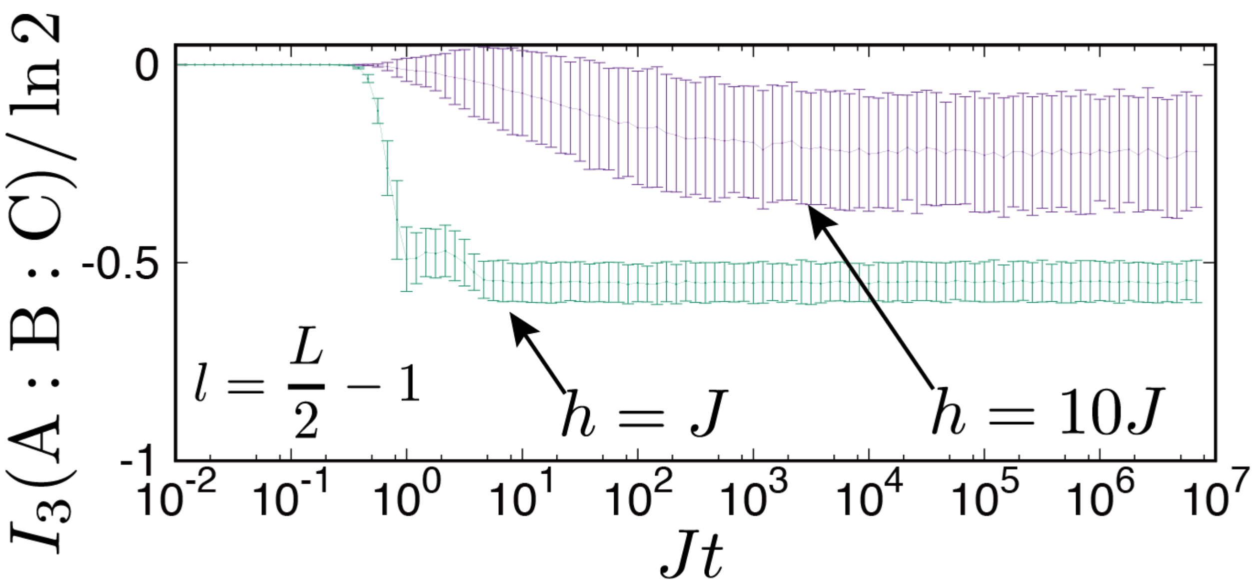
Many-body localization (MBL)

$L = 12$

$$H = J\sum_{<ij>}(\hat{\sigma}_i^x\hat{\sigma}_j^x + \hat{\sigma}_i^y\hat{\sigma}_j^y + \hat{\sigma}_i^z\hat{\sigma}_j^z) + \sum_i h_i\hat{\sigma}_i^z \qquad h_i \in [-h, +h]$$

Nearest neighbor

Random magnetic field



Initial state: Néel

126 samples

Scrambling occurs, but is quite slower in the MBL phase

Consistent with previous results on OTOC:

Y. Huang, Y-L. Zhang, and X. Chen, arXiv:1608.01091
Y. Chen, arXiv:1608.02765
R. Fan, P. Zhang, H. Shen, H. Zhai, arXiv:1608.01914
B. Swingle and D. Chowdhury, arXiv:1608.03280

Sachdev-Ye-Kitaev (SYK) model

Four-body, random, and all-connected interaction of fermions

$$\hat{H}_{\mathrm{SYK}} := \frac{1}{(2L)^{3/2}} \sum_{i,j,k,l} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$$

$c_i^\dagger (c_i)$: creation(anhillation) operator of a fermion

$J_{ij;kl}$: complex Gaussian with variance J^2 satisfying

$$J_{ij;kl} = -J_{ji;kl} = -J_{ij;lk} = J^*_{lk;ji}$$

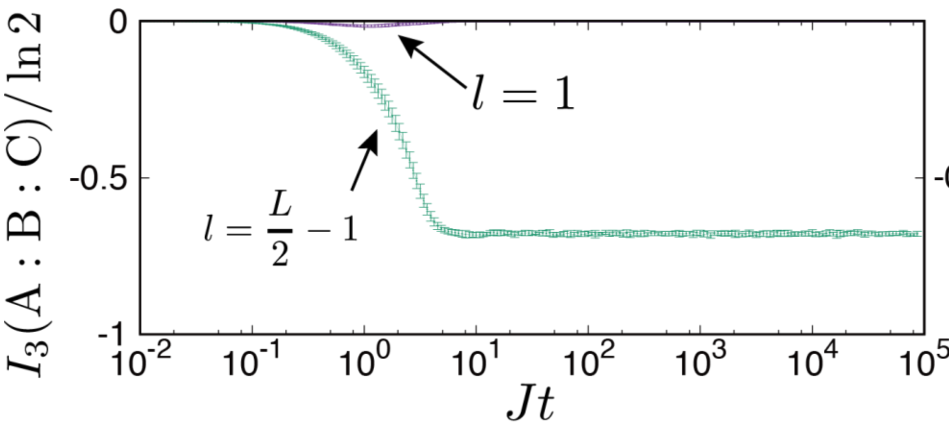
Maldacena and Stanford, PRD 94, 106002 (2016)
Sachdev and Ye, PRL 70, 3339 (1993)

Clean SYK model (for comparison)

$$J_{ij;kl} \equiv J \qquad i > j \quad k > l$$

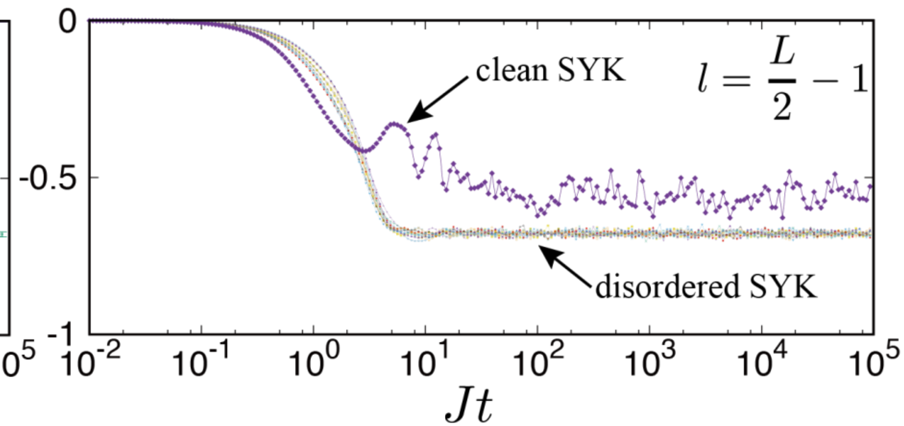
Scrambling of the SYK model

Disordered SYK



Ensemble average over 12 samples

Clean vs disordered



Typical trajectories of the disordered SYK

Initial state: Néel $L = 14$

Disorder does not make scrambling slower, but makes it smoother in the SYK model, in contrast to MBL of spin chains.

Summary of Part 1

E. Iyoda, T. Sagawa, arXiv:1704.04850

	Scrambled ($I_3 < 0$)	Not scrambled ($I_3 > 0$)
Non-integrable	XXX + n.n.n. (Néel) TFI + h_z (Néel,all-up)	XXX + n.n.n. (all-up)
Integrable	XXX (Néel) TFI (Néel,all-up) Clean SYK (Néel)	XXX (all-up) Clean SYK (all-up)
Disordered	XXX + disorder (Néel) SYK (Néel)	XXX + disorder (all-up) SYK (all-up)

Scrambling always occurs independently of integrability, except for a few initial states.

Scrambling is an independent property of the standard notion of quantum chaos.

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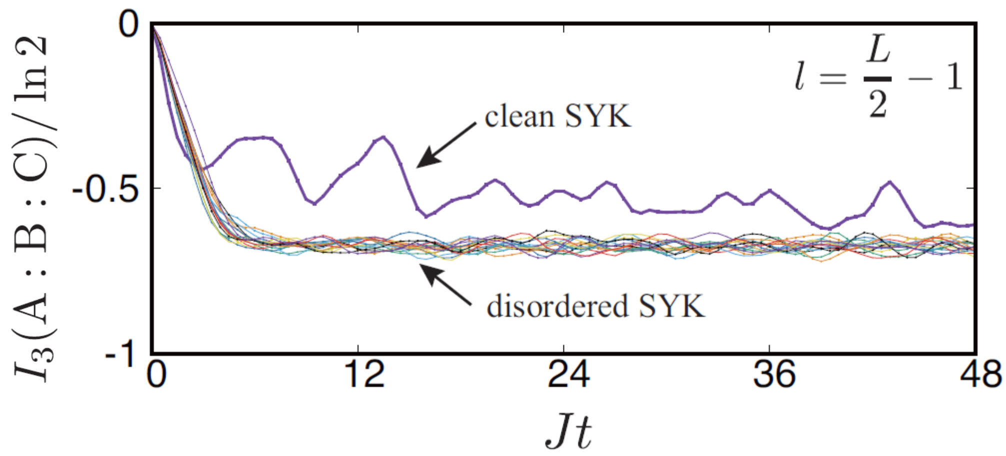
Motivation of Part 2

E. Iyoda, H. Katsura, T. Sagawa, in preparation.



In the SYK model, disorder makes scrambling smoother.

What is the origin of the large temporal fluctuation of the *clean* SYK model?
What is the role of disorder in the SYK model?

SYK model

Maldacena and Stanford, PRD 94, 106002 (2016)
Sachdev and Ye, PRL 70, 3339 (1993)

Four-body, random, and all-connected interaction of fermions

$$H_{\mathrm{SYK}} = \frac{1}{N^{3/2}} \sum_{\substack{1 \le j < i \le N \\ 1 \le k < l \le N}} J_{i,j;k,l} c_i^\dagger c_j^\dagger c_k c_l$$

c_i : complex fermion

$J_{i,j;k,l}$: complex Gaussian distribution with variance J^2

N : the number of fermion modes
N_P : the number of particles (conserved)

- ✓ Believed to be a dual of AdS_2
- ✓ Saturates the upper bound of “Lyapunov exponent”

Extensive zero-energy states in the clean SYK model

$$H_{\mathrm{cSYK}} = \frac{J}{N^{3/2}} \sum_{\substack{1 \le j < i \le N \\ 1 \le k < l \le N}} c_i^\dagger c_j^\dagger c_k c_l \quad \Rightarrow \quad H_{\mathrm{cSYK}} \propto Q^\dagger Q \ge 0$$

$$Q := \sum_{1 \le k < l \le N} c_k c_l$$

Q maps the (N_P+2)-particle subspace to the N_P-particle subspace

Below half-filling, the difference of their dimensions is the number of zero-energy states!

The total number is $\binom{N+1}{[(N+1)/2]}$ $\Rightarrow$ Extensive residual entropy

Expectation: This huge degeneracy leads to the large temporal fluctuation in scrambling dynamics.

"Wishart" SYK model

$$H_{\mathrm{wSYK}} = Q^\dagger Q$$

$$Q := \frac{1}{N} \sum_{1 \le k < l \le N} J_{k,l} c_k c_l$$

$J_{k,l}$: complex Gaussian distribution with variance J^2

(The name "Wishart" comes from the Wishart matrix in the random matrix theory)

The clean SYK model is a special case of the Wishart SYK model

Cf. $\mathcal{N}$=2 SUSY SYK

Fu, Gaiotto, Maldacena, Sachdev, PRD 95, 026009 (2017)
Kanazawa, Wettig, arXiv:1706.03044

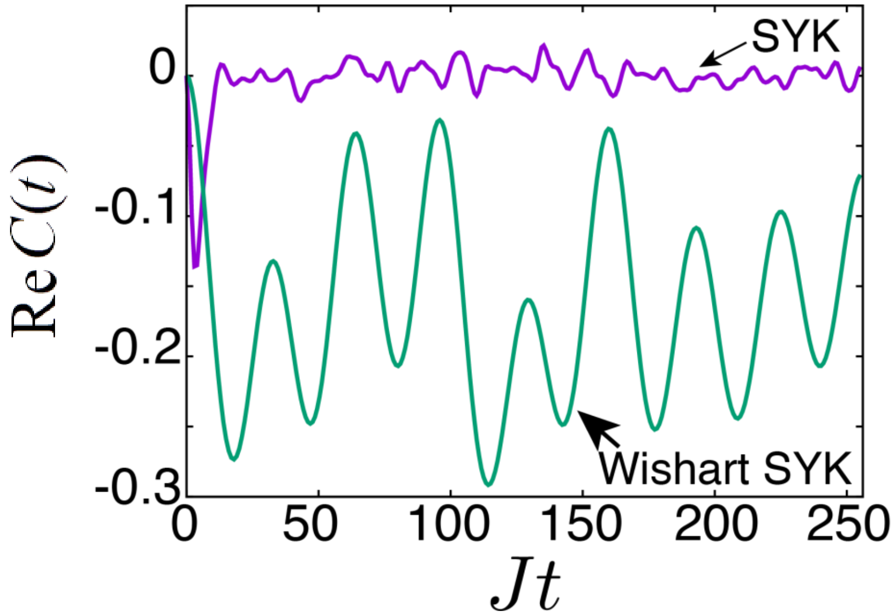
$$H_{\text{SUSY-SYK}} = Q^\dagger Q + Q Q^\dagger \qquad Q := \frac{i}{N^2} \sum_{1 \le k < l < m \le N} J_{k,l,m} c_k c_l c_m \qquad Q^2 = 0$$

Large temporal fluctuation

$$H_{\mathrm{wSYK}} = Q^{\dagger} Q \qquad Q := \frac{1}{N} \sum_{1 \le k < l \le N} J_{k,l} c_k c_l$$

The Wishart SYK model has the same number of zero-energy states as the clean SYK model

OTOC exhibits large temporal fluctuations!



$N = 12$

Temporal fluctuation and effective dimension (1/2)

OTOC $C(t) := \langle \Psi | c_0(t) c_0^\dagger(0) c_0^\dagger(t) c_0(0) | \Psi \rangle$

Time average $\overline{C} := \lim_{T\to\infty} \frac{1}{T} \int_0^T C(t)dt$

Temporal fluctuation $(\Delta C)^2 := \overline{\left(\mathrm{Re}\, C(t) - \mathrm{Re}\, \overline{C}\right)^2}$

Effective dimension $D_{\mathrm{eff}} := \left(\sum_i \langle \Psi | P_i | \Psi \rangle^2 \right)^{-1}$ $\quad H = \sum_i E_i P_i$

$E_i \neq E_j \;\; \text{for } i \neq j$

$1 \leq D_{\mathrm{eff}} \leq D$

Large D_{eff} ➡ Small temporal fluctuation of $\langle \Psi | A(t) | \Psi \rangle$

Reimann, PRL 101, 190403 (2008)
Short, Farrelly, New J. Phys. 14 013063 (2012)

Such a theorem is not know for OTOC,
but qualitatively the same relation is expected.

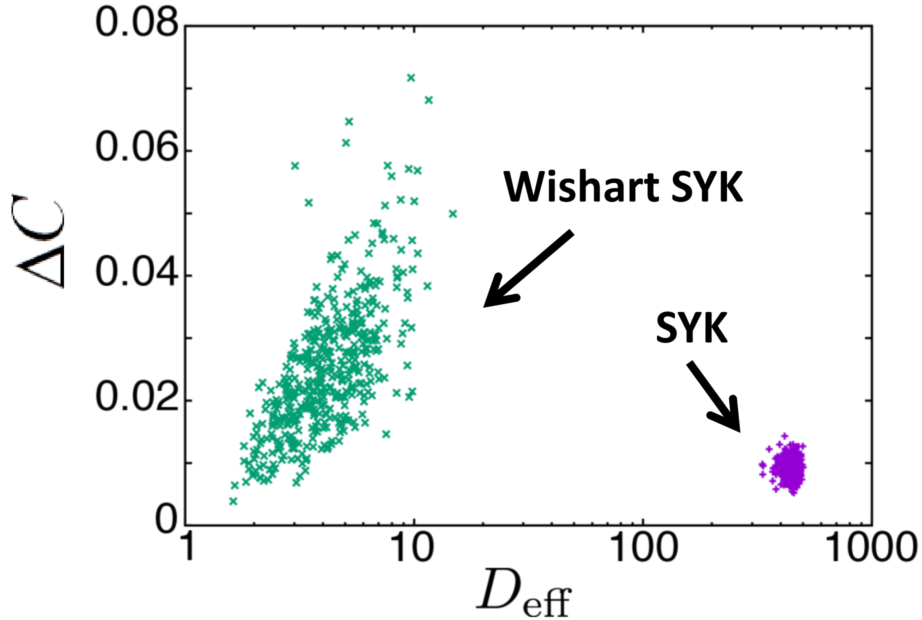
Temporal fluctuation and effective dimension (2/2)

Expected scenario for the Wishart (and clean) SYK model:

Extensive zero-energy states (huge degeneracy)

➡ Small D_{eff} ➡ Large temporal fluctuation of OTOC

We have confirmed this expectation numerically, by calculating all the states of the computational basis:



$N = 12$

Integrability of the Wishart SYK model (1/2)

> The Wishart (and thus clean) SYK model is integrable by the algebraic Bethe ansatz!

$$H_{\mathrm{wSYK}} = Q^\dagger Q \qquad Q := \frac{1}{N}\sum_{1\le k<l\le N} J_{k,l} c_k c_l = \frac{1}{2N}\sum_{1\le k,l\le N} J_{k,l} c_k c_l$$

For simplicity, we assume that N is even and $J_{k,l}$ is a real skew symmetric matrix.

Canonical form:

$$O^{\mathrm{T}} J O = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_{N/2} \\ & & & -\lambda_{N/2} & 0 \end{pmatrix}$$

New fermion operators:

$$\left(f_{1,\uparrow}, f_{1,\downarrow}, \cdots, f_{N/2,\uparrow}, f_{N/2,\downarrow}\right) := \left(c_1, c_2, \cdots, c_{N-1}, c_N\right) O$$

O: an orthogonal matrix

$$\Rightarrow \quad Q = \frac{1}{N}\sum_{j=1}^{N/2} \lambda_j f_{j,\uparrow} f_{j,\downarrow}$$

Integrability of the Wishart SYK model (2/2)

The Wishart SYK Hamiltonian then reduces to

$$H_{\mathrm{wSYK}} = \frac{1}{N^2}\left(\sum_{j=1}^{N/2} \lambda_j f_{j,\downarrow}^{\dagger} f_{j,\uparrow}^{\dagger}\right)\left(\sum_{j=1}^{N/2} \lambda_j f_{j,\uparrow} f_{j,\downarrow}\right)$$

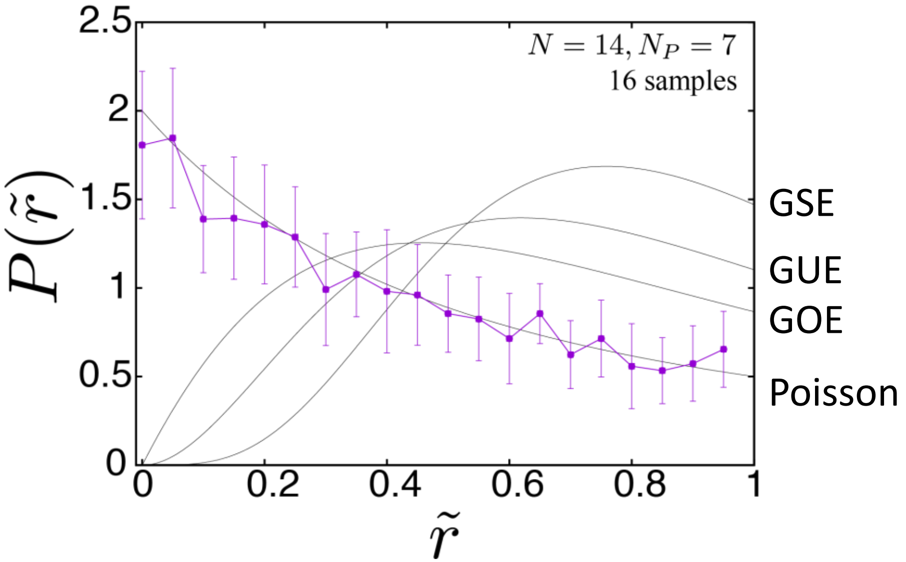
This is equivalent to **the Richardson-Gaudin model** (degenerate case), which is integrable by the algebraic Bethe ansatz.

Review: Y. Pehlivan, arXiv:0806.1810

Level statistics is Poisson:

$$\tilde{r}_i := \min(r_i, 1/r_i)$$

$$r_i := \frac{E_{i+1} - E_i}{E_i - E_{i-1}}$$



Summary of Part 2

E. Iyoda, H. Katsura, T. Sagawa, in preparation.

- Introduced the Wishart SYK model, which includes the clean SYK model as a special case.

$$H_{\mathrm{wSYK}} = Q^{\dagger} Q \qquad Q := \frac{1}{N} \sum_{1 \le k < l \le N} J_{k,l} c_k c_l$$

- Wishart SYK model has the extensive number of zero-energy states, which leads to large temporal fluctuations of OTOC.

- Wishart SYK model can be mapped to the Richardson-Gaudin model, which is integrable by the algebraic Bethe ansatz.

Thank you for your attention!

Part 1: Scrambling of quantum information

Scrambling is an independent property of integrability (and thus the standard notion of quantum chaos)

E. Iyoda, T. Sagawa, arXiv:1704.04850

Part 2: Wishart SYK model

Introduced a new variant of the SYK model, which exhibits large temporal fluctuations in scrambling dynamics

E. Iyoda, H. Katsura, T. Sagawa, in preparation