

# The Nash problem on families of arcs



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## The arc space of a variety

Let  $X$  be an algebraic variety over  $\mathbb{C}$ .

The **arc space**  $X_\infty$  of  $X$  is a scheme whose  $K$ -valued points are **arcs**

$$\alpha: \text{Spec } K[[t]] \rightarrow X$$

$$0 \in \text{Spec } K[[t]] \text{ closed point} \quad \eta \in \text{Spec } K[[t]] \text{ generic point}$$

Canonical projection

$$\pi = \pi_X: X_\infty \rightarrow X, \quad \alpha(t) \mapsto \alpha(0)$$

*Note:* Every arc  $\alpha$  defines a **valuation**

$$\text{val}_\alpha: \mathcal{O}_{X, \alpha(0)}^* \rightarrow K[[t]]^* \rightarrow \mathbb{Z}$$

If  $\alpha(\eta) \in X$  is the generic point of  $X$ , the valuation extends to  $X$

$$\text{val}_\alpha: \mathbb{C}(X)^* \rightarrow K((t))^* \rightarrow \mathbb{Z}$$

## Example

If  $X = \{f(x_1, \dots, x_n) = 0\} \subset \mathbb{A}^n$ , then  $X_\infty$  parametrizes  $n$ -ples of formal power series  $(x_1(t), \dots, x_n(t))$  such that

$$f(x_1(t), \dots, x_n(t)) \equiv 0$$

This condition describes  $X_\infty$  as a subscheme in an infinite dimensional affine space (the arc space of  $\mathbb{A}^n$ ), defined by *infinitely* many equations in *infinitely* many variables.

In general,  $X_\infty$  is not Noetherian and is not of finite type.

$X_\infty$  can also be described as the inverse limit of the **jet schemes** of  $X$ :

$$X_\infty = \varprojlim X_m, \quad \text{where} \quad X_m(K) = \{\gamma: \text{Spec } K[t]/t^{m+1} \rightarrow X\}$$

If  $X$  is **smooth**, then  $X_\infty$  is the projective limit of a system of affine bundles, and hence it is irreducible.

(In fact,  $\pi_X^{-1}(S)$  is irreducible for every irreducible set  $S \subset X$ .)

If  $X$  is **singular**, we take a resolution of singularities  $f: Y \rightarrow X$

$$\begin{array}{ccc} Y_\infty & \xrightarrow{f_\infty} & X_\infty \\ \pi_Y \downarrow & & \downarrow \pi_X \\ Y & \xrightarrow{f} & X \end{array}$$

By valuative criterion of properness and generic smoothness,  $f_\infty$  is dominant.

### Theorem (Kolchin 1973)

*The arc space  $X_\infty$  of any variety  $X$  is irreducible.*

*Note:*  $\pi_X^{-1}(S)$  may be reducible for an irreducible set  $S \subset X$ .

# Nash's Theorem

## Theorem (Nash 1968)

The set of arcs through the singular locus of  $X$  has a *finite decomposition* into irreducible components:

$$\pi_X^{-1}(\text{Sing } X) = \bigcup_{i \in I} C_i \quad \text{where } I \text{ is finite}$$

*Proof.* For every irreducible component  $E$  of  $f^{-1}(\text{Sing } X)$ , let

$$\pi_Y^{-1}(E) \subset Y_\infty \xrightarrow{f_\infty} X_\infty \supset f_\infty(\pi_Y^{-1}(E)) = N_E$$

Then

$$f^{-1}(\text{Sing } X) = \bigcup_{i \in J} E_i \quad \Rightarrow \quad \pi_X^{-1}(\text{Sing } X) = \bigcup_{i \in I} \overline{N_{E_i}}$$

where  $J$  is finite and  $I \subset J$ .

To every irreducible component  $C_i$  of  $\pi_X^{-1}(\text{Sing } X)$  there corresponds a valuation  $\text{val}_{\alpha_i}$  on  $X$ , defined by the generic point  $\alpha_i \in C_i$ .

- We call any valuation defined in this way a **Nash valuation** over  $X$ .

*Note:* Each Nash valuation is a (distinct) divisorial valuation: by further blowing up, we may assume that each  $E_i$  is a divisor, and we have

$$\text{val}_{\alpha_i} = \text{val}_{\tilde{\alpha}_i} = \text{val}_{E_i} \quad (i \in I)$$

Nash valuations are intrinsic to  $X$ .

### Question

Which divisorial valuations occur as Nash valuation?

# The Nash Problem

- A divisorial valuation  $\text{val}_E$  is **essential** over  $X$  if its center on every resolution  $g: X' \rightarrow X$  is an irreducible component of  $g^{-1}(\text{Sing } X)$ .

*Note:* Every Nash valuation over  $X$  is essential.

Define the **Nash map**  $\mathcal{N}_X$  by the diagram

$$\begin{array}{ccc} \{\text{components of } \pi^{-1}(\text{Sing } X)\} & & \\ \downarrow 1-1 & \searrow \mathcal{N}_X & \\ \{\text{Nash valuations over } X\} & \hookrightarrow & \{\text{essential valuations over } X\} \end{array}$$

By construction, the Nash map is injective.

## Nash Problem (1968)

Is the Nash map surjective?

## Results on the Nash Problem

- If  $\dim X = 2$ , then  $\text{val}_E$  is essential over  $X$  if and only if  $E$  is an exceptional divisor in the *minimal resolution* of singularities of  $X$ .

The 2-dimensional Nash problem asks if every such valuation  $\text{val}_E$  is a Nash valuation, i.e., comes from a maximal family of arcs through the singularities of  $X$ . The complexity of the equations of  $X_\infty$  makes the problem very hard, by a direct approach, even for relatively simple singularities.

**Theorem (Fernandez de Bobadilla and Pe Pereira 2012)**

*In dimension 2, the Nash map is surjective.*

The proof is a combination of a standard approach via a *Curve Selection Lemma* (Lejeune-Jalabert, Reguera) with new **topological methods**.



- If  $\dim X \geq 3$ , then the notion of essential valuation is more subtle ( $\nexists$  minimal resolutions,  $\exists$  small resolutions,...)

In principle, one needs to check the condition on the center of the valuation on *every* resolution of singularities.

**Theorem (Ishii–Kollár 2003, de Fernex 2013, Johnson–Kollár 2013)**

*There are examples in all dimensions  $\geq 3$  where the Nash map is not onto.*

Easiest examples are 3-dimensional  $cA_1$ -singularities:

$$X : x^2 + y^2 + z^2 + t^m = 0 \quad \text{where } m \geq 5 \text{ is odd}$$

Other examples show that the notion of **essential valuation** depends on the topology (Zariski vs analytic): in such an example the Nash map is surjective in the analytic category, but not in the algebraic category.

## Nash Problem (revisited)

Describe the image of the Nash map.

We are looking for additional conditions (apart from being essential) that ensure that a valuation is a Nash valuation.

- A divisorial valuation  $\text{val}_E$  is a **terminal valuation** over  $X$  if  $E$  is an exceptional divisor on a **minimal model**  $Y \rightarrow X$ .
  - ▶  $Y$  has terminal singularities
  - ▶  $K_Y$  is relatively nef over  $X$

*Note:* The set of terminal valuations corresponds to the set of exceptional divisors on *any given* minimal model  $Y \rightarrow X$ .

## Theorem (de Fernex and Docampo 2014)

*Every terminal valuation over  $X$  is a Nash valuation.*

We obtain inclusions (both may be strict)

$$\{\text{terminal valuations}\} \subset \{\text{Nash valuations}\} \subset \{\text{essential valuations}\}$$

This gives a new (purely algebro-geometric) proof of the theorem of Fernandez de Bobadilla and Pe Pereira.

Also get new examples where the Nash map is onto (e.g., when  $X$  admits a divisorial crepant resolution).

## Sketch of the proof

Theorem (de Fernex and Docampo 2014)

*Every terminal valuation over  $X$  is a Nash valuation.*

*Proof by contradiction:* Suppose there is a terminal valuation that is not a Nash valuation.

Then there is an exceptional divisor  $E$  on a minimal model  $f: Y \rightarrow X$  such that

$\overline{N_E} \subset \pi_X^{-1}(\text{Sing } X)$  is *not* an irreducible component (\*)

$$\begin{array}{ccc} \pi_Y^{-1}(E) \subset Y_\infty & \xrightarrow{f_\infty} & X_\infty \supset N_E = f_\infty(\pi_Y^{-1}(E)) \\ \pi_Y \downarrow & & \downarrow \pi_X \\ E \subset Y & \xrightarrow{f} & X \end{array}$$

$$\overline{N_E} \subset \pi_X^{-1}(\text{Sing } X)$$

Fix a very general point  $p \in E \subset Y$  of dimension  $n - 2$ , where  $n = \dim X$ .

### Curve Selection Lemma (Reguera 2006 + specialization)

$\exists$  finite extension  $K/\kappa(p)$

$\exists \Phi: \text{Spec } K[[s]] \rightarrow X_\infty$  (an arc on the arc space)

such that

- $\Phi(0) = \alpha_0 \in N_E$  is an arc on  $X$  with  $\tilde{\alpha}_0(0) = p$  and  $\text{ord}_{\tilde{\alpha}_0}(E) = 1$
- $\Phi(\eta) = \alpha_\eta \in \pi_X^{-1}(\text{Sing } X) \setminus \overline{N_E}$

Can regard  $\Phi$  as a *1-parameter family of arcs* on  $X$ , or as a map

$$\Phi: S = \text{Spec } K[[s, t]] \rightarrow X, \quad \Phi(s, t) =: \alpha_s(t)$$

We say that  $\Phi$  is a **wedge** on  $X$ .

*Note:* The curve selection lemma is a simple fact for Noetherian schemes, but it can fail in the non Noetherian setting.

### Example (The Whitney umbrella)

Let  $X = \{xy^2 = z^2\} \subset \mathbb{A}^3$  and

- $M = \pi_X^{-1}(0) \subset X_\infty$
- $N = \{(x(t), 0, 0) \in X_\infty \mid \text{ord}(x(t)) \geq 1\} \subset M$
- $\alpha = (t, 0, 0) \in N$

Then  $\exists \Phi: \text{Spec } \mathbb{C}[[s]] \rightarrow M$  with  $\Phi(0) = \alpha$  and  $\Phi(\eta) \in M \setminus N$ .

*Proof:* Suppose  $\Phi(\eta) \in M \setminus N$ . Then

$$\Phi(\eta) = (x_\eta(t), y_\eta(t), z_\eta(t)) \quad \text{with} \quad y_\eta(t), z_\eta(t) \neq 0$$

$$\text{odd} \quad \text{ord}(x_\eta(t)y_\eta(t)^2) = \text{ord}(z_\eta(t)^2) \quad \text{even}$$

since  $\text{ord}(x_\eta(t)) = 1$  by semicontinuity

Back to the proof: Have a wedge  $\Phi: \text{Spec } K[[s]] \rightarrow X_\infty$  such that

- (1)  $\Phi(0) = \alpha_0 \in N_E$ , with  $\tilde{\alpha}_0(0) = p$  and  $\text{ord}_{\tilde{\alpha}_0}(E) = 1$
- (2)  $\Phi(\eta) = \alpha_\eta \in \pi_X^{-1}(\text{Sing } X) \setminus \overline{N_E}$

$$\Phi: S = \text{Spec } K[[s, t]] \rightarrow X$$

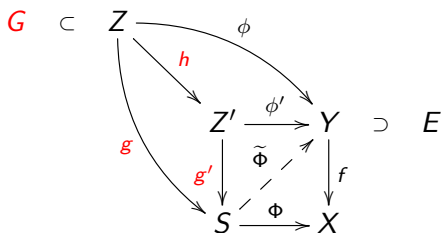
Note that  $S$  is a 2-dimensional scheme, but has 'dimension'  $n$  over  $\mathbb{C}$  ( $\text{tr.deg}(K/\mathbb{C}) = n - 2$ ).

Conditions (1) and (2) imply that the lift  $\tilde{\Phi}$  is not well-defined:

A commutative diagram with nodes  $S$ ,  $X$ , and  $Y$ . A solid arrow points from  $S$  to  $X$  and is labeled  $\Phi$ . A solid arrow points from  $X$  to  $Y$  and is labeled  $f$ . A dashed arrow points from  $S$  to  $Y$  and is labeled  $\tilde{\Phi}$  in red.

We take a resolution of indeterminacy of  $\tilde{\Phi}$ .

Two natural ways to resolve indeterminacies:



- $g'$  is the normalized blowup of the indeterminacy ideal of  $\tilde{\phi}$
- $g$  is the minimal sequence of point-blowups resolving indeterminacies
- $h$  contracts any  $g$ -exceptional curve that is contracted to a point by  $\phi$
- $G$  is the  $g$ -exc divisor intersecting the (proper transform of the)  $t$ -axis

Note:

- $Z$  is smooth and  $Z'$  has rational singularities, hence is  $\mathbb{Q}$ -factorial
- $\phi: Z \rightarrow Y$  is dominant and  $p \in \phi(G) \subset E$



Define **relative canonical divisors**

- $K_{Z/S}$  the relative canonical divisor of  $g: Z \rightarrow S$
- $K_{Z'/S}$  the relative canonical divisor of  $g': Z' \rightarrow S$
- $K_{Z'/Y}$  the relative canonical divisor of  $\phi': Z' \rightarrow Y$

Recall that  $f: Y \rightarrow X$  is a *minimal model* over  $X$ :

$$Y \text{ terminal singularities} \quad \Rightarrow \quad K_{Z'/Y} \geq 0$$

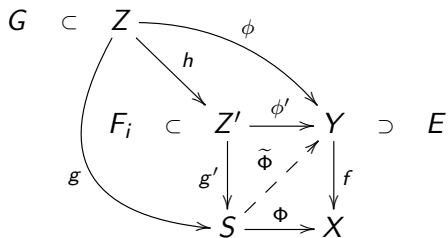
$$K_Y \text{ } f\text{-nef} \quad \Rightarrow \quad (\phi')^* K_Y \text{ } g'\text{-nef}$$

Decompose

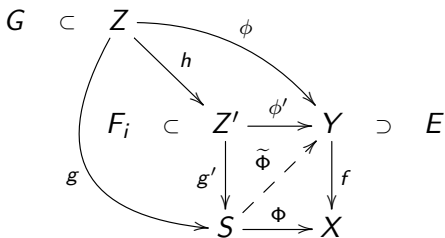
$$K_{Z'/Y} = K_{Z'/Y}^{g'\text{-exc}} + K_{Z'/Y}^{g'\text{-hor}}$$

Let  $F_1, \dots, F_m$  be the components of  $\text{Ex}(g')$  containing  $h(G) \subset Z'$ .

*Note:* Each  $F_i$  dominates  $E$  via  $\phi': Z' \rightarrow Y$ .

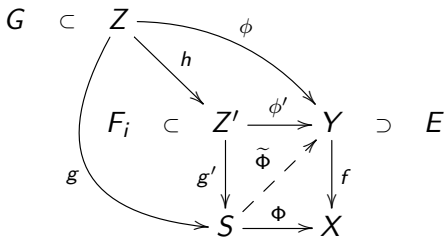


$$1 \leq \text{ord}_G(K_{Z/S}) \leq \text{ord}_G(h^*K_{Z'/S}) \leq \text{ord}_G(h^*K_{Z'/Y}^{g'-\text{exc}}) < \text{ord}_G(\phi^*E) = 1$$



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because  $S$  is smooth and  $G$  is  $g$ -exceptional.

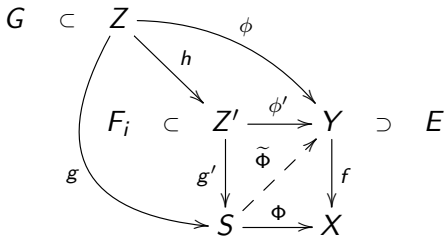


$$1 \leq \text{ord}_G(K_{Z/S}) \leq \text{ord}_G(h^*K_{Z'/S}) \leq \text{ord}_G(h^*K_{Z'/Y}^{g'-\text{exc}}) < \text{ord}_G(\phi^*E) = 1$$

because

$$K_{Z/S} - K_{Z'/S} = K_{Z/Z'}$$

is  $h$ -nef and  $h$ -exceptional ( $h$  is the minimal resolution)  
hence  $\leq 0$  (by Hodge Index Theorem).

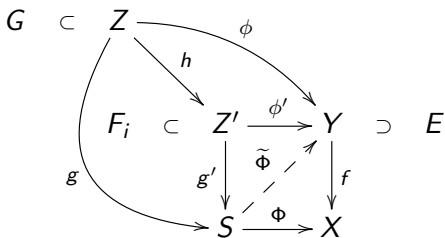


$$1 \leq \text{ord}_G(K_{Z/S}) \leq \text{ord}_G(h^*K_{Z'/S}) \leq \text{ord}_G(h^*K_{Z'/Y}^{g'-\text{exc}}) < \text{ord}_G(\phi^*E) = 1$$

because

$$K_{Z'/S} - K_{Z'/Y}^{g'-\text{exc}} \sim K_{Z'} - K_{Z'/Y} + K_{Z'/Y}^{g'-\text{hor}} \sim (\phi')^*K_Y + K_{Z'/Y}^{g'-\text{hor}}$$

is  $g'$ -nef and  $g'$ -exceptional ( $Y$  is a minimal model)  
hence  $\leq 0$  (by Hodge Index Theorem).



$$1 \leq \text{ord}_G(K_{Z/S}) \leq \text{ord}_G(h^*K_{Z'/S}) \leq \text{ord}_G(h^*K_{Z'/Y}^{g'-\text{exc}}) < \text{ord}_G(\phi^*E) = 1$$

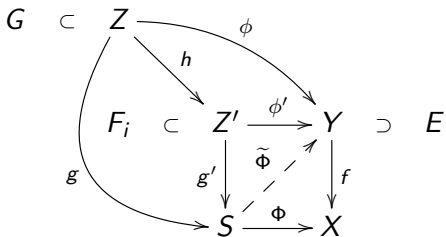
because

$$\text{ord}_{F_i}(K_{Z'/Y}) = \text{ord}_{F_i}((\phi')^*E) - 1 \quad \forall i$$

by Hurwitz-type computation, hence

$$\text{ord}_G(h^*K_{Z'/Y}) < \text{ord}_G(\phi^*E)$$

since  $Z'$  is  $\mathbb{Q}$ -factorial.



$$1 \leq \text{ord}_G(K_{Z/S}) \leq \text{ord}_G(h^*K_{Z'/S}) \leq \text{ord}_G(h^*K_{Z'/Y}^{g'-\text{exc}}) < \text{ord}_G(\phi^*E) = 1$$

because  $\phi$  maps the  $t$ -axis to an arc  $\tilde{\alpha}_0$  on  $Y$  with

$$\text{ord}_{\tilde{\alpha}_0}(E) = 1.$$

## Closing remarks

(1) **Minimal valuations** over  $X$  (in the valuative sense) are also examples of Nash valuations:

if  $\beta$  is a specialization of  $\alpha$  in  $X_\infty$ , then  $\text{val}_\beta \geq \text{val}_\alpha$ .

- For *surface singularities*:

$$\{\text{minimal}\} \subset \{\text{terminal}\} = \{\text{Nash}\} = \{\text{essential}\}$$

- For *toric singularities* (Ishii–Kollár 2003):

$$\{\text{terminal}\} \subset \{\text{minimal}\} = \{\text{Nash}\} = \{\text{essential}\}$$

(both sets can be described combinatorically)

### Question

For which class of varieties one has

$$\{\text{minimal valuations}\} \cup \{\text{terminal valuations}\} = \{\text{Nash valuations}\} ?$$



(2) There are no terminal valuations if  $X$  has terminal singularities

### Problem

Study Nash valuations over terminal singularities.

(3) Most of the proof extends to **positive characteristics**.

Two difficulties:

- the finite extension  $K/\kappa(p)$  may be inseparable
- the map  $\phi': Z' \rightarrow Y$  may have wild ramification

*Note:* The first issue does not occur in dimension 2.

### Problem

Gain control on the curve selection lemma to overcome these difficulties.