The Nash problem on families of arcs

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The arc space of a variety

Let X be an algebraic variety over $\mathbb{C}$.

The **arc space** X_∞ of X is a scheme whose K-valued points are arcs

$$\alpha\colon \operatorname{Spec} K[[t]] \to X$$

$$0 \in \operatorname{Spec} K[[t]] \text{ closed point} \qquad \eta \in \operatorname{Spec} K[[t]] \text{ generic point}$$

Canonical projection

$$\pi = \pi_X\colon X_\infty \to X, \qquad \alpha(t) \mapsto \alpha(0)$$

Note: Every arc α defines a valuation

$$\operatorname{val}_\alpha\colon \mathcal{O}^*_{X,\alpha(0)} \to K[[t]]^* \to \mathbb{Z}$$

If $\alpha(\eta) \in X$ is the generic point of X, the valuation extends to X

$$\operatorname{val}_\alpha\colon \mathbb{C}(X)^* \to K((t))^* \to \mathbb{Z}$$

Example

If $X = \{f(x_1, \dots, x_n) = 0\} \subset \mathbb{A}^n$, then X_∞ parametrizes n-ples of formal power series $\big(x_1(t), \dots, x_n(t)\big)$ such that

$$f\big(x_1(t), \dots, x_n(t)\big) \equiv 0$$

This condition describes X_∞ as a subscheme in an infinite dimensional affine space (the arc space of $\mathbb{A}^n$), defined by *infinitely* many equations in *infinitely* many variables.

In general, X_∞ is not Noetherian and is not of finite type.

X_∞ can also be described as the inverse limit of the **jet schemes** of X:

$$X_\infty = \varprojlim X_m, \quad \text{where} \quad X_m(K) = \{\gamma \colon \operatorname{Spec} K[t]/t^{m+1} \to X\}$$

If X is smooth, then X_∞ is the projective limit of a system of affine bundles, and hence it is irreducible.
(In fact, $\pi_X^{-1}(S)$ is irreducible for every irreducible set $S \subset X$.)

If X is singular, we take a resolution of singularities $f\colon Y \to X$

$$\begin{tikzcd} Y_\infty \arrow[r, "f_\infty"] \arrow[d, "\pi_Y"'] & X_\infty \arrow[d, "\pi_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

By valuative criterion of properness and generic smoothness, f_∞ is dominant.

Theorem (Kolchin 1973)

The arc space X_∞ of any variety X is irreducible.

Note: $\pi_X^{-1}(S)$ may be reducible for an irreducible set $S \subset X$.

Nash's Theorem

Theorem (Nash 1968)

The set of arcs through the singular locus of X has a finite decomposition into irreducible components:

$$\pi_X^{-1}(\operatorname{Sing} X) = \bigcup_{i \in I} C_i \quad \textit{where } I \textit{ is finite}$$

Proof. For every irreducible component E of $f^{-1}(\operatorname{Sing} X)$, let

$$\pi_Y^{-1}(E) \subset Y_\infty \xrightarrow{f_\infty} X_\infty \supset f_\infty(\pi_Y^{-1}(E)) = N_E$$

Then

$$f^{-1}(\operatorname{Sing} X) = \bigcup_{i \in J} E_i \quad \Rightarrow \quad \pi_X^{-1}(\operatorname{Sing} X) = \bigcup_{i \in I} \overline{N_{E_i}}$$

where J is finite and $I \subset J$.

To every irreducible component C_i of $\pi_X^{-1}(\operatorname{Sing} X)$ there corresponds a valuation val_{α_i} on X, defined by the generic point $\alpha_i \in C_i$.

- We call any valuation defined in this way a **Nash valuation** over X.

Note: Each Nash valuation is a (distinct) divisorial valuation: by further blowing up, we may assume that each E_i is a divisor, and we have

$$\mathrm{val}_{\alpha_i} = \mathrm{val}_{\widetilde{\alpha}_i} = \mathrm{val}_{E_i} \qquad (i \in I)$$

Nash valuations are intrinsic to X.

Question

Which divisorial valuations occur as Nash valuation?

The Nash Problem

- A divisorial valuation val_E is **essential** over X if its center on every resolution $g\colon X' \to X$ is an irreducible component of $g^{-1}(\operatorname{Sing} X)$.

Note: Every Nash valuation over X is essential.

Define the **Nash map** $\mathcal{N}_X$ by the diagram

$$\begin{tikzcd}
\{\text{components of } \pi^{-1}(\operatorname{Sing} X)\} \arrow[d, equal, "1-1"'] \arrow[drr, "\mathcal{N}_X"] & & \\
\{\text{Nash valuations over } X\} \arrow[rr, hook] & & \{\text{essential valuations over } X\}
\end{tikzcd}$$

By construction, the Nash map is injective.

Nash Problem (1968)

Is the Nash map surjective?

Results on the Nash Problem

- If $\dim X = 2$, then val_E is essential over X if and only if E is an exceptional divisor in the *minimal resolution* of singularities of X.

The 2-dimensional Nash problem asks if every such valuation val_E is a Nash valuation, i.e., comes from a maximal family of arcs through the singularities of X. The complexity of the equations of X_∞ makes the problem very hard, by a direct approach, even for relatively simple singularities.

Theorem (Fernandez de Bobadilla and Pe Pereira 2012)

In dimension 2, the Nash map is surjective.

The proof is a combination of a standard approach via a *Curve Selection Lemma* (Lejeune-Jalabert, Reguera) with new topological methods.

- If $\dim X \geq 3$, then the notion of essential valuation is more subtle ($\nexists$ minimal resolutions, $\exists$ small resolutions,...)

In principle, one needs to check the condition on the center of the valuation on *every* resolution of singularities.

Theorem (Ishii–Kollár 2003, de Fernex 2013, Johnson–Kollár 2013)

There are examples in all dimensions ≥ 3 where the Nash map is not onto.

Easiest examples are 3-dimensional cA_1-singularities:

$$X : \quad x^2 + y^2 + z^2 + t^m = 0 \quad \text{where } m \geq 5 \text{ is odd}$$

Other examples show that the notion of **essential valuation** depends on the topology (Zariski vs analytic): in such an example the Nash map is surjective in the analytic category, but not in the algebraic category.

Nash Problem (revisited)

Describe the image of the Nash map.

We are looking for additional conditions (apart from being essential) that ensure that a valuation is a Nash valuation.

- A divisorial valuation val_E is a **terminal valuation** over X if E is an exceptional divisor on a minimal model $Y \to X$.
  - Y has terminal singularities
  - K_Y is relatively nef over X

Note: The set of terminal valuations corresponds to the set of exceptional divisors on *any given* minimal model $Y \to X$.

Theorem (de Fernex and Docampo 2014)

Every terminal valuation over X is a Nash valuation.

We obtain inclusions (both may be strict)

$$\{\text{terminal valuations}\} \subset \{\text{Nash valuations}\} \subset \{\text{essential valuations}\}$$

This gives a new (purely algebro-geometric) proof of the theorem of Fernandez de Bobadilla and Pe Pereira.

Also get new examples where the Nash map is onto (e.g., when X admits a divisorial crepant resolution).

Sketch of the proof

Theorem (de Fernex and Docampo 2014)

Every terminal valuation over X is a Nash valuation.

Proof by contradiction: Suppose there is a terminal valuation that is not a Nash valuation.

Then there is an exceptional divisor E on a minimal model $f\colon Y \to X$ such that

$$\overline{N_E} \subset \pi_X^{-1}(\operatorname{Sing} X) \text{ is } \textit{not} \text{ an irreducible component} \qquad (*)$$

$$\begin{array}{ccccccc} \pi_Y^{-1}(E) & \subset & Y_\infty & \xrightarrow{f_\infty} & X_\infty & \supset & N_E = f_\infty(\pi_Y^{-1}(E)) \\ & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} & & \\ E & \subset & Y & \xrightarrow{f} & X & & \end{array}$$

$$\overline{N_E} \subset \pi_X^{-1}(\operatorname{Sing} X)$$

Fix a very general point $p \in E \subset Y$ of dimension $n-2$, where $n = \dim X$.

Curve Selection Lemma (Reguera 2006 + specialization)

$\exists$ *finite* extension $K/\kappa(p)$
$\exists$ $\Phi\colon \operatorname{Spec} K[[s]] \to X_\infty$ (an arc on the arc space)
such that

- $\Phi(0) = \alpha_0 \in N_E$ is an arc on X with $\widetilde{\alpha}_0(0) = p$ and $\operatorname{ord}_{\widetilde{\alpha}_0}(E) = 1$
- $\Phi(\eta) = \alpha_\eta \in \pi_X^{-1}(\operatorname{Sing} X) \smallsetminus \overline{N_E}$

Can regard Φ as a *1-parameter family of arcs* on X, or as a map

$$\Phi\colon S = \operatorname{Spec} K[[s,t]] \to X, \quad \Phi(s,t) =: \alpha_s(t)$$

We say that Φ is a **wedge** on X.

Note: The curve selection lemma is a simple fact for Noetherian schemes, but it can fail in the non Noetherian setting.

Example (The Withney umbrella)

Let $X = \{xy^2 = z^2\} \subset \mathbb{A}^3$ and

- $M = \pi_X^{-1}(0) \subset X_\infty$
- $N = \{(x(t), 0, 0) \in X_\infty \mid \mathrm{ord}(x(t)) \geq 1\} \subset M$
- $\alpha = (t, 0, 0) \in N$

Then $\nexists\, \Phi \colon \operatorname{Spec} \mathbb{C}[[s]] \to M$ with $\Phi(0) = \alpha$ and $\Phi(\eta) \in M \smallsetminus N$.

Proof: Suppose $\Phi(\eta) \in M \smallsetminus N$. Then

$$\Phi(\eta) = (x_\eta(t), y_\eta(t), z_\eta(t)) \quad \text{with} \quad y_\eta(t), z_\eta(t) \not\equiv 0$$

$$\textit{odd} \quad \mathrm{ord}(x_\eta(t) y_\eta(t)^2) = \mathrm{ord}(z_\eta(t)^2) \quad \textit{even}$$

since $\mathrm{ord}(x_\eta(t)) = 1$ by semicontinuity

Back to the proof: Have a wedge $\Phi\colon \operatorname{Spec} K[[s]] \to X_\infty$ such that

(1) $\Phi(0) = \alpha_0 \in N_E$, with $\widetilde{\alpha}_0(0) = p$ and $\operatorname{ord}_{\widetilde{\alpha}_0}(E) = 1$
(2) $\Phi(\eta) = \alpha_\eta \in \pi_X^{-1}(\operatorname{Sing} X) \smallsetminus \overline{N_E}$

$$\Phi\colon S = \operatorname{Spec} K[[s,t]] \to X$$

Note that S is a 2-dimensional scheme, but has 'dimension' n over $\mathbb{C}$ $(\mathrm{tr.deg}(K/\mathbb{C}) = n - 2)$.

Conditions (1) and (2) imply that the lift $\widetilde{\Phi}$ is not well-defined:

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ S \arrow[ur, dashed, "\widetilde{\Phi}"] \arrow[r, "\Phi"] & X \end{tikzcd}$$

We take a resolution of indeterminacy of $\widetilde{\Phi}$.

Two natural ways to resolve indeterminacies:

$$\begin{tikzcd}
G & \subset & Z \arrow[dr, "h"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "g"'] & & & & \\
& & & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \arrow[d, "f"] & \supset & E \\
& & & S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X & &
\end{tikzcd}$$

- g' is the normalized blowup of the indeterminacy ideal of $\widetilde{\Phi}$
- g is the minimal sequence of point-blowups resolving indeterminacies
- h contracts any g-exceptional curve that is contracted to a point by ϕ
- G is the g-exc divisor intersecting the (proper transform of the) t-axis

Note:

- Z is smooth and Z' has rational singularities, hence is $\mathbb{Q}$-factorial
- $\phi\colon Z \to Y$ is dominant and $p \in \phi(G) \subset E$

Define relative canonical divisors

- $K_{Z/S}$ the relative canonical divisor of $g\colon Z \to S$
- $K_{Z'/S}$ the relative canonical divisor of $g'\colon Z \to S$
- $K_{Z'/Y}$ the relative canonical divisor of $\phi'\colon Z' \to Y$

Recall that $f\colon Y \to X$ is a *minimal model* over X:

$$\begin{aligned} Y \text{ terminal singularities} \quad &\Rightarrow \quad K_{Z'/Y} \geq 0 \\ K_Y \ f\text{-nef} \quad &\Rightarrow \quad (\phi')^* K_Y \ g'\text{-nef} \end{aligned}$$

Decompose

$$K_{Z'/Y} = K_{Z'/Y}^{g'\text{-exc}} + K_{Z'/Y}^{g'\text{-hor}}$$

Let $F_1, \dots, F_m$ be the components of $\mathrm{Ex}(g')$ containing $h(G) \subset Z'$.

Note: Each F_i dominates E via $\phi'\colon Z' \to Y$.

$$
\begin{tikzcd}
G \subset Z \arrow[dr, "h"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "g"'] & & \\
F_i \subset & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \supset E \arrow[d, "f"] \\
& S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X
\end{tikzcd}
$$

$$1 \leq \operatorname{ord}_G(K_{Z/S}) \leq \operatorname{ord}_G(h^* K_{Z'/S}) \leq \operatorname{ord}_G(h^* K^{g'\text{-exc}}_{Z'/Y}) < \operatorname{ord}_G(\phi^* E) = 1$$

$$\begin{tikzcd}
G & \subset & Z \arrow[drr, "h"] \arrow[drrr, bend left, "\phi"] \arrow[ddrr, bend right, "g"'] & & & & & \\
& & F_i & \subset & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \arrow[d, "f"] & \supset & E \\
& & & & S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X & &
\end{tikzcd}$$

$$1 \leq \mathrm{ord}_G(K_{Z/S}) \leq \mathrm{ord}_G(h^* K_{Z'/S}) \leq \mathrm{ord}_G(h^* K^{g'\text{-exc}}_{Z'/Y}) < \mathrm{ord}_G(\phi^* E) = 1$$

because S is smooth and G is g-exceptional.

$$
\begin{tikzcd}
G & \subset & Z \arrow[drr, "h"] \arrow[drrr, bend left, "\phi"] \arrow[ddrr, bend right, "g"'] & & & & \\
 & & F_i & \subset & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \arrow[d, "f"] & \supset & E \\
 & & & & S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X & &
\end{tikzcd}
$$

$$1 \leq \mathrm{ord}_G(K_{Z/S}) \leq \mathrm{ord}_G(h^* K_{Z'/S}) \leq \mathrm{ord}_G(h^* K^{g'\text{-exc}}_{Z'/Y}) < \mathrm{ord}_G(\phi^* E) = 1$$

because

$$K_{Z/S} - K_{Z'/S} = K_{Z/Z'}$$

is h-nef and h-exceptional (h is the minimal resolution)
hence ≤ 0 (by Hodge Index Theorem).

$$
\begin{tikzcd}
G & \subset & Z \arrow[drr, "h"] \arrow[drrr, bend left, "\phi"] \arrow[ddrr, bend right, "g"'] & & & & & \\
& & F_i & \subset & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \arrow[d, "f"] & \supset & E \\
& & & & S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X & &
\end{tikzcd}
$$

$$
1 \leq \mathrm{ord}_G(K_{Z/S}) \leq \mathrm{ord}_G(h^* K_{Z'/S}) \leq \mathrm{ord}_G(h^* K^{g'\text{-exc}}_{Z'/Y}) < \mathrm{ord}_G(\phi^* E) = 1
$$

because

$$
K_{Z'/S} - K^{g'\text{-exc}}_{Z'/Y} \sim K_{Z'} - K_{Z'/Y} + K^{g'\text{-hor}}_{Z'/Y} \sim (\phi')^* K_Y + K^{g'\text{-hor}}_{Z'/Y}
$$

is g'-nef and g'-exceptional (Y is a minimal model)
hence ≤ 0 (by Hodge Index Theorem).

$$
\begin{tikzcd}
G & \subset & Z \arrow[drr, "h"] \arrow[drrr, bend left, "\phi"] \arrow[ddrr, bend right, "g"'] & & & & \\
 & & F_i & \subset & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \arrow[d, "f"] & \supset & E \\
 & & & & S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X & &
\end{tikzcd}
$$

$$1 \le \operatorname{ord}_G(K_{Z/S}) \le \operatorname{ord}_G(h^* K_{Z'/S}) \le \operatorname{ord}_G(h^* K^{g'\text{-exc}}_{Z'/Y}) < \operatorname{ord}_G(\phi^* E) = 1$$

because

$$\operatorname{ord}_{F_i}(K_{Z'/Y}) = \operatorname{ord}_{F_i}((\phi')^* E) - 1 \quad \forall i$$

by Hurwitz-type computation, hence

$$\operatorname{ord}_G(h^* K_{Z'/Y}) < \operatorname{ord}_G(\phi^* E)$$

since Z' is $\mathbb{Q}$-factorial.

$$
\begin{tikzcd}
G & \subset & Z \arrow[drr, "h"] \arrow[drrr, bend left, "\phi"] \arrow[ddrr, bend right, "g"'] & & & & & \\
& & F_i & \subset & Z' \arrow[r, "\phi'"] \arrow[d, "g'"'] & Y \arrow[d, "f"] & \supset & E \\
& & & & S \arrow[r, "\Phi"] \arrow[ur, dashed, "\widetilde{\Phi}"] & X & &
\end{tikzcd}
$$

$$1 \le \mathrm{ord}_G(K_{Z/S}) \le \mathrm{ord}_G(h^* K_{Z'/S}) \le \mathrm{ord}_G(h^* K^{g'\text{-exc}}_{Z'/Y}) < \mathrm{ord}_G(\phi^* E) = 1$$

because ϕ maps the t-axis to an arc $\widetilde{\alpha}_0$ on Y with

$$\mathrm{ord}_{\widetilde{\alpha}_0}(E) = 1.$$

Closing remarks

(1) **Minimal valuations** over X (in the valuative sense) are also examples of Nash valuations:

$$\text{if } \beta \text{ is a specialization of } \alpha \text{ in } X_\infty, \text{ then } \mathrm{val}_\beta \geq \mathrm{val}_\alpha.$$

- For *surface singularities*:

$$\{\text{minimal}\} \subset \{\text{terminal}\} = \{\text{Nash}\} = \{\text{essential}\}$$

- For *toric singularities* (Ishii–Kollár 2003):

$$\{\text{terminal}\} \subset \{\text{minimal}\} = \{\text{Nash}\} = \{\text{essential}\}$$

(both sets can be described combinatorically)

Question

For which class of varieties one has

$$\{\text{minimal valuations}\} \cup \{\text{terminal valuations}\} = \{\text{Nash valuations}\} \ ?$$

(2) There are no terminal valuations if X has terminal singularities

Problem

Study Nash valuations over terminal singularities.

(3) Most of the proof extends to positive characteristics.

Two difficulties:

- the finite extension $K/\kappa(p)$ may be inseparable
- the map $\phi'\colon Z' \to Y$ may have wild ramification

Note: The first issue does not occur in dimension 2.

Problem

Gain control on the curve selection lemma to overcome these difficulties.